ISSN 0348-7652

REPORTS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF STOCKHOLM
SWEDEN

SEMINAR ON SUPERMANIFOLDS NO 26

edited by

D. Leites

1988 - No 8

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Non-commutative Affine Semischemes and Schemes

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Contents

Introduction

1. Recall the main concepts and facts, participating in the construction of affine schemes, to which this paper calls - implicitly or explicitly.

Localizations. To a multiplicative subset S of a ring R the quotient ring $S^{-1}R$ is assigned, as well as to each R-module M the quotient module $S^{-1}M$. The ring $S^{-1}R$ is the "minimal" R-algebra (universal arrow) among the R-algebras $f:R \longrightarrow R'$ such that $f(s)$ is invertible for each $s \in S$. The isomorphism $S^{-1}M \simeq S^{-1}R \otimes_R M$ takes place.

Spectrum. The set of points of the space SpecR consists of the prime ideals of R, i.e. of ideals p such that their complement S_p is a multiplicative set. The closed sets are the sets $V(\mathfrak{B}) = \{p \in \mathrm{Spec}R \mid \mathfrak{B} \subset p$, where $\mathfrak{B}$ runs through the set IR of all the ideals of $R\}$. It is easy to see that the sets $U(s) = \mathrm{Spec}R \setminus V(Rs)$, $s \in R$, constitute a basis of open neighbourhoods of the topology of the space SpecR.

Globalization. For each $t \in R$ set $(t) = \{t^k \mid k \geq 1\}$.

Theorem. For each unitary R-module M there exists a unique sheaf $\widetilde{M}$ of R-modules over SpecR such that $\widetilde{M}(U(s)) = \Gamma(U(s), \widetilde{M}) = (s)^{-1}M$ for each $s \in R$.

The stalk $\widetilde{M}_p$ of the sheaf $\widetilde{M}$ at a point $p \in \mathrm{Spec}R$ is isomorphic to the localization $M_p = S_p^{-1}M$ of M.

The sheaf $\widetilde{M}$ is a sheaf of R-modules.

All the stalks $\widetilde{R}_p = S_p^{-1}R$ of the structural sheaf $\widetilde{R}$ are local rings.

The correspondence $M \longmapsto \widetilde{M}$ defines the equivalence beween the category of R-modules and the category of quasicoherent sheaves of R-modules.

The correspondence $R \longmapsto (\mathrm{Spec}R, \widetilde{R})$ extends in a natural way to contravariant full and faithful functor from the category of commutative rings with unit into the category of locally ringed spaces.

How do the analogues of these concepts and results look like (and do they exist) for non-commutative rings?

The search for an answer in the direction of the straightforward generalizations leads to the following:

Localizations. It is also possible here to try to assign to a multiplicative set $S \subset R$ the quotient ring. But from the very beginning the attention should be paid not to confuse the left with the right.

A pair (B,f), where $f: R \longrightarrow B$ is a ring morphism, is called the left quotient ring of R, if the following conditions are satisfied:

1) $f(s)$ is invertible for all $s \in S$;

2) any element of B is of the form $f(s)^{-1}f(r)$, where $s \in S$, $r \in R$;

3) if $f(r)=0$, then $sr=0$ for some $s \in S$.

The necessary and sufficient conditions for the existence of the left quotient ring with the denominators in S are the following left Ore conditions:

(O1) for each $s \in S$, $r \in R$ there exist $s' \in S$ and $r' \in R$ such that $s'r=r's$;

(O2) if $r \in R$, $s \in S$ and $rs=0$, then $tr=0$ for some $t \in S$.

Spectrum. A straightforward analogue of prime ideals is the completely prime ideals. Recall that a two-sided ideal p in R is called completely prime if the set $S_p = R \setminus p$ is multiplicative.

Now note that the elements $t \in R$ such that the multiplicative set $(t)=\{t^k \mid k \geqslant 1\}$ satisfies the Ore conditions (O1), (O2) are, in general, rare; for "almost all" rings the completely prime ideals form a rather meagre set.

The fiasco at the first two steps makes futher progress meaningless -- there is no hope to recover the rings and modules from the values at the rare points of the seldom existing quotient rings and modules. The way out of this situation, inspired by the remarkable paper by Gabriel [1], is to modify the notion of localization.

Gabriel's localizations. In the Gabriel theory the multiplicative sets of elements are replaced by the idempotent topologizing sets $\mathfrak{F}$ of the left ideals of R (otherwise called radical sets or radical filters). They are described by the following axioms:

(1) $R \in \mathfrak{F}$;

(2) if $m \in \mathfrak{F}$, then the ideal $(m:x)=\{z \in R \mid zx \in m\}$ also belongs to $\mathfrak{F}$ for each $x \in R$;

(3) if m and n are left ideals of R such that $m \in \mathfrak{F}$ and $(n:x) \in \mathfrak{F}$ for each $x \in m$, then $n \in \mathfrak{F}$.

To a radical filter $\mathfrak{F}$ the Gabriel functor $G_{\mathfrak{F}}: R\text{-mod} \longrightarrow R\text{-mod}$ and the natural transformation $j_{\mathfrak{F}} = \{j_{\mathfrak{F},M} : M \longrightarrow G_{\mathfrak{F}} M\}$ of the identity functor into $G_{\mathfrak{F}}$ correspond. The module $G_{\mathfrak{F}} R$ turns out to be a ring with unit, $j_{\mathfrak{F},R}$ a morphism of unitary rings, and for each R-module M the structure R-module on $G_{\mathfrak{F}}M$ extends naturally to a structure $G_{\mathfrak{F}}R$-module

A connection with the quotient rings is realised in the following way: to any multiplicative set $S \subset R$ there is associated the set F_S of left ideals m such that $(m:x) \cap S \neq \emptyset$ for each $x \in R$. It is not difficult to show (exercise 22 to chapter 2 in [3]) that F_S is a radical filter, and the following conditions are equivalent:

(a) R has a left quotient ring with denominators in S;

(b) the canonical morphism $j_{F_S,R}: R \longrightarrow G_{F_S}R$ sends the elements from S into invertible elements of the ring $G_{F_S}R$;

(c) the pair $(G_{F_S}R, j_{F_S,R})$ is a left quotient ring of R with denominators in S.

If R is commutative, then the functor G_{F_S} is isomorphic to the functor S^{-1}.

For any radical filter $\mathcal{F}$ the functor $G_{\mathcal{F}}$ takes values in the full subcategory R-mod$_{\mathcal{F}}$ of R-mod, formed by all the modules M, for which $j_{\mathcal{F},M}$ is an isomorphism. The induced functor $\mathcal{F}^{-1}$:R-mod$\longrightarrow$R-mod$_{\mathcal{F}}$ is called the <u>localizing functor</u> defined by a filter $\mathcal{F}$ or the <u>$\mathcal{F}$-localization</u>. The main properties of the functor $\mathcal{F}^{-1}$: it is exact and left-adjoint to the embedding R-mod$_{\mathcal{F}} \hookrightarrow$ R-mod. The latter means that any morphism $M \longrightarrow N$ of R-modules, where N belongs to R-mod , may be uniquely represented as a composition $M \longrightarrow G_{\mathcal{F}}M \longrightarrow N$. Conversely, any full subcategory C of the category R-mod such that the embedding $C \hookrightarrow$ R-mod has an exact left-adjoint functor, coincides with R-mod$_{\mathcal{F}}$ for a uniquely defined radical filter $\mathcal{F}$. This means that any attempt to construct localizations of modules with nice functorial properties -- the commutability with colimits and exactness -- leads inevitably to a Gabriel localization.

So, the notion of the Gabriel localization is the initial point this work. The interpretation of the other notions -- spectrum and globalization -- belongs to its essential part, to which we pass now.

II. Contents. The following informal remark seems to me essential for understanding the essence of what will follow.

The geometry of a commutative ring R is needed for the construction in a category (preorder) of its ideals IR (inclusions serving as morphisms). If R is non-commutative, then at the first glance the role of IR is *usurped* by the category of two-sided ideals denoted by the same symbol IR (this is actually the case in the unique known to me monograph in non-commutative algebraic geometry [4]) unless the category $I_\ell R$ of the left ideals of R (the inclusions also serve as morphisms) takes over. From the point of view of this paper this is not so. The left inheritor of IR is the category of preorder $I_\ell^{\succ} R$ whose objects are all the left ideals and the arrows (ordering) are determined as follows: $m \rightarrow n$ if either $m \subset n$ or there exists a finite subset $x \subset R$ such that $(m:x) \overset{\text{def}}{=} \{ \lambda \in R \mid \lambda x \subset m \}$ belongs to n.

Now describe the results. Unless otherwise stated, the modules and ideals are supposed to be left ones and rings non-unitary.

The main notions dealt with in § 1 : a uniform filter of left ideals, Gabriel's multiplication of sets of left ideals and, the most important, the left spectrum.

A uniform filter $\mathcal{F}$ is a filter in $I_\ell^{\succ} R$; i.e. $[n \in I_\ell R, m \in \mathcal{F}$ and $m \rightarrow n] \Longrightarrow [n \in \mathcal{F}]$.
The multiplication is determined as follows: $\mathcal{F} \circ \mathcal{G} = \{ n \in I_\ell R \mid (n:x) \in \mathcal{F}$ for any $x \in \mathcal{P}(m)$ where m is an ideal from $\mathcal{G} \}$; here $\mathcal{P}(m)$ is a family of family of finitely generated Z-submodules in m. Notice

that uniform filters may be determined as the filters $\mathcal{F}$ of left ideals ($[m \in \mathcal{F}$ and $m \subset n] \Rightarrow [n \in \mathcal{F}]$) such that $\mathcal{F} = \mathcal{F} \circ \{R\}$; and the radical filters are the filters $\mathcal{F}$ which are idempotents: $\mathcal{F} = \mathcal{F} \circ \mathcal{F}$.

With every left ideal m of R a uniform filter $\mathcal{F}_m = \{n \in I_\ell R \mid n \not\to m\}$ is connected. The left spectrum $Spec_\ell R$ is the set of all the left ideals m for which $\mathcal{F}_m$ is radical. An ideal m belongs to $Spec_\ell R$ if and only if any of the following statement hold:

a) $m \notin \mathcal{F}_m \circ \mathcal{F}_m$;

b) $[m \in \mathcal{F} \circ \mathcal{G}] \Rightarrow [m \in \mathcal{F} \cup \mathcal{G}]$ for any pair $\mathcal{F}, \mathcal{G}$ of uniform filters;

c) if n is a left ideal and $(m:x) \not\to m$ for any $x \in \mathcal{P}(n)$, then $n \to m$

An important role in the majority of this paper is played by the subset $\widehat{Spec}_\ell R$ of the left spectrum formed by all the left ideals p of R such that $(p:x) \simeq p$ for any $x \in R \setminus p$

We prove (Proposition 1.6) that for any $m \in Spec_\ell R$ the set $\widehat{m} = \{\lambda \in R \mid (m:\lambda) \not\to m\}$ is an ideal from $\widehat{Spec}_\ell R$ and the inclusion $m \subset \widehat{m}$ is an isomorphism in the category $I_\ell^{\succ} R$. In particular $\mathcal{F}_m = \mathcal{F}_{\widehat{m}}$ for any $m \in Spec_\ell R$. We have the following "estimate from below" for $\widehat{Spec}_\ell R$: the set $Max_\ell^{reg} R$ of the maximal regular left ideals of R (annihilators of the non-zero elements of irreducible R-modules) belongs to $\widehat{Spec}_\ell R$

In §2 we list the data on localizations of the category of modules. They include the above-mentioned known facts on Gabriel's functors, the formulas for localizing functors modulo non-radical filters (we start from the notion of localization modulo an arbitrary set of left ideals) and corollaries of these formulas which describe localizations of modules modulo filters generated by a family of ideals of finite type. Besides we study the relations of radical filters and a prime spectrum, the behaviour of $\mathrm{Spec}_\ell R$ and its subset $\widehat{\mathrm{Spec}}_\ell R$ under localizations, the properties of inductive limits of localizations.

The following part of the paper (§§ 3 and 4) are devoted to globalization in the context of $\amalg$-semischeme. $\amalg$-semischemes are the pairs $(R, \mathcal{T})$ where R is a ring and $\mathcal{T}$ a category of radical filters (inclusions serving as morphisms) such that $\mathcal{T}$ contains together with every pair of filters $\mathcal{F}$ and $\mathcal{G}$ their intersection $\mathcal{F} \cap \mathcal{G}$ and the co-product $\mathcal{F} \amalg \mathcal{G}$ (equal to intersection of all the filters from $\mathcal{T}$ containing $\mathcal{F}$ and $\mathcal{G}$). The category $\mathcal{T}$ is interpreted as a preorder of a closed sets in a "topology". The structure of the topology is given the co-coverings (co since we are speaking about closed sets): a co-covering of a filter $\mathcal{F}$ from $\mathcal{T}$ is a family $\{\mathcal{F} \hookrightarrow \mathcal{F}_i \mid i \in I\} \subset \mathcal{T}$ such that $\bigcap\{\mathcal{F}_i \mid i \in I\} = \mathcal{F}$. The family of co-coverings will be denoted by $\widehat{Cov}\,\mathcal{T}$ or $\mathcal{T}$. A pair

$(\mathcal{T}, \widehat{Cov}\,\mathcal{T})$ resembles in every respect the structure of closed sets of a topological space except one, but, perhaps, the most essential: the co-restriction of a co-covering is not a co-covering in general, i.e. $(\bigcap_{i\in I}\mathcal{F}_i)\amalg \mathcal{G} \subset \bigcap_{i\in I}(\mathcal{F}_i \amalg \mathcal{G})$ is usually a strict embedding.

Though exotic, the "topology" $\underline{\mathcal{T}} = (\mathcal{T}, \widehat{Cov}\,\mathcal{T})$ possesses all what is needed to translate directly the notions of presheaf and sheaf from topological spaces. Presheaves on $(\mathcal{T}, \widehat{Cov}\,\mathcal{T})$ with values in a category $\mathcal{C}$ are arbitrary functors from $\mathcal{T}$ into $\mathcal{C}$; and a presheaf $F: \mathcal{T} \to \mathcal{C}$ is called a sheaf if the canonical diagram

$$F(\mathcal{G}) \longrightarrow \prod_{i\in I} F(\mathcal{G}_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(\mathcal{G}_i \amalg \mathcal{G}_j)$$

is exact for any $\{\mathcal{G} \hookrightarrow \mathcal{G}_i \mid i\in I\} \in \widehat{Cov}\,\mathcal{T}$.

The local data of a module M is a set of localizations $G_{\mathcal{F}}M \simeq \mathcal{F}^{-1}M$, $\mathcal{F} \in \mathcal{T}$, which, clearly, constitutes a presheaf on $(\mathcal{T}, \widehat{Cov}\,\mathcal{T})$ that will be denoted by $M_{\mathcal{T}}$.

The globalization is recovering the modules $G_{\mathcal{F}}M$ from the diagrams

$$G_{\mathcal{F}_i}M \longrightarrow G_{\mathcal{F}_i\amalg\mathcal{F}_j}M \longleftarrow G_{\mathcal{F}_j}M$$

where $\{\mathcal{F} \hookrightarrow \mathcal{F}_i \mid i\in I\} \in \widehat{Cov}\,\mathcal{T}$. When is such a recovering possible? Or, equivalently, when a presheaf $M_{\mathcal{T}}$ is a sheaf?

The key to the answer to this question is the following

Theorem 4.1. *Let* $\{\mathcal{F}_i \mid i \in I\}$ *be a finite family of radical filters, and* $\mathcal{F} = \bigcap\{\mathcal{F}_i \mid i \in I\}$. *For any R-module M the canonical diagram*

$$G_{\mathcal{F}} M \to \prod_{i \in I} G_{\mathcal{F}_i} M \rightrightarrows \prod_{(i,j) \in I \times I} G_{\mathcal{F}_i} G_{\mathcal{F}_j} M$$

is exact.

It is not difficult to derive from here that

If $M_{\mathcal{T}}$ is a sheaf, then

(#) For any pair of filters $\mathcal{F}$, $\mathcal{G}$ from $\mathcal{T}$ the $\mathcal{F} \circ \mathcal{G}$-torsion submodule $(\mathcal{F} \circ \mathcal{G})M = \{\xi \in M \mid m \cdot \xi = 0$ for some $m \in \mathcal{F} \circ \mathcal{G}\}$ coincides with the $\mathcal{G} \amalg \mathcal{F}$-torsions of module M.

If every co-covering from $\widehat{\mathrm{Cov}}\,\mathcal{T}$ possesses a finite sub-cocovering, then (#) means that M is a sheaf.

The adjacent text is devoted to deciphering the condition (#).

The further efforts in solving the globalization problem are connected with the notion of the *spectrum* of a $\amalg$-semischeme. The *spectrum of* $\amalg$*-semischeme* $(R, \mathcal{T})$ is the topological space $\mathrm{Spec}_\ell(R, \mathcal{T})$ whose points are all the left ideals p such that $[p \in \mathcal{F} \amalg \mathcal{G}] \Rightarrow [p \in \mathcal{F} \cup \mathcal{G}]$ for any two filters $\mathcal{F}$, $\mathcal{G}$ from $\mathcal{T}$; and the topology on $\mathrm{Spec}_\ell(R, \mathcal{T})$ is the weakest of the topologies for which all the sets $V_{\mathcal{F}} \overset{\mathrm{def}}{=} \mathrm{Spec}_\ell(R, \mathcal{T}) \cap \mathcal{F}$, $\mathcal{F} \in \mathcal{T}$, are closed. There is a canonical morphism $\varphi_{\mathcal{T}}$ from the "topology" $\underline{\mathrm{Spec}_\ell(R, \mathcal{T})}$ of closed sets of the topological space $\mathrm{Spec}_\ell(R, \mathcal{T})$ into the "topology" $\underline{\mathcal{T}} = (\mathcal{T}, \widehat{\mathrm{Cov}}\,\mathcal{T})$. We find out the conditions under which $\varphi_{\mathcal{T}}$ is an isomorphism (then the categories of presheaves and sheaves on $\mathrm{Spec}_\ell(R, \mathcal{T})$ and $\underline{\mathcal{T}}$ are isomorphic) and the relations of the prime spectrum $\mathrm{Spec}\, R$ with the space $\mathrm{Spec}_\ell(R, \mathcal{T})$.

The most satisfactory is the case when all the filters from $\mathcal{T}$ are symmetric and of bifinite type (i.e. contain a cofinal subset of finitely generated two-sided ideals). In this case $\varphi_{\mathcal{T}}$ is an isomorphism, $\mathrm{Spec}\, R \subset \mathrm{Spec}_\ell(R, \mathcal{T})$ and, moreover, for every closed subset $W \subset \mathrm{Spec}_\ell(R, \mathcal{T})$ the intersection $W \cap \mathrm{Spec}\, R$ is dense in W. Besides,

the embedding $\mathrm{Spec}\, R \hookrightarrow \mathrm{Spec}_\ell(R, \mathcal{T})$ is continuous if and only if $\mathcal{T}$ consists of the radical filters generated by two-sided ideals;

any radical filter generated by a two-sided ideal which is finitely generated as a left ideal is symmetric and of bifinite type.

This makes it clear that the most convenient for applications are left Noetherian rings.

To a left Noetherian ring R the "canonical" semi-scheme $(R, \mathcal{T}_{IR})$, where $\mathcal{T}_{IR}$ consists of radical filters generated by all the two-sided ideals of R, corresponds. The embedding $\mathrm{Spec}\, R \hookrightarrow \mathrm{Spec}_\ell(R, \mathcal{T}_{IR})$

is a quasihomeomorphism (induces an isomorphism of the categories of closed sets) and, therefore, the "topology" $\underline{\mathcal{T}_{IR}}$ is isomorphic to the "topology" of closed sets of the prime spectrum $\mathrm{Spec}\,R$. With the help of this isomorphism the structure presheaves $\mathcal{M}_{\mathcal{T}_{IR}}$ corresponding to R-module M are transfered onto $\mathrm{Spec}\,R$. Thus obtained geometric picture may be described directly:

To every closed set $\mathcal{V}(\alpha)=\{p\in \mathrm{Spec}\,R \mid \alpha\subset p\}$ assign the "radical closure" $\widehat{\{\alpha\}}$ of an ideal α; i.e. the intersection of the radical filters containing α (it is shown that $\widehat{\{\alpha\}}=\widehat{\{\alpha'\}}$ iff $\mathcal{V}(\alpha)=\mathcal{V}(\alpha')$); to every R-module M a presheaf $\widetilde{M}$ corresponds which sends $\mathcal{V}(\alpha)$ into $G_{\widehat{\{\alpha\}}}\mathcal{M}$. Among others the following statements are proved:

If R is a semiprime left Noetherian ring with unit, then a presheaf of rings $\widetilde{R}$ is a sheaf and the canonical morphism of R into the ring $\Gamma\widetilde{R}$ of the global sections of $\widetilde{R}$ is an isomorphism.

(Recall that a ring R is called semiprime if its lower Bair radical $\mathfrak{B}(R)\overset{\mathrm{def}}{=}\bigcap\{p \mid p\in \mathrm{Spec}\,R\}$ is zero.)

If M is a unitary module over a left Noetherian ring R and for any $\xi\in\mathcal{M}\smallsetminus\{0\}$ the annihilator of ξ contains Bair's radical of its symmetric part, i.e. $\mathrm{Ann}\,\xi\supset\mathfrak{B}((\mathrm{Ann}\,\xi)_s)\overset{\mathrm{def}}{=}\bigcap\{p \mid p\in\mathcal{V}((\mathrm{Ann}\,\xi)_s)\}$

(such modules are called here semiprim), then $\widetilde{M}$ is a sheaf and the canonical R-module morphism $\mathcal{M} \to \Gamma \widetilde{\mathcal{M}}$ an isomorphism.

The strongest results obtained so far in the same direction suggest that R is prime (i.e. 0 is a prime ideal), in addition to the left Noetherian property, and $(\mathrm{Ann}\,\xi)_s = \mathrm{Ann}\,R\xi = 0$ for all $\xi \in \mathcal{M} \setminus \{0\}$.

In §5 we study the left spectrum rigged with topologies and structure presheaves and sheaves. On $\mathrm{Spec}_\ell R$ we consider three topologies: $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$. The closure in $\mathfrak{T}_0$ assigns an arbitrary $W \subset \mathrm{Spec}_\ell R$ the set ${}^{\mathfrak{T}_0}\overline{W} = \{ p \mid \mu \to p$ for some $\mu \in W\}$; $\mathfrak{T}_1$ has for a base of closed sets the family $\{ V_\ell(n) \mid n \in I_\ell R \}$, where $V_\ell(n)$ is a family of all the $p \in \mathrm{Spec}_\ell R$ such that $n \nrightarrow p$; and the closed sets of the topology $\mathfrak{T}$ are all the $V_\ell(\alpha)$ where α runs the family IR of the two-sided ideals of R.

To every set $W \subset \mathrm{Spec}_\ell R$ we associate the radical filter $\mathcal{F}_W = \bigcap \{ \mathcal{F}_p \mid p \in W^\perp \overset{\mathrm{def}}{=} \mathrm{Spec}_\ell R \setminus W \}$, and to an arbitrary R-module M the presheaf ${}^0\mathcal{O}_{\mathcal{M}}$ on $(\mathrm{Spec}_\ell R, \mathfrak{T}_0)$ sending a closed set W into $G_{\mathcal{F}_W} \mathcal{M}$.
The restrictions of ${}^0\mathcal{O}_{\mathcal{M}}$ onto the topologies $\mathfrak{T}_1$ and $\mathfrak{T}$ are denoted by ${}^1\mathcal{O}_{\mathcal{M}}$ and $\mathcal{O}_{\mathcal{M}}$ respectively. Let us discuss

the peculiarities of the local behaviour of the associated sheaves ${}^{o}\mathcal{O}^{a}_{\mathcal{M}}$, ${}^{1}\mathcal{O}^{a}_{\mathcal{M}}$ and $\mathcal{O}^{a}_{\mathcal{M}}$.

The fibre of the sheaf ${}^{o}\mathcal{O}^{a}_{\mathcal{M}}$ associated with ${}^{o}\mathcal{O}_{\mathcal{M}}$ at $p \in \mathrm{Spec}_{\ell} R$ is isomorphic to $G_{\mathfrak{F}_p}\mathcal{M}$. Under certain "finiteness" conditions (e.g. $\mathfrak{F}_p$ contains a cofinite subset of left ideals of finite type, the condition which is clearly satisfied if R is commutative or left Noetherian; or if the torsion submodule $\mathfrak{F}_p\mathcal{M}$ is finitely generated) the some is true for the fibre at p of the sheaf ${}^{1}\mathcal{O}^{a}_{\mathcal{M}}$ associated with ${}^{1}\mathcal{O}_{\mathcal{M}}$; i.e. ${}^{1}\mathcal{O}^{a}_{\mathcal{M},p} \simeq G_{\mathfrak{F}_p}\mathcal{M}$.

The conditions similar to those listed in parentheses for the filter $\mathfrak{F}_{\langle p_s\rangle} \overset{def}{=} \bigcap\{\mathfrak{F}_{\mu} \mid \mu \in \mathrm{Spec}_{\ell} R,\ \mu_s \subset p_s\}$ lead to an isomorphism $\mathcal{O}^{a}_{\mathcal{M},p} \xrightarrow{\sim} G_{\mathfrak{F}_{p_s}}\mathcal{M}$.
In general there are natural embeddings
${}^{1}\mathcal{O}^{a}_{\mathcal{M},p} \rightarrowtail G_{\mathfrak{F}_p}\mathcal{M}$ and
$\mathcal{O}_{\mathcal{M},p} \rightarrowtail G_{\mathfrak{F}_{\langle p_s\rangle}}\mathcal{M}$ which differ but slightly from isomorphisms.

Notice that $G_{\mathfrak{F}_p}R$ and $G_{\mathfrak{F}_{\langle p_s\rangle}}R$ possess properties close to locality.

The first ring is left quasilocal, meaning that there exists a left ideal $\mu\ (= G_{\mathfrak{F}_p}p)$ such that for any left ideal n of $G_{\mathfrak{F}_p}R$ either $n \not\subset \mu$ or the natural map $G_{\mathfrak{F}_p}R \to \mathrm{Hom}_{G_{\mathfrak{F}_p}R}(n, G_{\mathfrak{F}_p}R)$ is a bijection (in the latter case n is called q-nonproper).

$G_{\mathfrak{F}_{\langle p_s \rangle}} R$ is symmetrically quasilocal, meaning that there exists a proper two-sided ideal $m\ (= (G_{\mathfrak{F}_{p_s}})_s)$ such that for any two-sided ideal n of $G_{\mathfrak{F}_{\langle p_s \rangle}} R$ either $n \subset m$ or n is a q-nonproper.

The second half of the section concentrates around the statements which are analogues of Jacobson's theorems on "main homeomorphisms" of the structural spaces (= the spaces of primitive ideals).

Proposition 5.9. Let α be a two-sided ideal of R.

1) The map $p \mapsto p/\alpha$ determines a homeomorphism of the closed subspace $V_\ell(\alpha) \subset (Spec_\ell R, \mathfrak{T})$ onto $(Spec_\ell R/\alpha, \mathfrak{T})$.

2) The map $p \mapsto p \cap \alpha$ is a quasihomeomorphism of an open subspace $U_\ell(\alpha) = Spec_\ell R \smallsetminus V_\ell(\alpha) = \{p \in Spec_\ell R \mid \alpha \not\subset p\}$ into $Spec_\ell \alpha$, and the homeomorphism of $\widehat{U}_\ell(\alpha) \overset{def}{=} U_\ell(\alpha) \cap \widehat{Spec}_\ell R$ onto $\widehat{Spec}_\ell \alpha$ with respect to the topologies $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$.

(Even this statement alone

Proposition 5.11. Let e be a non-zero idempotent in R and $\widehat{\mathcal{U}}_\ell(eRe)=\{p\in \widehat{Spec}_\ell R \mid eRe \not\subset p\}$

The map $p \mapsto p\cap eRe$

determines a homeomorphism of the subspace

$\widehat{\mathcal{U}}_\ell(eRe)$ of the space $(Spec_\ell R, \mathfrak{T})$

onto the subspace $\widehat{Spec}_\ell\, eRe$ of $(Spec_\ell eR, e, \mathfrak{T})$.

One of corollaries of Proposition 5.9, having a "classical" prototype in the theory of structural spaces, is the following one:

The map $p \longmapsto p\cap R$ determines a homeomorphism with respect to either of the three topologies $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$ of the subspace $\widehat{Spec}_\ell R^{(1)} \setminus Spec\, Z$ onto $\widehat{Spec}_\ell R$, where $R^{(1)}$ is the ring obtained by adjoining the unit to R, $Spec Z$ is identified with $\widehat{V}_\ell(R)=\{p\in \widehat{Spec}_\ell R^{(1)} \mid R\subset p\}$.

Among the corollaries of Proposition 5.11 are the having well-known prototypes statements on the relations of the structure of central idempotents of R and the structure of the open-closed subsets of $(Spec_\ell R, \mathfrak{T})$

One more important character of this section is the left radical $rad_\ell = rad_\ell^R$ the function assigning to every left ideal n of R the intersection $\bigcap\{p \mid p\in V_\ell(n)\}$, provided $V_\ell(n)=\{p\in Spec_\ell R \mid n \to p\}$ is non-empty, and R otherwise. It is easily verified that rad_ℓ^R is a functor from $I_\ell^{\succ} R$ into IR. With the help of Proposition 5.9 cited above we prove that the function $\widehat{rad}_\ell$ assigning to an associative ring R the two-sided ideal $rad^R(0)=\bigcap\{p \mid p\in Spec_\ell R\}$ is torsion; i.e. for any R, a ring morphism $\varphi: R \longrightarrow R'$

and a two-sided ideal α of R we have
$\varphi(\widehat{rad}_\ell(R)) \subset \widehat{rad}_\ell(\varphi(R))$, $\widehat{rad}_\ell(R/\widehat{rad}_\ell(R)) = 0$,
$\widehat{rad}_\ell(\alpha) = \widehat{rad}_\ell(R) \cap \alpha$.

The main happening of § 6 is appearance of left quasi-schemes and schemes. A left Affine quasi-scheme is a ringed space isomorphic to the ringed space $(\overline{Spec}\, R, \overline{\mathcal{O}}_R^a)$ for an associative ring R; here $\overline{Spec}$ R is the subspace of the prime spectrum formed by all the prime ideals which are intersections of ideals from $Spec_\ell R$, i.e. all the $p \in$ $\in$ Spec R satisfying an equation $p = rad_\ell(p)$, and $\overline{\mathcal{O}}_R$ the presheaf assigning to a closed subset $\overline{V}(\alpha) = \{p \mid \alpha \subset p\}$ of $\overline{Spec}$ R the ring $G\mathcal{F}_{V_\ell(\alpha)} R$, $\overline{\mathcal{O}}_R^a$ the sheaf associated with $\overline{\mathcal{O}}_R$. A left Affine quasi-scheme $(X, \mathcal{O})$ is called a left Affine scheme if $(X, \mathcal{O}) \simeq (\overline{Spec}\, R, \overline{\mathcal{O}}_R^a)$ where R is a ring with a unit. Left Affine quasi-schemes and schemes are the ringed spaces locally isomorphic to left Affine quasi-schemes and schemes respectively.

Notice that the map assigning to a left ideal p the two-sided ideal $p_s = p \cap (p : R)$ determines a quasi-homeomorphism $(Spec_\ell R, \mathfrak{T}) \longrightarrow \overline{Spec}\, R$, which naturally extends to a quasi-isomorphism of ringed spaces $(Spec_\ell R, \mathcal{O}_R^a) \to (\overline{Spec}\, R, \overline{\mathcal{O}}_R^a)$. The preference showed to Spec R is caused by its social advantages of above the same kind as those of Spec A as compared with the space Max A of the closed points even if A is the ring of regular functions on an Affine variety: isomorphism of the pre-order $cl\, X$ of the closed sets of a topological space X onto $cl\, \overline{Spec}\, R$ uniquely determines a quasi-homeomorphism $X \to \overline{Spec}\, R$.

We will verify that for any associative ring R and its two-sided ideal α the canonical bijection $\widehat{U}_\ell(\alpha) \xrightarrow{\sim}$ $\xrightarrow{\sim} \widehat{Spec}_\ell \alpha$, $p \longmapsto p \cap \alpha$, naturally induces isomorphisms of pre-ringed spaces

$$(\widehat{U}_\ell(\alpha), {}^\circ\mathcal{O}_R|\widehat{U}_\ell(\alpha)) \xrightarrow{\sim} (\widehat{Spec}_\ell \alpha, {}^\circ\mathcal{O}_\alpha),$$

$$(\widehat{U}_\ell(\alpha), {}^1\mathcal{O}_R|\widehat{U}_\ell(\alpha)) \xrightarrow{\sim} (\widehat{Spec}_\ell \alpha, {}^1\mathcal{O}_\alpha),$$

$$(\widehat{U}_\ell(\alpha), \mathcal{O}_R|\widehat{U}_\ell(\alpha)) \xrightarrow{\sim} (\widehat{Spec}_\ell \alpha, \mathcal{O}_\alpha),$$

$$(\overline{U}(\alpha), \overline{\mathcal{O}}_R|\overline{U}(\alpha)) \xrightarrow{\sim} (\overline{Spec}\,\alpha, \overline{\mathcal{O}}_\alpha).$$

This implies, besides other corollaries, that an open subspace of a left Affine quasi-scheme is a left Affine quasi-scheme:

$$(\overline{U}(\alpha), \overline{\mathcal{O}}{}^a_R|\overline{U}(\alpha)) \xrightarrow{\sim} (\overline{Spec}\,\alpha, \overline{\mathcal{O}}{}^a_\alpha)$$

One of the main statements of the section is the following one:

if R is a $\widehat{rad}_\ell$ -semisimple (i.e. $\widehat{rad}_\ell(R) = 0$) ring with unit, then the canonical morphism $R \longrightarrow \Gamma\overline{\mathcal{O}}{}^a_R$ is an isomorphism.

This fact is a corollary of a more general result:

Let M be a unitary module over an arbitrary associative ring with unit R such that $rad_\ell(Ann\,\xi) \subset Ann\,\xi$ for any $\xi \in \mathcal{M}$.
Then the canonical R-module morphism $\mathcal{M} \longrightarrow \Gamma\overline{\mathcal{O}}{}^a_{\mathcal{M}}$ is an isomorphism (c.f. with the statements of § 4 on semiprime rings and modules).

A ringed space $(X, \mathcal{O})$ is called reduced (or $\widehat{rad}_\ell$ -reduced) if $\mathcal{O}$ is a sheaf of rad_ℓ -semisimple rings. It is not difficult to verify that a left Affine scheme $(\overline{\mathrm{Spec}}R\ ,\ \bar{\mathcal{O}}^{a}_{R})$ is reduced if and only if R is $\widehat{rad}_{\ell}$ -semisimple.

§ 7 is devoted to geometrization whose destination is to serve irreducible modules, modules of finite length and rings semisimple in the sense of Jacobson (the rings with zero Jacobson radical). Actually we imitate the main stages of Sections 5 and 6 starting this time not from the left spectrum, but from the set $Max^{reg}_{\ell}R$ of regular left maximal ideals of R.

First of all notice that the ordering $\longrightarrow$ is expressed on $Max^{reg}_{\ell}R$ in extremely nice terms:

$[\mu \to \mu'] \Leftrightarrow [\mu \simeq \mu'] \Leftrightarrow [\mu' = (\mu : t)$ for some $t \in R \setminus \mu] \Longleftrightarrow$ [the simple R-modules R/μ and R/μ' are isomorphic].

To an arbitrary subset $X \subset Max^{reg}_{\ell}R$ we assign the radical filter $\dot{\mathfrak{F}}_X = \bigcap\{\mathfrak{F}_{\mu} \mid \mu \in Max^{reg}_{\ell}R \setminus X\}$; any topology τ on $\mathrm{Spec}_{\ell}R$ induces a topology $\hat{\tau}$ on $Max^{reg}_{\ell}R$, and to any R-module M the presheaf on $(Max^{reg}_{\ell}R\ ,\ \hat{\tau})$ corresponds which sends a closed set W into $G_{\dot{\mathfrak{F}}_W}M$. Thus, to the topology $\hat{\mathfrak{T}}_1$ we assign the presheaf ${}^{1}\hat{\mathcal{O}}_M$ and to the topology $\hat{\mathfrak{T}}$ the presheaf $\hat{\mathcal{O}}_M$ (the topology $\hat{\mathfrak{T}}_0$ is of hardly any interest since $(Max^{reg}_{\ell}R\ ,\ \hat{\mathfrak{T}}_0)$ is quasi-homeomorphic to the discrete space $\sim Max^{reg}_{\ell}R$ of isomorphicy classes of ideals from $Max^{reg}_{\ell}R$). The local behaviour of the associated sheaves ${}^1\widehat{\mathcal{O}}^a_{\mathcal{M}}$ and $\widehat{\mathcal{O}}^a_{\mathcal{M}}$ is similar to that of ${}^1\mathcal{O}^a_{\mathcal{M}}$ and $\mathcal{O}^a_{\mathcal{M}}$; only the rings ${}^1\widehat{\mathcal{O}}^a_{R,\mu}$ and $\widehat{\mathcal{O}}^a_{R,\mu}$ are a trifle "closer" to the local rings (e.g. the canonical left ideals ${}^1\widehat{\mathcal{O}}^a_{\mathcal{M},\mu} \subset {}^1\mathcal{O}^a_{R,\mu}$ and $\widehat{\mathcal{O}}^a_{\mathcal{M},\mu} \subset \widehat{\mathcal{O}}^a_{R,\mu}$ are maximal).

To any topological space X we may assign the maximal quasi-homeomorphic to it space irrX of "irreducible components" (the points of irrX are all the irreducible closed subsets of X) together with the canonical quasi-homeomorphism $X \longrightarrow \mathrm{irr}X$ which sends every point x into its closure. If X=SpecR, then irrX is homeomorphic to $\overline{\mathrm{Spec}}\, R$, and if $X = (Max_\ell^{reg}, \widehat{\mathfrak{T}})$, then irrX is homeomorphic to the subspace $\mathcal{P}$SpecR of the prime spectrum formed by all the prime ideals which are intersections of the families of ideals from $Max_\ell^{reg} R$.
The canonical arrow $c : Max_\ell^{reg} R \longrightarrow \mathcal{P}Spec\, R$ assigns to every ideal from $Max_\ell^{reg} R$ its symmetric part, i.e. $c(\mu) = \mu_s = \mu \cap (\mu : R)$ so that the image of c is the space of primitive ideals of R.

For any two-sided ideal α of R the map $\mu \mapsto \mu \cap \alpha$ determines isomorphism of preringed spaces:

$$(\dot{U}_\ell(\alpha), {}^1\widehat{\mathcal{O}}_R|_{\dot{U}_\ell(\alpha)}) \xrightarrow{\sim} (Max_\ell^{reg}\alpha, {}^1\widehat{\mathcal{O}}_\alpha)$$

$$(\dot{U}_\ell(\alpha), \widehat{\mathcal{O}}_R|_{\dot{U}_\ell(\alpha)}) \xrightarrow{\sim} (Max_\ell^{reg}\alpha, \widehat{\mathcal{O}}_\alpha)$$

$$(\mathcal{P}\mathcal{U}(\alpha), \mathcal{P}\mathcal{O}_R|_{\mathcal{P}\mathcal{U}(\alpha)}) \xrightarrow{\sim} (\mathcal{P}\mathrm{Spec}\,\alpha, \mathcal{P}\mathcal{O}_\alpha)$$

Here $\dot{\mathcal{U}}_\ell(\alpha) = \mathrm{Max}_\ell^{reg} R \cap \mathcal{U}_\ell(\alpha)$, $\mathcal{P}\mathcal{U}(\alpha) = \mathcal{U}(\alpha) \cap \mathcal{P}\mathrm{Spec}\,R$; $\mathcal{P}\mathcal{O}_R$ is the direct image of the presheaf $\widehat{\mathcal{O}}_R$ with respect to the quasi-homeomorphism $(\mathrm{Max}_\ell^{reg} R, \mathfrak{F}) \to \mathcal{P}\mathrm{Spec}\,R$.

Under the passage from $\mathrm{Spec}_\ell R$ to $\mathrm{Max}_\ell^{reg} R$ the place of the left radical takes the "left extension of the Jacobson radical" $J_\ell : n \mapsto \bigcap\{\mu \mid \mu \in \mathrm{Max}_\ell^{reg} R,\ n \to \mu\}$. The following fact takes place:

If M is a unitary R-module such that

$\mathrm{Ann}\,\xi \supset J_\ell(\mathrm{Ann}\,\xi)$ for any $\xi \in \mathcal{M}$,

then the canonical arrow $\mathcal{M} \mapsto \Gamma\widehat{\mathcal{O}}^a_{\mathcal{M}}$ is an isomorphism. In particular, $R \longrightarrow \Gamma\widehat{\mathcal{O}}^a_R$ is an isomorphism if R is a semisimple ring with unit.

In § 8 we demonstrate the responsibility of the preorder category $\mathrm{I}_\ell^{\succ} R$ for the good and vice aspects of the left geometry and "left algebra" of R. The starting point is the following simply established fact:

Let $\mathfrak{F}$ be the radical filter of left ideals of R. Any ideal from $\mathrm{Max}(\mathrm{I}_\ell^{\succ} R \setminus \mathfrak{F})$ (the maximality is understood in the sense of the pre-order $\to$) belongs to $\mathrm{Spec}_\ell R$.

This implies that $\mathfrak{F} = \bigcap\{\mathfrak{F}_\mu \mid \mu \in \mathrm{Spec}_\ell R \setminus \mathfrak{F}\} = \mathfrak{F}_{V_\ell(\mathfrak{F})}$, if for every $n \in \mathrm{I}_\ell R \setminus \mathfrak{F}$ there exists an arrow $n \longrightarrow \mu$ where $\mu \in \mathrm{Max}(\mathrm{I}_\ell^{\succ} R \setminus \mathfrak{F})$. The equality $\mathfrak{F} = \mathfrak{F}_{V_\ell(\mathfrak{F})}$ implies that every ideal from $\mathrm{I}R \setminus \mathfrak{F}$ is contained in an ideal from $\mathrm{S}\overline{\mathrm{pec}}\,R \setminus \mathfrak{F}$.

We find out when the above condition is satisfied for every radical filters and every radical symmetric filters of bifinite type. As a result we distinguish a class $\mathcal{I}_\ell^{\vdash}$ Rings of uniformly left Noetherian rings (briefly, $\mathcal{I}_\ell^{\vdash}$-Noetherian) formed by all the rings R for which $\mathcal{I}_\ell^{\vdash} R$ is Noetherian, and the corresponding class $\mathcal{L}_s$ Rings formed by the rings R such that

for every linearly ordered chain $\{n^i \mid i \in I\}$ from $\mathcal{I}_\ell^{\vdash} R$ there exists a left ideal m, such that $n^i \longrightarrow m$ for all $i \in I$ and $\sup\{n_s^i \mid i \in I\} = m_s$.

There exists, however, a more convenient property of the rings which guarantees an equality $\mathcal{F} \cap IR = \mathcal{F}_{V_\ell(\mathcal{F})} \cap IR$ for every symmetric radical filter of a bifinite type:

for every prime ideal p of R the set of left ideals $\{(p : x) \mid x \in R - p\}$ possesses a maximal (with respect to $\longrightarrow$) element.

To the class distinguished by this property (and denoted by S_1Rings) belong, in particular, all the rings R such that

(4) the quotient of R modulo any prime ideal is a left Goldi ring.

Recall that a ring R is a left Goldi ring if it satisfies the maximality condition for left annihilators, i.e. the ideals of the form $(0 : w)$, $w \subset R$, and does not contain infinite direct sums of non-zero left ideals.

The following facts hold:

1) If R is from $R \in Ob\, S_\ell$ Rings, then $\mathcal{S}pec\, R = \widetilde{\mathcal{S}pec}\, R = \{\mu_s \mid \mu \in \hat{\mathcal{S}pec}_\ell R\}$ and for any two-sided ideal α, finitely generated as

a left ideal, the filter $\mathrm{I}R\cap{}^{\alpha}\hat{\mathcal{F}}$ (${}^{\alpha}\hat{\mathcal{F}}$ is the radical closure of $\{\alpha\}$) coincide with $\mathcal{F}_{V_\ell(\alpha)}\cap \mathrm{I}R$.

2) If R satisfies $(\natural)$, then $\mathrm{Spec}\,R \subset \widehat{\mathrm{Spec}_\ell}\,R$.

3) If the quotient R modulo the lower Baire radical is a PI-algebra (i.e. an algebra satisfying a polynomial identity), then R satisfies $(\natural)$.

Since any left Noetherian ring is, obviously, a left Goldi ring, then left Noetherian rings satisfy $(\natural)$ and, therefore, belong to $\mathcal{S}_\ell\,\mathrm{Rings}$. This and (1) implies the coincidence of the base (topological) spaces of Van Oystaeyen's and Vershoren's Affine schemes (geometrization of prime spectrum of left Noetherian rings with unit; see the description of results of § 4) with the base spaces of left Affine schemes of the corresponding rings; more exactly $\overline{\mathrm{Spec}}\,R = \mathrm{Spec}\,R \subset \widehat{\mathrm{Spec}_\ell}\,R$. For any R-module M the canonical presheaves $\mathcal{O}_M$ and $\widetilde{M}$ coincide, if $R \in \mathcal{B}_s\,\mathrm{Rings}$.

The majority of the section (both in volume and in meaning) is the extension onto non-commutative rings and modules over them one of the most "commutative" departments of the commutative algebra - the theory of associative ideals and primary decomposition. The role of simple ideals is given to the elements of the left spectrum: the set $\mathrm{Ass}(M) = \mathrm{Ass}_R(M)$ of the ideals associated with a module M consists of all $\mathfrak{p} \in \mathrm{Spec}_\ell R$ such that $\mathfrak{p} \simeq \mathrm{Ann}\,\xi$ for some $\xi \in M$. This notion starts to break down when the localizations enter; besides the relations of Ass(M) with the support M (by definition Supp(M) consists of all $\mathfrak{p} \in \mathrm{Spec}_\ell R$ such that $\mathcal{F}_{\mathfrak{p}} M \neq M$ or equivalently

$G_{\mathcal{F}_p} M \neq 0$) are much less sturdy than in the commutative case.

These gaps are filled in with the help of the parametrized by the radical filters family "spectra" $\mathrm{Spec}_\ell^{\mathcal{F}} R$ and the extension $\mathrm{Spec}_\ell^* R$ of the left spectrum (here $\mathrm{Spec}_\ell^{\mathcal{F}} R$ consists of all the left ideals $m \in I_\ell R \smallsetminus \mathcal{F}$ such that $\hat{m} = m_{\mathcal{F}}$,
where $\hat{m} = \{\lambda \in R \mid (m:\lambda) \not\to m\}$, $m_{\mathcal{F}} = \{\lambda \in R \mid (m:\lambda) \in \mathcal{F}\}$;
$\mathrm{Spec}_\ell^* R = \{m \in I_\ell R \mid \hat{m} \in \widehat{\mathrm{Spec}}_\ell R$ and $[\lambda \in \hat{m}] \Rightarrow$
$\Rightarrow [(m:\lambda) \not\to m]\}$.

If R is commutative, then $\mathrm{Spec}_\ell^* R = \mathrm{Spec} R$; in general case $\widehat{\mathrm{Spec}}_\ell R \smallsetminus \mathcal{F} \subset \mathrm{Spec}_\ell^{\mathcal{F}} R$ and $\mathrm{Spec}_\ell^* R = \bigcup_{\mathcal{F}} \mathrm{Spec}_\ell'^{\mathcal{F}} R = \bigcup \{\mathrm{Spec}_\ell^{\mathcal{F}_p} R \mid p \in \widehat{\mathrm{Spec}}_\ell R\}$.
The sets $\mathrm{Ass}^{\mathcal{F}}(M) = \mathrm{Ass}_R^{\mathcal{F}}(M)$ and $\mathrm{Ass}^*(M) = \mathrm{Ass}_R^*(M)$ are obviously defined. For us the subsets of $\widehat{\mathrm{Spec}}_\ell R$ connected with them:

$$\widehat{\mathrm{Ass}^{\mathcal{F}}}(M) \overset{\mathrm{def}}{=} \{\hat{p} \mid p \in \mathrm{Ass}^{\mathcal{F}}(M)\}, \quad \widehat{\mathrm{Ass}^*}(M) = \{\hat{p} \mid p \in \mathrm{Ass}^*(M)\}$$

are more important; namely their elements play the role of the "additional" associated ideals.

As a model for exposition we used [3]; practically all the however distinguished results of Chapter IV of this book (and also some other ones) got here a left sided image. To emphasize the similarity of formulations (but not proofs!) still more, the place of the commutative Noetherian rings is offered their natural heirs
the uniform left Noetherian rings. However everywhere (and also in the commutative case) we may take the rings from a wider class $\mathrm{Rings}_{(\ell)}$ formed by all the rings

R such that:

for any pair of left ideals m, n of R the set of ideals $\{(m : x) \mid x \in R \smallsetminus n\}$ possesses a maximal (with respect to preordering $\rightarrow$) element.

Notice that if R is a ring from $Rings_{(\ell)}$, then for any radical filter $\mathcal{F}$ of its left ideals we have $\mathcal{F} = \bigcap \{\mathcal{F}_p \mid p \in \widehat{Spec}_\ell R \smallsetminus \mathcal{F}\}$.

Studying the associated ideals and primary decompositions of certain classes of modules we may sometimes considerably weaken the requirements to the ring of scalars. Thus, if for example, we confine our interest to modules of finite length, then there is no need to impose any restrictions onto a ring. If we consider submodules of products of families of projective R-modules (and in particular the ring R itself as a left module over itself) it suffices to assume that R is a semiprime left Goldi ring as the following statement shows:

Let R be a semiprime left Goldi ring with unit, M a non-zero submodule of the product of a family of projective R-modules. Then $Ass(M) \neq \emptyset$, $Ass(M) \subset Ass(R)$, and every ideal from $Ass(R)$ is isomorphic to a prime ideal; the set $Ass(M) \cap Spec\, R$ is finite and M possesses a primary decomposition.

§ 9 is devoted to the "social contacts" of the constructed geometrizations. It begins with the investigation of the conditions for the ring morphisms $f : R \longrightarrow R'$ and radical filters $\mathcal{F} \subset I_\ell R$ and $\mathcal{G} \subset I_\ell R'$ that guarantee the existence of the continuation of f to a localization morphism $f_{\mathcal{F},\mathcal{G}} : G_{\mathcal{F}} R \longrightarrow G_{\mathcal{G}} R'$

One of the variants of necessary and sufficient conditions is the following one:

(b) for any $m \in \mathfrak{F}$ the left ideal $(R', f(m))$ of the ring R', generated by the image $f(m)$ of an ideal m, belongs to $\mathcal{G}$ and the R'-module $Tor_1(R', R/m)$ coincides with its $\mathcal{G}$-torsion

$$Tor_1(R', R/m) = {}_{\mathcal{G}}Tor_1(R', R/m).$$

A $\coprod$-semischeme morphism $(R, \mathfrak{T}) \to (R', \mathfrak{T}')$ is a pair (f, ψ), where f is a ring morphism $R \to R'$, ψ a functor (a function monotonous with respect to inclusions) sending the intersections of filters (which are nothing but the products in $\mathfrak{T}$) into intersections and such that $(\mathfrak{F}, f, \psi\mathfrak{F})$ satisfies (b) for any $\mathfrak{F} \in \mathfrak{T}$.

We verify that to a $\coprod$-semischeme morphism a morphism of the corresponding ringed "topologies" corresponds, and under a natural additional condition the map $(R, \mathfrak{T}) \mapsto Spec_\ell(R, \mathfrak{T})$ canonically extends to a functor from the category of $\coprod$-semischemes into the category of ringed topological spaces.

In general case (unlike a commutative one) far from any ring morphism induces a morphism of the corresponding left spectra. The harmony is recovered if we confine ourselves to the morphisms $\varphi : R \longrightarrow R'$ satisfying the following natural condition:

$$(*) \quad [\, p' \in Spec_\ell R',\ m \in I_\ell R' \text{ and } m \to p' \,] \Rightarrow [\, \varphi^{-1}m \to \varphi^{-1}p' \,]$$

The property $(*)$ distinguished a subcategory in the category of rings which is denoted by $\widetilde{Rings}_\ell$

If $\varphi : R \to R'$ is a morphism from $\widetilde{Rings}_\ell$, then the map $p \mapsto \varphi^{-1}p$ determines maps $\varphi_\ell : U_\ell(\varphi(R)) \longrightarrow Spec_\ell R$ and $\hat{\varphi}_\ell : \hat{U}_\ell(\varphi(R)) \to \widehat{Spec}_\ell R$ continuous with respect to topologies $\mathfrak{T}_0$ and $\mathfrak{T}$.

An important subcategory of $\widetilde{Rings}_\ell$ is the category $Rings_\ell$ formed by all the ring morphisms $\varphi : R \longrightarrow R'$ for which the map $n \longrightarrow \varphi^{-1}n$ determines a functor from $I_\ell^{>} R'$ into $I_\ell^{>} R^{(1)}$. The examples of the arrows from $Rings_\ell$ are provided with a left normal morphisms which are by definition the morphisms $\varphi : R \longrightarrow R'$ such that $\varphi(R)$ and $\mathcal{N}_\ell(\varphi) \overset{def}{=} \{ z \in R' \mid \varphi(x)z \subset (R', \varphi(x))$ for any $x \in R \}$ generate R'. Particular cases of the left normal morphisms - the central extensions - are the arrows $\varphi : R \longrightarrow R'$ such that $R' = \varphi(R) Z(\varphi)$ and $Z(\varphi) = \{ z \in R' \mid z\varphi(x) = \varphi(x)z$ for any $x \in R \}$.

Notice that the central extensions induce continuous maps of the prime spectra and behave functorially with respect to some of the geometrizations of Van Oyestaeyen and Vershoren.

An example of subcategory of $\widetilde{Rings}_\ell$ which does not in general belong to $Rings_\ell$ is provided with a family of ring morphisms $\varphi : R \to R'$ satisfying the following condition

there exists a finite chain $R_0 \subset R_1 \subset \ldots \subset R_{k+1}$ of the subrings R such that $R_0 = \varphi(R)$, $R_{k+1} = R'$ and R_i is a two-sided ideal in R_{i+1} for $0 \leq i \leq k$.

The search of invariant with respect to $\overline{Spec}\, R$ ring morphisms leads to the subcategory $\overline{Rings}_\ell$ formed by all the ring morphisms $\psi : R \longrightarrow R'$ such that

$rad_\ell(\mathfrak{f}^{-1}n) \subset \mathfrak{f}^{-1} rad_\ell(n)$ for any $n \in I_\ell R'$

We establish that $\widetilde{Rings}_\ell \subset \overline{Rings}_\ell$ and for any morphism $\varphi: R \to R'$ from $\overline{Rings}_\ell$ the map $p \mapsto \varphi^{-1}p$ determines a continuous map $\bar{\varphi}_\ell : \overline{U}(\varphi(R)) \to \overline{Spec}_\ell R$ where $\overline{U}(\varphi(R))$ is the subspace of $\overline{Spec}\, R'$ formed by all the $p' \in \overline{Spec}\, R'$ such that $\varphi(R) \not\subset p'$ (if φ is a morphism of rings with unit, then $\overline{U}(\varphi(R)) = \overline{Spec}\, R'$ evidently).

The morphisms from $\widetilde{Rings}_\ell$ induce morphisms of left spectra but do not, generally, extend to morphisms of the structure (pre)sheaves. For such a continuation to take place one should additionally require that

For any $p \in Spec_\ell R'$ and any $n \in \mathcal{F}_{\varphi^{-1}p}$ the kernel $K_{\varphi,n}$ of the canonical morphism $R' \otimes_R n \longrightarrow R'$ coincides with its $\mathcal{F}_p$ -torsion: $K_{\varphi,n} = \mathcal{F}_p K_{\varphi,n}$.

The morphisms from $\widetilde{Rings}_\ell$ satisfying this condition form a subcategory $\widetilde{Rings}_\ell^1$, and the map $R \mapsto (Spec_\ell R, {}^\circ\mathcal{O}_R)$ extends to a functor from $\widetilde{Rings}_\ell^1$ into the category of preringed spaces.

We similarly distinguish the subcategory $\overline{Rings}_\ell^1$ formed by the morphisms $\varphi: R \to R'$ from $\overline{Rings}_\ell$ such that $rad_\ell(n) \subset \varphi^{-1} rad_\ell(Ann\, \xi)$ for any left ideal n of R and an arbitrary $\xi \in K_{\varphi,n}$.
The map $R \mapsto (\overline{Spec}\, R, \bar{\mathcal{O}}_R^a)$ extends to a functor from $\overline{Rings}_\ell^1$ into the category of left Affine quasischemes.

The main text is appended with the following results.

Appendix 1 is devoted to proof of the following fact:

The torsion $\widehat{rad}_\ell(R) \overset{def}{=} \bigcap\{p \mid p \in Spec_\ell R\}$ coincides with the locally nilpotent radical $\mathcal{L}(R)$ for any associative ring R. (Recall that an ideal m is called locally nilpotent if any finite subset of elements of m generates a nilpotent subring; a locally nilpotent radical or the Levitzky radical of R is the maximal locally nilpotent ideal $\mathcal{L}(R)$, i.e. the sum of all the locally nilpotent ideals of R.)

Let us list several corollaries.

1) The left torsion $\widehat{rad}_\ell(\cdot)$ coincides with the symmetrically determined right torsion $\widehat{rad}_r(\cdot)$.

2) For any associative ring R the set $\overline{Spec}\, R$ consists of all the prime ideals p such that R/p has not non-zero locally nilpotent ideals. It is clear from this that the base space $\overline{Spec}\, R$ of a left Affine quasi-scheme of R coincides with the base space $\overline{Spec}_r\, R$ of its rith Affine quasi-scheme.

3) So important in the non-commutative algebraic geometry $\widehat{rad}_\ell$-semiprime rings (see the description of results of § 6 above) are exactly the rings without non-zero left locally nilpotent ideals.

In Appendix 2 we study the connections between local and global properties of the modules. The properties we discuss are finiteness of type, projectiveness, coherentness, flatness, local freedom.

III. Perspectives. At least three of the possible continuations of this paper seem to be sound

- constructing of non-commutative projective spectra and, it goes without saying, their study;

- extension of results obtained here and the notions onto the graded case; in particular, construction of a super-noncommutative geometry;

- geometrization of rings and modules with filtration required by the means of

At the first glance all these three directions diverge. But this is not so. The point is that the constructions and statements of this paper are translated into the algebras and modules in categories with product.

(Recall that a category with product is a pair $(\mathcal{C}, T)$ where $\mathcal{C}$ is a category and $T : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ a "multiplication functor". An algebra in $(\mathcal{C}, T)$ is a pair (R, ξ) where R is an object of $\mathcal{C}$ and $\xi : T(R,R) \to R$ a morphism. Given an "associativity", a functor morphism $\alpha : T(\cdot, T(-,\prime)) \longrightarrow T(T(\cdot,-),\prime)$ we can determine modules as the pairs $(\mathcal{M}, \xi)$, where $\mathcal{M} \in Ob\,\mathcal{C}$ and ξ is a morphism $T(R,\mathcal{M}) \longrightarrow \mathcal{M}$ such that the diagram

$$\begin{array}{ccccc} T(R, T(R, \mathcal{M})) & & \xrightarrow{\alpha} & & T(T(R,R), \mathcal{M}) \\ {\scriptstyle T(1_R, \xi)} \downarrow & & & & \downarrow {\scriptstyle T(\xi, 1_{\mathcal{M}})} \\ T(R, \mathcal{M}) & \xrightarrow{\xi} & \mathcal{M} & \xleftarrow{\xi} & T(R, \mathcal{M}) \end{array}$$

commutes. The usual rings and modules are the algebras and modules in the "classical" category with multiplication $(Z\text{-mod}, \otimes_Z)$ and α the standard associativity $x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z$.)

The fact, that may look astounding, is that we do not require additivity for a category with multiplication. In other words, the possibility of an algebra-geometric approach is actually based on the multiplicative structure given by the multiplication functor. For example, an algebraic geometry may be constructed for monoids and their left actions.

The work is written so as to prepare the reader to the non-commutative algebraic geometry in categories with multiplications. The specifics of certain constructions and proofs is explained by this hidden aim. If for a category with multiplication we take the category of graded Z-modules with graded tensor product we get via the same way the graded Affine (quasi) schemes. From them, in turn, we, by analogy with the commutative case, construct projective spectra. Starting from the category of Z-modules with filtration (of a fixed type) and the corresponding tensor products we arrive to non-commutative semischemes and (quasi)schemes, but now for the rings with filtration.

IV. The main facts on Affine $\mathfrak{U}$-semischemes (in categories with multiplication) were obtained in 80-81 and delivered from time to time here and there starting from the summer school on operator theory on Baikal in '81. The work was resumed

four years later thanks to a stimulating interest of L. A. Bocut, to whom I am glad to express my sincere gratitude.
As a result, the contours of the other characters were outlined: the left spectrom, filters $\mathcal{F}_p$, $\widetilde{Spec}$, affine (quasi)schemes and related notions.

This text is due to the great extent by its appearance and shape to D. Leites, whose advice I used as far as I could understand it . The main advice - to write clearly and with details - enabled me to get rid of a number of mistakes and vague statements (I am afraid that not of all of them) and discover a few new facts. This does not exhaust all the reasons for my hartily thanks.

It is pleasure also to express my acknowledgements to S. Prishchepionok for useful comments.

Concluding this introduction (written mainly for those who don't read anything except introductions; it is also one of Leites' suggestions) I cannot but mention once more P. Gabriel whose remarkable work [1] enabled the existence of this paper.

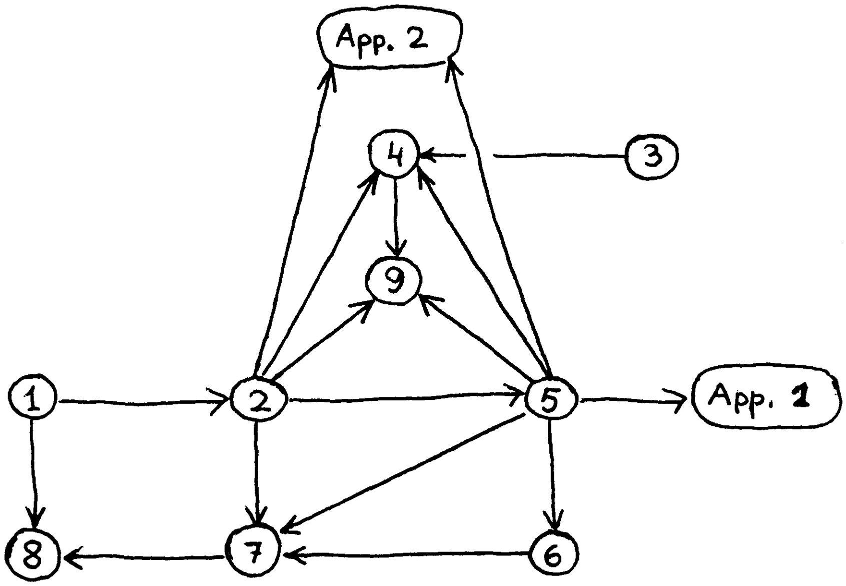
Grothendieck's name seldom appears in the text explicitly, but implicitly it is present in practically every line.

Recommendations to the reader

Sections 1, 4, 5 are the central ones. Section 2 contains not only an exposition of Gabriel's results, which deserve to be read carefully, but, also, a number of "technical" statements repeatedly used in what follows. Therefore we advice to read the second half of this section not too assiduously, and return to it in the sequel as needed. Section 3 is connected, as is shown on the diagram below, only with Section 4 (its results play much more important role in continuations of this paper; in particular, they are the source of [17]). Therefore the reader can only brose the definitions, formulations, the statements and examples, believing in the possibility of transition to the associated sheaves on exotic topologies of Section 4, unless he wants to verify this directly.

The paper is written almost self-contained (modulo preliminary data on rings and categories) and, as far as I could, elementary. Nevertheless, for a better understanding of the hints, it is desirable to be acquainted with sheaves and schemes (say, the second half of [15] suffices) and also with non-commutative rings (here it is difficult to suggest anything nicer than [16]). I also highly recommend to go through Exercises 17-25 to Ch.II of [3], which reflect the important for this paper Gabriel's results.

Scheme of the logical dependence of Sections



§1. Uniform, topologizing and radical sets of ideals

1. <u>Conventions and notations</u>. Here R is an associative ring, $I_\ell R$ a family of left ideals of R. For an arbitrary $m \in I_\ell R$ and a subset x of elements of R denote by (m: x) the ideal consisting of all λ such that $\lambda \cdot x \subset m$. Denote by $\mathcal{P}(R)$ the family of the finitely generated Z-submodules of R and let $\mathcal{P}(\nu) = \{\gamma \in \mathcal{P}(R) \mid \gamma \subset \nu\}$ for any ideal (or $\mathbb{Z}$-sub-module) ν of R.

On $I_\ell R$, a natural category structure (with embeddings as morphisms) will be assumed and the subsets of the set $Ob\, I_\ell R$ will be identified with the corresponding full subcategories of the category $I_\ell R$. Similarly, the set $2^{I_\ell R}$ of the subsets of $I_\ell R$ will be sometimes considered as a category with inclusions as arrows.

2. <u>Multiplication on $2^{I_\ell R}$</u>. For any set $\mathcal{F}$ of left ideals denote by $\overline{\mathcal{F}}$ $\{n \in I_\ell R \mid m \subset n \text{ for some } m \in \mathcal{F}\}$ the filter spanned by $\mathcal{F}$. On $2^{I_\ell R}$, determine a multiplication setting $\mathcal{F} \circ \mathcal{G} = \{n \mid$ there exists $m \in \mathcal{G}$ such that $(n : x) \in \overline{\mathcal{F}}$ for any $x \in \mathcal{P}(m)\}$ for every pair $\mathcal{F}, \mathcal{G}$ of sets of left ideals. Clearly, $\mathcal{F} \circ \mathcal{G} = \bigcup \{\mathcal{F} \circ \{m\} \mid m \in \mathcal{G}\}$.

<u>Proposition</u>. 1) $\mathcal{F} \circ \mathcal{G} = \overline{\mathcal{F} \circ \mathcal{G}} = \overline{\mathcal{F}} \circ \mathcal{G}$ <u>for any</u> $\{\mathcal{F}, \mathcal{G}\} \subset 2^{I_\ell R}$.

2) <u>If</u> $\mathcal{F}$ <u>and</u> $\mathcal{G}$ <u>are cofilters then so is</u> $\mathcal{F} \circ \mathcal{G}$.

3) <u>For any</u> $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\} \subset 2^{I_\ell R}$ <u>we have the inclusion</u> $\mathcal{F}_1 \circ (\mathcal{F}_2 \circ \mathcal{F}_3) \subset (\mathcal{F}_1 \circ \mathcal{F}_2) \circ \mathcal{F}_3$ <u>which turns into the identity if</u> $\mathcal{F}_1$ <u>is a cofilter</u>.

Proof. 1) is obvious.

2) Let n_1, n_2 be ideals of $\mathfrak{F}\circ\mathcal{G}$; i.e. $(n_i : x_i)\in\overline{\mathfrak{F}}$ for any $x_i\in\mathcal{P}(m_i)$, $m_i\in\mathcal{G}$, $i=1, 2$. Since $\mathcal{G}$ is a cofilter, there exists an ideal m in $\mathcal{G}$ belonging to $m_1\cap m_2$. Clearly,

$[\mathfrak{F}$ is cofilter$]\Longleftrightarrow[$ with every pair of ideals $\overline{\mathfrak{F}}$ contains their intersection$]$.

Therefore, $(n_1\cap n_2 : x)=(n_1:x)\cap(n_2:x)\in\overline{\mathfrak{F}}$ for any $x\in\mathcal{P}(m)$.

3) $(\mathfrak{F}_1\circ\mathfrak{F}_2)\circ\{\nu\}=\{n\,|\,(n:x)\in\mathfrak{F}_1\circ\mathfrak{F}_2$ for any $x\in\mathcal{P}(\nu)\}=\{n\,|$ for any $x\in\mathcal{P}(\nu)$ there exists $m_x\in\mathfrak{F}_2$ such that $(n:yx)=((n:x):y)\in\overline{\mathfrak{F}}_1$ if $y\in\mathcal{P}(\nu)\}$ for an arbitrary ideal $\nu\in I_l R$.

On the other hand,

$[\mathfrak{F}_1\circ(\mathfrak{F}_2\circ\{\nu\})\ni n]\Longleftrightarrow[$there exists a left ideal m such that $(m:x)\in\overline{\mathfrak{F}}_2$ for any $x\in\mathcal{P}(\nu)$ and $(n:z)\in\overline{\mathfrak{F}}_1$ for any $z\in\mathcal{P}(m)]$

in particular, $((n:x):y)=(n:yx)\in\overline{\mathfrak{F}}_1$ for any $x\in\mathcal{P}(\nu)$ and $y\in\mathcal{P}((m:x))$. This implies that $\mathfrak{F}_1\circ(\mathfrak{F}_2\circ\{\nu\})\subset(\mathfrak{F}_1\circ\mathfrak{F}_2)\circ\{\nu\}$.

If $\mathfrak{F}_1$ is a cofilter, then

$[(n:y_i x_i)\in\overline{\mathfrak{F}}_1,\ i=1,2]\Rightarrow[(n:y_1x_1+y_2x_2)=(n:y_1x_1)\cap(n:y_2x_2)\in\overline{\mathfrak{F}}_1]$

for $n\in I_l R$ and for any $\{x_1,x_2\}\subset\mathcal{P}(\nu)$, $y_i\in\mathcal{P}(m_{x_i})$, $i=1,2$. This implies that $(n:z)\in\overline{\mathfrak{F}}_1$ for any $z\in\mathcal{P}(\sum_{x\in\mathcal{P}(\nu)} m_x x)$. Clearly, $\sum_{x\in\mathcal{P}(\nu)} m_x x\in\mathfrak{F}_2\circ\{\nu\}$.

3. Definitions. 1) A set $\mathfrak{F}$ of left ideals will be called uniform if $\mathfrak{F}\subset\mathfrak{F}\circ\{R\}$

2) A uniform set $\mathfrak{F}$ will be called topologizing if $\mathfrak{F}$ is a cofilter and radical if $\mathfrak{F}\circ\mathfrak{F}\subset\overline{\mathfrak{F}}$.

Proposition. 1) If $\{\mathfrak{F}, \mathfrak{G}\} \subset 2^{I_e R}$ and $\mathfrak{F}$ is uniform, then $\mathfrak{G} \circ \mathfrak{F}$ is uniform.

2) If $\mathfrak{F}$ is uniform and $\{n, m\} \subset \mathfrak{F}$, then $n \cap m \in \mathfrak{F} \circ \mathfrak{F}$.

Proof. 1) If $\mathfrak{F}$ is uniform, then by Proposition 2
$\mathfrak{G} \circ \mathfrak{F} \subset \mathfrak{G} \circ (\mathfrak{F} \circ \{R\}) \subset (\mathfrak{G} \circ \mathfrak{F}) \circ \{R\}$.

2) $(n \cap m : a) = (n : a) \cap (m : a) = (m : a)$ for any $a \in \mathcal{P}(n)$. Therefore $m \cap n \in \mathfrak{F} \circ \mathfrak{F}$ since $(m : a) \in \mathfrak{F}$.

Corollary. 1) For any two topologizing sets $\mathfrak{F}$, $\mathfrak{G}$ the set $\mathfrak{F} \circ \mathfrak{G}$ is topologizing.

2) Radical sets are topologizing.

Proof. The first statement follows from the first heading of Proposition 3 and the first heading of Proposition 2; the second statement follows from the second heading of Proposition 3. □

Remark. For rings with unit the above definitions of topologizing and radical sets of ideals differ from conventional ones (in the essence, not the form) only in the lack of the condition " $\mathfrak{F}$ is a filter", i.e. the condition $\mathfrak{F} = \overline{\mathfrak{F}}$. □

4. Examples.

4.1. Sets ${}^{m}\mathfrak{F}$ and weakly regular ideals. With a left ideal m one can associate the set ${}^{m}\mathfrak{F} = \{n \in I_e R \mid m \subset n$ or $(m : x) \subset n$ for some $x \in \mathcal{P}(R)\}$. It is easy to verify that ${}^{m}\mathfrak{F}$ is minimal among uniform filters containing m.

A left ideal m will be called weakly regular if $(m : x) \subset m$ for some $x \in \mathcal{P}(R)$.

Clearly, regular ideals are weakly regular. In fact, by definition m is regular if $x - xa \in m$ for some $a \in R$ and any $x \in R$. Obviously, $[x \in m] \Longleftrightarrow [xa \in m]$ in this case, i.e. $(m : a) = m$. If R contains a right unit then all the left ideals of R are regular, hence weakly regular.

Proposition. The following conditions are equivalent:

1) ${}^m\mathfrak{F}$ is a topologizing set;

2) m is two-sided and/or weakly regular ideal.

Proof. 1) a) If m is a two-sided ideal then ${}^m\mathfrak{F} = \{n \mid m \subset n\}$ is a cofilter.

b) If $(m : x) \subset m$ for some $x \in \mathcal{P}(R)$ then ${}^m\mathfrak{F} = \{n \mid (m:y) \subset n$ for some $y \in \mathcal{P}(R)\}$. Therefore, if $(m: x_i) \subset n_i$, $i = 1, 2$ then $(m : x_1 + x_2) = (m : x_1) \cap (m : x_2) \subset n_1 \cap n_2$.

2) Conversely, let ${}^m\mathfrak{F}$ be a cofilter. Then $m \cap (m : x) \in {}^m\mathfrak{F}$ for any $x \in \mathcal{P}(R)$, i.e. either $m \subset m \cap (m : x)$ for all $x \in \mathcal{P}(R)$ or $(m : y) \subset m \cap (m : x)$ for some $\{x, y\} \subset \mathcal{P}(R)$. In the first case $m \subset (m : x)$ for all $x \in \mathcal{P}(R)$ and therefore m is a two-sided ideal; in the second case $(m : y) \subset m$. $\square$

Denote by $I_\ell^w R$ the set of all weakly regular left ideals of R. Clearly, $\mathfrak{F} \circ \{R\} \cap I_\ell^w R \subset \overline{\mathfrak{F}}$ for any subset $\mathfrak{F}$ of $I_\ell R$.

4.2. Categories $\mathfrak{F}_m$ and the left spectrum. For an arbitrary left ideal m denote by $\mathfrak{F}_m$ the set complementary to ${}^m\mathfrak{F}$ in a natural sense: $\mathfrak{F}_m = \{n \in I_\ell R \mid m \notin {}^n\mathfrak{F}$; i.e. $n \not\subset m \not\supset (n : x)$ for all $x \in \mathcal{P}(R)\}$. Clearly, $\mathfrak{F}_m$ is the maximal of uniform filters that do not contain m. The simplest properties of the sets $\mathfrak{F}_m$:

a) $[m \subset m'] \Rightarrow [\mathfrak{F}_{m'} \subset \mathfrak{F}_m]$;

b) $\mathfrak{F}_{(m:t)} \subset \mathfrak{F}_m$ and $[\mathfrak{F}_{(m:t)} = \mathfrak{F}_m] \Leftrightarrow [(m:t) \notin \mathfrak{F}_m]$ for any $t \in \mathcal{P}(R)$

The first property is obvious. Let us verify the second one. Let $n \notin \mathfrak{F}_m$; i.e. either $n \subset m$ or $(n:x) \subset m$. In the first case $(n:t) \subset (m:t)$, in the second one we have $((n:x):t) = (n:tx) \subset (m:t)$. Both mean that $n \notin \mathfrak{F}_{(m:t)}$.

If $(m:t) \notin \mathfrak{F}_m$ then $\mathfrak{F}_m \subset \mathfrak{F}_{(m:t)}$ thanks to the maximality of $\mathfrak{F}_{(m:t)}$ among the uniform filters that do not contain $(m:t)$. Clearly, $(m:t) \notin \mathfrak{F}_m$, if $\mathfrak{F}_m = \mathfrak{F}_{(m:t)}$.

As a rule, $\mathfrak{F}_m$ is not a cofilter. If e.g. there exists a pair of two-sided ideals α and β such that $\alpha \not\subset m$, $\beta \not\subset m$ and $\alpha \cap \beta \subset m$, then, obviously, $\{\alpha, \beta\} \subset \mathfrak{F}_m$ and $\alpha \cap \beta \notin \mathfrak{F}_m$.

Proposition. The following conditions on a left proper ideal m are equivalent:

1) $m \notin \mathfrak{F}_m \circ \mathfrak{F}_m$;

2) $\mathfrak{F}_m$ is a radical filter;

3) $[m \in \mathcal{G} \circ \mathcal{G}'] \Rightarrow [m \in \overline{\mathcal{G}} \cup \overline{\mathcal{G}'}]$ for any two uniform sets $\mathcal{G}$ and $\mathcal{G}'$.

4) $[\, n \in I_e R$ and $(m:x) \not\subset m$ for any $x \in \mathcal{P}(n)] \Rightarrow$ $\Rightarrow [(n:y) \subset m$ for some $y \in \mathcal{P}(R)$ or $n \subset m]$.

Proof. 1) $\Longrightarrow$ 2). Since $\mathfrak{F}_m \circ \mathfrak{F}_m$ is a uniform filter, then $[m \notin \mathfrak{F}_m \circ \mathfrak{F}_m] \Longrightarrow [\mathfrak{F}_m \circ \mathfrak{F}_m \subset \mathfrak{F}_m]$. The converse implication is obvious.

3) $\Longrightarrow$ 1). It suffices to set $\mathcal{G} = \mathcal{G}' = \mathfrak{F}_m$.

1) $\Longrightarrow$ 4). The implication 4) can be rewritten in the form $[\mathfrak{F}_m \circ \{n\} \ni m] \Rightarrow [n \notin \mathfrak{F}_m]$.

4) $\Longrightarrow$ 3). Let $m \in \mathcal{G} \circ \{n\}$ for some $n \in \mathcal{G}'$, i.e. $(m:x) \in \overline{\mathcal{G}}$ for any $x \in \mathcal{P}(n)$. If $m \notin \overline{\mathcal{G}}$, then $(m:x) \not\subset m$ for any $x \in \mathcal{P}(R)$. By (4) this means that either $n \subset m$ or $(n:y) \subset m$ for some $y \in \mathcal{P}(R)$.

Since $\mathcal{Y}'$ is uniform, then in either of these cases $m \in \overline{\mathcal{Y}}'$. □

The collection of left ideals of R satisfying the equivalent conditions 1) - 4) will be denoted by $\mathrm{Spec}_\ell R$ and will be called the left spectrum of R.

Corollary. All the ideals of $\mathrm{Spec}_\ell R$ are weakly regular.

Proof. In fact, if $m \in \mathrm{Spec}_\ell R$ is such that $(m:x) \not\subset m$ for any $x \in \mathcal{P}(R)$, then by 4) $R \subset m$, i.e. $m = R$, which is impossible. □

4.3. The sets $\hat{\mathcal{F}}_m$ and maximal left ideals. For an arbitrary $m \in I_\ell R$ consider $\hat{\mathcal{F}}_m = \{n \in I_\ell R \mid n \not\subset m\}$, the maximal filter not containing m, which is almost always non-uniform. There is a statement on the relation of m with $\hat{\mathcal{F}}_m$ with the first four headings similar to the corresponding headings of Proposition 4.2:

Proposition. The following conditions on a left proper ideal m are equivalent:

1) $m \notin \hat{\mathcal{F}}_m \circ \hat{\mathcal{F}}_m$;

2) $\hat{\mathcal{F}}_m \circ \hat{\mathcal{F}}_m \subset \hat{\mathcal{F}}_m$;

3) $[m \in \mathcal{Y} \circ \mathcal{Y}'] \Rightarrow [m \in \overline{\mathcal{Y}} \cup \overline{\mathcal{Y}}']$ for any pair of subsets $\mathcal{Y}$, $\mathcal{Y}'$ of $I_\ell R$;

4) $[n \in I_\ell R$ and $(m:x) \not\subset m$ for any $x \in \mathcal{P}(n)] \Rightarrow [n \subset m]$;

5) $\mathcal{F}_m = \mathcal{F}_{(m:t)}$ for all $t \in \mathcal{P}(R) \setminus \mathcal{P}(m)$.

Proof. As it has been done in 4.2, we establish the validity of implications:

$$\begin{tikzcd} & 1) \arrow[r, Leftrightarrow] \arrow[dr, Rightarrow] & 2) \\ 3) \arrow[ur, Rightarrow] & & 4) \arrow[ll, Rightarrow] \end{tikzcd}$$

4) $\Longrightarrow$ 5). Let $t \in \mathcal{P}(R)$ and $(m:t) \in \mathfrak{F}_m$; i.e. $(m:t) \not\subset m \not\supset (m:xt)$ for any $x \in \mathcal{P}(R)$. This means that $(m:s) \not\subset m$ for every $s \in \mathcal{P}((R,t))$, where (R,t) is the left ideal generated by t. By 4) this implies $(R,t) \subset m$. In particular, $t \in \mathcal{P}(m)$. By the property b) in 4.2 $(m:t) \in \mathfrak{F}_m$ if and only if $\mathfrak{F}_{(m:t)} \neq \mathfrak{F}_m$.

5) $\Longrightarrow$ 4). Now let $n \in I_\ell R$ and $(m:x) \not\subset m$ for any $x \in \mathcal{P}(n)$. This means that $(m:x) \in \mathfrak{F}_m$ and therefore $\mathfrak{F}_{(m:x)} \neq \mathfrak{F}_m$ for all $x \in \mathcal{P}(n)$. By condition 5) this implies $\mathcal{P}(n) \subset \mathcal{P}(m)$ which is obviously equivalent to $n \subset m$. $\square$

The set of ideals satisfying the equivalent conditions 1)-5) will be denoted by $\widehat{\mathrm{Spec}}_\ell R$. The comparison of the condition 3) of the just proved proposition 4.3 with condition 3) of Proposition 4.2 shows that $\widehat{\mathrm{Spec}}_\ell R \subset \mathrm{Spec}_\ell R$. In particular, all the ideals of $\widehat{\mathrm{Spec}}_\ell R$ are weakly regular. More impressive is the fact that weakly regular maximal left ideals belong to $\widehat{\mathrm{Spec}}_\ell R$, i.e.

$$\mathrm{Max}_\ell R \cap I_\ell^w R \subset \widehat{\mathrm{Spec}}_\ell R.$$

In fact, if $n \in I_\ell R$ and $(m:x) \not\subset m$ for any $x \in \mathcal{P}(n)$, then $(m:x) \not\subset m$ for any $x \in \mathcal{P}(m+n)$. If, moreover, $m \in \mathrm{Max}_\ell R$ and $n \not\subset m$, then $m+n=R$ and therefore m is not weakly regular.

4.4. The sets F_S. Let S be a subset of $\mathcal{P}(R)$. Set $F_S = \{n \in I_\ell R \mid \mathcal{P}(n) \cap S \neq \emptyset$ and $\mathcal{P}((n:x)) \cap S \neq \emptyset$ for any $x \in \mathcal{P}(R)\}$. It is not difficult to verify that F_S is a uniform set which turns out to be a radical filter if S is a monoid, i.e. if $\{s,t\} \subset S$ implies $st \in S$, or, more generally, S satisfies $[\{s,t\} \subset S] \Rightarrow [s' \subset st$ for some $s' \in S]$.

Let $\mathcal{U}$ be a subset of elements of R, $\mathcal{U}^+$ the set of $\mathbb{Z}$-submodules of R generated by $\mathcal{U}$. Clearly, the multiplicativity of $\mathcal{U}$ implies the multiplicativity of $\mathcal{U}^+$ and the radical filter $F_{\mathcal{U}^+}$ coincides with the mentioned in Introduction idempotent topologizing set $F_{\mathcal{U}}$ by now classical Gabriel's construction (for unitary R).

4.5. Completely prime spectrum. For an arbitrary left ideal m denote by S_m the set $\mathcal{P}(R) \setminus \mathcal{P}(m)$. Clearly, $\mathcal{F}_m = F_{S_m}$, since by definition $F_{S_m} = \{n \in I_e R \mid \mathcal{P}(n) \cap S_m \neq \emptyset$ and $\mathcal{P}((n:x)) \cap S_m \neq \emptyset$ for any $x \in \mathcal{P}(R)\}$. Denote by $\widehat{\widehat{Spec}}_e R$ the set of ideals m such that S_m is multiplicative. Clearly, $\widehat{\widehat{Spec}}_e R \subset Spec_e R$. As a rule, $\widehat{\widehat{Spec}}_e R$ is considerably poorer than $Spec_e R$: the regular maximal ideals must not necessarily belong to $\widehat{\widehat{Spec}}_e R$. Therefore, the difference between $Spec_e R$ and $\widehat{\widehat{Spec}}_e R$ is a source of a number of examples of radical filters of the form F_S, where S is a subset of $\mathcal{P}(R)$ that does not satisfy the conditions of Example 4.4.

The two-sided ideals of $\widehat{\widehat{Spec}}_e R$ are exactly completely prime ideals of R, and we retain the same name for one-sided ideals.

4.6. Radical filters of finite type.

A set of left ideals $\mathcal{E}$ will be called a set of finite type if it has a cofinal subset of ideals of finite type; i.e. every ideal in $\mathcal{E}$ contains a finite type ideal from $\mathcal{E}$.

Proposition. Any radical filter of finite type is of the form F_S, where S is a multiplicative subset of $\mathcal{P}(R)$ satisfying the following condition:

For any $a \in \mathcal{P}(R)$ and $s \in S$ there exists $t \in S$ such that $ta \subset (R,s)$. (#)

Proof. 1) Clearly, the ideals of finite type are exactly the ideals of the form (R,s) (i.e. generated by s), where $s \in \mathcal{P}(R)$. The condition (#) means that (R,s) belongs to F_S for any $s \in S$. It is also clear that the ideals (R,s), where s runs through S, constitute a cofinal subset in F_S.

2) Now let $\mathfrak{F}$ be an uniform set of finite type. It is not difficult to see that the set $S_{\mathfrak{F}}$ of all $s \in \mathcal{P}(R)$, such that (R,s) belongs to $\mathfrak{F}$, satisfies (#). Besides, (R,st), where $\{s, t\} \in \mathcal{P}(R)$, satisfies the following condition:

if $x \in \mathcal{P}((R,t))$ and $y \in \mathcal{P}(R)$ are such that $x \subset yt$, then

$$((R,s):y) \subset (((R,st):t):y) = ((R:st):yt) \subset ((R:st):x).$$

Therefore, if $\mathfrak{F}$ is a radical filter, then the following implication holds:

$$[(R,t),\ (R,s) \in \mathfrak{F}] \Longrightarrow [\ (R,st) \in \mathfrak{F}]\ ;$$

i.e. $S_{\mathfrak{F}}$ is multiplicative. □

5. Arrows of category $I_\ell^{\succ} R$. Recall that $I_\ell^{\succ} R$ is a preorder category, the objects of which are all the left ideals of the ring R; the existence of an arrow $m \to n$ means, that either $m \subset n$ or $(m:x) \subset n$ for some $x \in \mathcal{P}(R)$ (or, equivalently, $(m:x') \subset n$ for some finite subset x' of the elements of the ring R).

It is obvious, that if α is a two-sided ideal and n -arbitrary left ideal, then $[\alpha \to n] \Leftrightarrow [\alpha \subset n]$. In particular, the preorder of the category $I_\ell^{\succ} R$ induces the usual preorder(morphisms being the inclusions) on the set IR of all the two-sided ideals of the ring R.

Many concepts and constructions of the present paper acquire especially convenient form if we use the language of arrows from $I_\ell^{\succ} R$. It can be confirmed by the following reformulations, which are going to be often used in the sequel.

The uniform filter $\mathcal{F}$ is a filter in the category $I_\ell^{\succ} R$; i.e. $[n \in \mathcal{F}$ and $n \to m] \Rightarrow [m \in \mathcal{F}]$.

The uniform filters ${}^m\mathcal{F}$ and $\mathcal{F}_m$ are ${}^m\mathcal{F} = \{ n \in I_\ell R \mid m \to n \}$; $\mathcal{F}_m = \{ n \in I_\ell R \mid n \not\to m \}$.

The left spectrum $Spec_\ell R$ consists of all such $p \in I_\ell R$, that $[n \in I_\ell R$ and $(p:x) \not\to p$ for all $x \in \mathcal{P}(n)] \Rightarrow [n \to p]$.

$\widehat{Spec}_\ell R$ consists of all such $\mu \in I_\ell R$, that $[x \in R$ and $(\mu : x) \not\to \mu] \Rightarrow [x \in \mu]$.

The fact that the left ideal p belongs to the completely prime left spectrum $\widehat{\widehat{Spec}}_\ell R$ is equivalent to the implication: $[x \in R$ and $(p:x) \not\subset p] \Rightarrow [x \in p]$.

We leave to the reader the checking of the equivalence of the new definitions to the old ones.

6. $\mathrm{Spec}_\ell R$ and $\widehat{\mathrm{Spec}}_\ell R$. For each $n \in I_\ell R$ denote by $\hat{n}$ the set $\{z \in R \mid (n:z) \not\to n\}$. It is clear, that for each proper ideal n the set $\hat{n}$ contains n, and $\lambda z \subset \hat{n}$ for each $z \in \hat{n}$ and $\lambda \in \mathcal{P}(R)$. Nevertheless, it happens quite seldom that for a given $n \in I_\ell R$ the set $\hat{n}$ turns out to be a left ideal.

Denote $I_\ell^* R$ the collection of all the proper left ideals of the ring R enjoying the property:

if x_1, x_2 are such elements of the ring R, that $(n:x_i) \not\to n$, $i = 1, 2$, then $(n:\{x_1, x_2\}) = (n:x_1) \cap (n:x_2) \not\to n$.

Proposition. I) For each $n \in I_\ell^* R$ the set $\hat{n}$ is an ideal from $\widehat{\mathrm{Spec}}_\ell R \cup \{R\}$.

2) The following properties of an proper ideal m are equivalent:

(a) $\hat{m} \in I_\ell R$ and $\hat{m} \to m$ (consequently the ideals m and $\hat{m}$ are isomorphic in $I_\ell^{\succ} R$).

(b) $m \in \mathrm{Spec}_\ell R$.

3) The following properties of an ideal $\mu \in I_\ell R \setminus \{R\}$ are equivalent:

(c) $\mu = \hat{\mu}$;

(d) $\mu \in \widehat{\mathrm{Spec}}_\ell R$.

Proof. I) Suppose $n \in I_\ell^* R$. For any pair x, y from $\hat{n}$ the relations

$$(n: x+y) \supset (n:x) \cap (n:y) \not\to n$$

hold; they show that the set $\hat{n}$ is closed with respect to addition and is, therefore, a left ideal in the ring R.

Let us show that $\hat{n} \in \widehat{\mathrm{Spec}}_\ell R \cup \{R\}$. According to the definition of $\widehat{\mathrm{Spec}}_\ell R$ from paragraph 5, the fact that $\hat{n}$ belongs to the set $\widehat{\mathrm{Spec}}_\ell R \cup \{R\}$ is equivalent to the implication

$$[z\in R,\ (\hat{n}:z)\not\to\hat{n}]\Rightarrow[z\in\hat{n}].$$

Lemma. Let $n\in I_\ell^* R$, w be a finite subset in R, and $(\hat{n}:w)\not\subset\hat{n}$. Then $(n:w)\not\subset n$.

Proof. Suppose, that $(n:w)\subset n$. Then for any $a_w\in$ $\in(\hat{n}:w)\setminus\hat{n}$ the relations hold

$$(n:a_w w)=((n:w):a_w)\subset(n:a_w)\to n$$

On the other hand, since $a_w w$ is a finite subset from $\hat{n}$ and $n\in I_\ell^* R$, then $(n:a_w w)\not\to n$. This is the contradiction. □

From the lemma just proved and from the definition of arrows of the category $I_\ell^\gamma R$ the implications follow:

$[(\hat{n}:z)\not\to\hat{n}]\Rightarrow[(\hat{n}:z)\not\subset n$ and $(\hat{n}:xz)\not\subset\hat{n}$ for each finite subset $x\subset R]\Rightarrow[(n:z)\not\subset n$ and $(n:xz)\not\subset n$ for each finite subset $x\subset R]\Leftrightarrow[(n:z)\not\to n$; i.e. $z\in\hat{n}]$.

This is what we had to prove.

2) $(a)\Longrightarrow(b)$. For each $n\in I_\ell R$ the implications take place: $[(m:x)\not\subset m$ for each $x\in\mathcal{P}(n)]\Rightarrow[n\subset\hat{m}$ (by definition of $\hat{m}$)$]\Rightarrow[n\to m]$, since $\hat{m}\to m$ by the condition.

$(b)\Longrightarrow(a)$. Any left ideal ν, such that $\mathcal{F}_\nu$ is a topologising filter, belongs to $I_\ell^* R$: $[\{z_1,z_2\}\subset\hat{\nu}]\overset{def}{\Longleftrightarrow}$ $\Leftrightarrow[(\nu:z_i)\in\mathcal{F}_\nu,\ i=1,2]\Rightarrow[(\nu:\{z_1,z_2\})=(\nu:z_1)\cap(\nu:z_2)\in\mathcal{F}_\nu]$. In particular, $Spec_\ell R\subset I_\ell^* R$, and, according to the first statement, the set $\hat{p}$ turns out to be an ideal (from $\widehat{Spec_\ell R}$) for each $p\in Spec_\ell R$. Since $p\in I_\ell^* R$, $(p:x)\not\subset p$ for each $x\in\mathcal{P}(\hat{p})$. Consequently, $\hat{p}\to p$.

3) The equivalence of (c) and (d) was announced in the paragraph 5; $(d)\Rightarrow(c)$ follows from definition of $Spec_\ell R$; the implication $(c)\Rightarrow(d)$ is obvious. □

Corollary 1. *If* $n \in I_\ell^* R \cap I_\ell^{w} R$, *then* $\hat{n} \in \widehat{Spec}_\ell R$.

Indeed, if $n \in I_\ell^* R$, then, as it was already mentioned during the proof of the implication (b) $\Longrightarrow$ (a) of Proposition 6, $(n:x) \not\subset n$ for any $x \in \mathcal{P}(\hat{n})$. On the other hand, if $n \in I_\ell^{w} R$, then $(n:y) \subset n$ for some $y \in \mathcal{P}(R)$. Therefore the equality $\hat{n} = R$ is impossible. □

Corollary 2. *Any ideal from* $Spec_\ell R$ *is isomorphic to an ideal from* $\widehat{Spec}_\ell R$; *this isomorphism is given by an inclusion.*

Note that for an arbitrary pair of left ideals n, m their isomorphicy is equivalent to the equality $\mathcal{F}_n = \mathcal{F}_m$.

7. Categories of topologizing and radical filters. Full subcategories of the category $2^{I_\ell R}$ formed by topologizing and radical filters will be denoted by $\mathcal{T}_\ell R$ and $\mathcal{T}I_\ell R$ respectively. It is easy to verify that $\mathcal{T}_\ell R$ and $\mathcal{T}I_\ell R$ are closed with respect to intersections (products) of arbitrary families of filters (objects).

For any pair of radical filters $\mathcal{F}$, $\mathcal{G}$ denote by $\mathcal{F} \vee \mathcal{G}$ their coproduct in $\mathcal{T}I_\ell R$. It is easy to see that $\mathcal{F} \vee \mathcal{G}$ coincides with the intersection of the set of all the radical filters containing $\mathcal{F}$ and $\mathcal{G}$.

§2. Gabriel's functor and localizations

1. Functors $\mathcal{F}^1$, $H_{\mathcal{F}}$ and $G_{\mathcal{F}}$. Let $\mathcal{F}$ be a uniform set of left ideals of R. Then for any R-modules M the set $\mathcal{F}M = \{x \in M \mid \mathrm{Ann}(x) \in \overline{\mathcal{F}}\}$ is a submodule of M. Following the tradition, call the submodule $\mathcal{F}M$ the $\mathcal{F}$-*torsion of* M. Clearly, the correspondence $M \mapsto \mathcal{F}M$ is functorial.

Denote by $\mathcal{F}^1 M$ the quotient of M modulo $\mathcal{F}M$. The map $M \mapsto \mathcal{F}^1 M$ extends up to a functor of R-mod into R-mod and the set of "projections" $\varphi_M^{\mathcal{F}} : M \to \mathcal{F}^1 M$, $M \in Ob\, R\text{-mod}$, turns out to be the morphism of functors $\varphi^{\mathcal{F}} : Id_{R\text{-mod}} \longrightarrow \mathcal{F}^1$.

Proposition. 1) *The map assigning the colimit* $\varinjlim_{m \in \mathcal{F}} Hom_R(m, M)$ *to* M *extends up to a functor* $H_{\mathcal{F}} : R\text{-mod} \to R\text{-mod}$. *The canonical morphisms* $\rho_M(\nu) : M \to Hom_R(\nu, M)$ *corresponding to the action* $\nu \otimes M \longrightarrow M$ *determine the morphism of functors*

$$\tau_{\mathcal{F}} : Id_{R\text{-mod}} \longrightarrow H_{\mathcal{F}} .$$

2) *If* $\mathcal{F}$ *is a cofilter, then* $\tau_{\mathcal{F}}$ *presents in the form of the composition* $Id_{R\text{-mod}} \xrightarrow{\varphi^{\mathcal{F}}} \mathcal{F}^1 \xrightarrow{\xi^{\mathcal{F}}} H_{\mathcal{F}}$, *where* $\xi^{\mathcal{F}}$ *is a monomorphism.*

Proof. 1) Let x be an arbitrary element of R and m a left ideal. Multiplying $(m : x)$ from the right by $x : (m : x) \longrightarrow m$ induces a morphism of Z-modules $Hom_R(m, M) \xrightarrow{\theta_{x,m}} Hom_R((m : x), M)$. Since $\mathcal{F}$ is a uniform set and $H_{\mathcal{F}} = H_{\overline{\mathcal{F}}}$, then the family of maps $\{\theta_{x,m} \mid m \in \mathcal{F}\}$ determines a morphism of Z-modules $\lambda_x : H_{\mathcal{F}}M \longrightarrow H_{\mathcal{F}}M$, i.e. the action of x. It is not difficult to verify that the family of maps $\{\lambda_x \mid x \in R\}$ determines an R-module structure on $H_{\mathcal{F}}M$.

2) Now let $\mathcal{F}$ be a topologizing set of ideals. Clearly, the kernel of the canonical morphism $M \to Hom_R(m, M)$

consists of all $x \in M$ such that $m \cdot x = 0$. This and the fact that $\mathcal{F}$ is codirected implies that the kernel of $\tau_{\mathcal{F}}$ coincides with $\mathcal{F}M$. Therefore there exists a monomorphism $\zeta^{\mathcal{F}}_M : \mathcal{F}^1 M \longrightarrow H_{\mathcal{F}} M$ uniquely determined by the identity $\tau_{\mathcal{F},M} = \zeta^{\mathcal{F}}_M \circ \varphi^{\mathcal{F}}_M$. $\square$

The composition of functors $\mathcal{F}^1$ and $H_{\mathcal{F}}$ is Gabriel's functor $G_{\mathcal{F}} = H_{\mathcal{F}} \circ \mathcal{F}^1$ and the composition of $\tau_{\mathcal{F}}$ and $H_{\mathcal{F}} \varphi^{\mathcal{F}}$ is the morphism $j_{\mathcal{F}} : Id_{R\text{-mod}} \longrightarrow G_{\mathcal{F}}$.

2. Functors $H^{\infty}_{\mathcal{F}}$ and $G^{\infty}_{\mathcal{F}}$. Denote by $H^{\infty}_{\mathcal{F}}$ and $G^{\infty}_{\mathcal{F}}$ colimits of inductive systems $\widetilde{H}_{\mathcal{F}} = \{ H^n_{\mathcal{F}} \xrightarrow{\tau_{\mathcal{F}} H^n_{\mathcal{F}}} H^{n+1}_{\mathcal{F}} \}_{n \geq 1}$ and $\widetilde{G}_{\mathcal{F}} = \{ G^n_{\mathcal{F}} \xrightarrow{j_{\mathcal{F}} G^n_{\mathcal{F}}} G^{n+1}_{\mathcal{F}} \}_{n \geq 1}$ respectively. The morphisms of functors $H_{\mathcal{F}} \varphi^{\mathcal{F}} : H_{\mathcal{F}} \to G_{\mathcal{F}}$ induce the morphism $\varphi^{\mathcal{F}}_{\infty} : H^{\infty}_{\mathcal{F}} \longrightarrow G^{\infty}_{\mathcal{F}}$.

Proposition. For any topologizing set $\mathcal{F}$ of left ideals of R the arrow $\varphi^{\mathcal{F}}_{\infty}$ is a functor isomorphism.

Sketch of the proof. We have the diagram

$$\begin{tikzcd}
H_{\mathcal{F}} \arrow[r, "\tau_{\mathcal{F}} H_{\mathcal{F}}"] \arrow[dr, "H_{\mathcal{F}} \varphi^{\mathcal{F}}"] & H^2_{\mathcal{F}} \\
Id \arrow[u, "\tau_{\mathcal{F}}"] \arrow[r, "j_{\mathcal{F}}"] & G_{\mathcal{F}} \arrow[u, "H_{\mathcal{F}} \zeta^{\mathcal{F}}"']
\end{tikzcd} \qquad (1)$$

Definition of $j_{\mathcal{F}}$ implies commutativity of its lower triangle and that of the upper triangle is equivalent to the identity $\tau_{\mathcal{F}} H_{\mathcal{F}} = H_{\mathcal{F}} \tau_{\mathcal{F}}$ which holds in a more general situation (see [17], Proposition II.1.1). The reader is invited to verify it independently.

Diagram (1)"extends" up to a family of evidently determined commuting diagrams

$$\begin{tikzcd}
H^{2^n}_{\mathcal{F}} \arrow[r, "\tau^{(n)}"] \arrow[dr, "\lambda^{(n)}"] & H^{2^{n+1}}_{\mathcal{F}} \\
G^{2^{n-1}}_{\mathcal{F}} \arrow[u, "\zeta^{(n)}"] \arrow[r, "j^{(n)}"] & G^{2^n}_{\mathcal{F}} \arrow[u, "\zeta^{(n+1)}"']
\end{tikzcd} \qquad (2)$$

Since $\{H_{\mathcal{F}}^{2^n} \xrightarrow{\tau(n)} H_{\mathcal{F}}^{2^{n+1}}\}_{n\geqslant 1}$ and $\{G_{\mathcal{F}}^{2^{n-1}} \xrightarrow{j(n)} G_{\mathcal{F}}^{2^n}\}_{n\geqslant 1}$ are final subsystems of inductive systems $\widetilde{H}_{\mathcal{F}}$ and $\widetilde{G}_{\mathcal{F}}$ respectively, then their colimits coincide with $H_{\mathcal{F}}^{\infty}$ and $G_{\mathcal{F}}^{\infty}$. The isomorphicy of $\varphi_{\infty}^{\mathcal{F}}$ follows directly from the existence of "diagonal" arrows in (2). $\square$

Concluding this section, list several properties of $H_{\mathcal{F}}$ and $G_{\mathcal{F}}$:

1) $H_{\mathcal{F}}$ is left exact. It is exact if $\mathcal{F}$ contains a cofinal subset (subcategory) of projective ideals.

2) Obviously, $H_{\mathcal{F}} = H_{\overline{\mathcal{F}}}$ and $G_{\mathcal{F}} = G_{\overline{\mathcal{F}}}$.

3) The identity $\tau_{\mathcal{F}} H_{\mathcal{F}} = H_{\mathcal{F}} \tau_{\mathcal{F}}$, playing the key role in the proof of isomorphicy of $\varphi_{\infty}^{\mathcal{F}}$, implies similar identities for $\mathcal{F}^1$ and $G_{\mathcal{F}}$: a) $\mathcal{F}^1 \varphi^{\mathcal{F}} = \varphi^{\mathcal{F}} \mathcal{F}^1$;
b) $G_{\mathcal{F}} j_{\mathcal{F}} = j_{\mathcal{F}} G_{\mathcal{F}}$.

4) Denote $R\text{-mod}^{\mathcal{F}}$ the full subcategory of the category R-mod of left R-modules formed by all the modules M such that $\mathcal{F}M = 0$ (such modules are called $\mathcal{F}$-torsion-free or just $\mathcal{F}$-free). The same identity $\tau_{\mathcal{F}} H_{\mathcal{F}} = H_{\mathcal{F}} \tau_{\mathcal{F}}$ implies the invariance of $R\text{-mod}^{\mathcal{F}}$ with respect to $H_{\mathcal{F}}$, since by Proposition 1 $\mathrm{Ob}\, R\text{-mod}^{\mathcal{F}}$ consists exactly of the R-modules M for which the canonical arrow $\tau_{\mathcal{F},M}: M \to H_{\mathcal{F}}M$ is a monomorphism. Clearly, the restriction of $G_{\mathcal{F}}$ onto $R\text{-mod}^{\mathcal{F}}$ coincides with $H_{\mathcal{F}}|_{R\text{-mod}^{\mathcal{F}}}$.

5) Let $\mathcal{F}$ be a topologizing set of finite type. Then $H_{\mathcal{F}}$ and all its iterations (including $H_{\mathcal{F}}^{\infty}$) commute with colimits of inductive systems $\{M_\alpha \xrightarrow{i_{\alpha\beta}} M_\beta\}_{\beta>\alpha}$ where all the arrows are monomorphisms. In particular, the functors $H_{\mathcal{F}}$ and $H_{\mathcal{F}}^{\infty}$ commute with colimits of modules.

6) The following implications hold:

[$H^{\infty}_{\mathcal{F}}$ transforms epimorphisms into epimorphisms] $\Longleftrightarrow$ [$H^{\infty}_{\mathcal{F}}$ is exact]

$\Uparrow$

[$G_{\mathcal{F}}$ transforms epimorphisms to epimorphisms] $\Longleftrightarrow$ [$G_{\mathcal{F}}$ is exact]

[$H_{\mathcal{F}}$ transforms epimorphisms to epimorphisms] $\Longleftrightarrow$ [$H_{\mathcal{F}}$ is exact] $\Longrightarrow$ [$G_{\mathcal{F}} = H_{\mathcal{F}}$].

All the implications but the last are the corollaries of the existence of the isomorphism $H^{\infty}_{\mathcal{F}} \simeq G^{\infty}_{\mathcal{F}}$ and the following general fact: a left exact additive functor is exact if and only if it sends epimorphisms into epimorphisms. The last implication is the result of applying the (exact) functor $H_{\mathcal{F}}$ to the exact sequence

$$0 \longrightarrow \mathcal{F}M \longrightarrow M \longrightarrow \mathcal{F}^{1}M \longrightarrow 0$$

If $\mathcal{F}$ contains a cofinal subset of projective ideals, then $H_{\mathcal{F}}$ is exact.

7) Let $\mathcal{F}$ be a topologizing set of finite type. Then [$H^{\infty}_{\mathcal{F}}$ sends epimorphisms to epimorphisms] $\Longleftrightarrow$ [$H^{\infty}_{\mathcal{F}}$ commutes with colimits] $\Longleftrightarrow$ [$H^{\infty}_{\mathcal{F}}$ has a right-adjoint functor].

This follows from 5), 6) (the exactness of $H^{\infty}_{\mathcal{F}}$) and the following general facts:

- an additive functor commuting with coproducts and cokernels commutes with arbitrary colimits;

- a functor $R\text{-}mod \to R\text{-}mod$ commuting with colimits possesses a right-adjoint.

Denote by $R^{(1)}$ the ring obtained of R by adjoining the unit. Any functor $F : R\text{-}mod \longrightarrow R\text{-}mod$ possessing a right adjoint is isomorphic to the functor $FR^{(1)} \otimes_R -$

This is proved as the similar statement in the unitary case: a) first, note that $FR^{(1)} \simeq FR^{(1)} \otimes_R R^{(1)}$; b) for every R-module M the canonical morphism $M \longrightarrow Hom_R(R^{(1)}, M)$ is a bijection; c) for any R-module M b) implies the existence of the exact sequence $\coprod_{\alpha \in \mathfrak{A}} R_\alpha^{(1)} \longrightarrow \coprod_{\gamma \in \Gamma} R_\gamma^{(1)} \longrightarrow M \longrightarrow 0$ where $R_\alpha^{(1)} = R_\gamma^{(1)} = R^{(1)}$ for all $(\alpha, \gamma) \in \mathfrak{A} \times \Gamma$; d) applying F and $FR^{(1)} \otimes_R -$ to the exact sequence of c) we get the desired isomorphism $FM \simeq FR^{(1)} \otimes_R M$ which can be obtained, however, as a corollary of the corresponding statement for the category of unitary modules making use of the existence of the canonical isomorphism of the category R-modules onto the category of unitary $R^{(1)}$-modules.

Therefore, if $\mathcal{F}$ is of finite type and $H_{\mathcal{F}}^{\infty}$ sends epimorphisms into epimorphisms, then $H_{\mathcal{F}}^{\infty} \simeq H_{\mathcal{F}}^{\infty} R^{(1)} \otimes_R -$ and therefore $H_{\mathcal{F}}^{\infty} R^{(1)}$ is a flat R-module.

If a topologizing set $\mathcal{F}$ is of finite type and contains a cofinal subset of projective ideals, then $H_{\mathcal{F}} \simeq G_{\mathcal{F}}$ is an exact functor with right adjoint, the same as $H_{\mathcal{F}}^{\infty} \simeq G_{\mathcal{F}}^{\infty}$.

The non-obvious in the above properties the reader may consider as a simple exercises.

3. $H_{\mathcal{F}}R$ -action on $H_{\mathcal{G}}M$.

Proposition. Let $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$ be topologizing sets of left ideals of R. For an arbitrary R-module M the canonical morphism $\xi_M^{\mathcal{F},\mathcal{G}} : H_{\mathcal{F}}R \otimes H_{\mathcal{G}}M \longrightarrow H_{\mathcal{G}\circ\mathcal{F}}M$ is determined such that the diagrams

$$\begin{tikzcd}
R \otimes M \arrow[r, "\xi_M"] \arrow[d, "\tau_{\mathcal{F},R} \otimes \tau_{\mathcal{G},M}"'] & M \arrow[d, "\tau_{\mathcal{G}\circ\mathcal{F},M}"] \\
H_{\mathcal{F}}R \otimes H_{\mathcal{G}}M \arrow[r, "\xi_M^{\mathcal{F},\mathcal{G}}"] & H_{\mathcal{G}\circ\mathcal{F}}M
\end{tikzcd}$$

$$\begin{tikzcd}
H_{\mathcal{F}'}R \otimes H_{\mathcal{F}}R \otimes H_{\mathcal{G}}M \arrow[rr] \arrow[d] & & H_{\mathcal{F}'}R \otimes H_{\mathcal{G}\circ\mathcal{F}}M \arrow[d] \\
 & H_{\mathcal{G}\circ(\mathcal{F}\circ\mathcal{F}')}M \arrow[dr] & \\
H_{\mathcal{F}\circ\mathcal{F}'}R \otimes H_{\mathcal{G}}M \arrow[ur] & & H_{(\mathcal{G}\circ\mathcal{F})\circ\mathcal{F}'}M
\end{tikzcd}$$

commute. Here $\zeta_M : R \otimes M \to M$ is the R-module structure on M.

Proof. 1) Let $\mathfrak{m}$ and $\mathfrak{n}$ be left ideals of R and $u \in Hom_R(\mathfrak{n}, R)$. If $\mathfrak{m} \in \mathcal{F} \in 2^{I_e R}$ and $\mathcal{F}$ is uniform, then the preimage $u^{-1}(\mathfrak{m})$ of m in $\mathfrak{n}$ belongs to $\mathcal{F} \circ \{\mathfrak{n}\}$.

In fact, for every subset x of $\mathfrak{n}$ we have

$$(u^{-1}(\mathfrak{m}) : x) = (\mathfrak{m} : u(x)).$$

2) Since the bifunctor $\otimes$ of tensor product commutes with colimits with respect to each argument, then the canonical arrow

$$\varinjlim_{\mathfrak{n} \in \mathcal{F}} \varinjlim_{\mathfrak{m} \in \mathcal{G}} \mathfrak{n}^*_R \otimes Hom_R(\mathfrak{m}, M) \longrightarrow H_{\mathcal{F}} R \otimes H_{\mathcal{G}} M$$

(here $\mathfrak{n}^*_R = Hom_R(\mathfrak{n}, R)$) is an isomorphism. Therefore it suffices to determine for any $\mathfrak{n} \in \mathcal{F}$ and $\mathfrak{m} \in \mathcal{G}$ the morphism $\zeta_M^{\mathfrak{n},\mathfrak{m}} : \mathfrak{n}^*_R \otimes Hom_R(\mathfrak{m}, M) \to H_{\mathcal{G} \circ \mathcal{F}} M$ functorially depending on $\mathfrak{n}$ and m.

To $u \in \mathfrak{n}^*_R$ the map $\hat{u}_{\mathfrak{m}} : Hom_R(\mathfrak{m}, M) \to Hom_R(u^{-1}(\mathfrak{m}), M)$ corresponds that assigns to a morphism $f : \mathfrak{m} \to M$ the composition of f and $u\Big|^{\mathfrak{m}}_{u^{-1}(\mathfrak{m})}$. Since $u^{-1}(\mathfrak{m}) \in \mathcal{G} \circ \{\mathfrak{n}\} \subset \mathcal{G} \circ \mathcal{F}$, it is possible to consider the composition $\hat{u}$ of $\hat{u}_{\mathfrak{m}}$ with the coprojection $Hom_R(u^{-1}(\mathfrak{m}), M) \to H_{\mathcal{G} \circ \mathcal{F}} M$.

The family of morphisms $\{\hat{u} \mid u \in \mathfrak{n}^*_R\}$ determine, as is not difficult to see, the action $\zeta_M^{\mathfrak{n},\mathfrak{m}} : \mathfrak{n}^*_R \otimes Hom_R(\mathfrak{m}, M) \to H_{\mathcal{G} \circ \mathcal{F}} M$; the morphisms $\{\zeta_M^{\mathfrak{n},\mathfrak{m}} \mid \mathfrak{n} \in \mathcal{F}, \mathfrak{m} \in \mathcal{G}\}$ uniquely determine the morphism $\zeta_M^{\mathcal{F},\mathcal{G}} : H_{\mathcal{F}} R \otimes H_{\mathcal{G}} M \to H_{\mathcal{G} \circ \mathcal{F}} M$, as is noted above. The verification of commutativity of the diagrams is straightforward. $\square$

Corollary 1. Let $\mathcal{F}$ be the radical set of left ideals of R. Then $H_{\mathcal{F} \circ \mathcal{F}} = H_{\mathcal{F}}$ and the morphisms $\zeta^{\mathcal{F}} \overset{\text{def}}{=} \zeta_R^{\mathcal{F},\mathcal{F}}$ and $\zeta_M^{\mathcal{F}} = \zeta_M^{\mathcal{F},\mathcal{F}}$ determine a ring structure on $H_{\mathcal{F}} R$ and the structure of a left $H_{\mathcal{F}}R$-module on $H_{\mathcal{F}}M$ for any R-module M. The ring $H_{\mathcal{F}}R=(H_{\mathcal{F}}R, \xi^{\mathcal{F}})$ is associative and unitary, and the canonical morphism $\tau_{\mathcal{F},R}: R \to H_{\mathcal{F}}R$ sends each right unit (if any) into the unit of $H_{\mathcal{F}}R$. The $H_{\mathcal{F}}R$-module $H_{\mathcal{F}}M$ is unitary for every $M \in Ob\, R\text{-mod}$.

Proof. The fact that $\xi^{\mathcal{F}} = \xi_R^{\mathcal{F},\mathcal{F}}$ is the structure of an associative ring, and $\xi_M^{\mathcal{F}} = \xi_M^{\mathcal{F},\mathcal{F}}$ is the structure of an $H_{\mathcal{F}}R$-mo-module is expressed by the commutativity of the diagram (1) in the particular case: $\mathcal{F}' = \mathcal{e}_J = \mathcal{F}$, $\mathcal{F} \circ \mathcal{F} = \mathcal{F}$.

Let $e_{\mathcal{F}}$ denotes the image of id_R under the canonical morphism $Hom_R(R,R) \longrightarrow H_{\mathcal{F}}R$. Clearly, for each $n \in I_\ell R$ the action $(\widehat{id})_n$ on $Hom_R(n, M)$ is identical (see the definition of the maps $\widehat{u}_n$ in the second paragraph of the proof of Proposition 3). This means that $e_{\mathcal{F}}$ acts on $H_{\mathcal{F}}M$ identically. In particular, (since the R-module M is arbitrary in these constructions) $e_{\mathcal{F}}$ is a left unit of $H_{\mathcal{F}}R$. On the other hand, for any left ideal m and any morphism $u \in Hom_R(m, R)$ we have $u^{-1}(R) = m$, and the map $\widehat{u}_R: Hom_R(R,M) \to Hom_R(m,R)$ is nothing else than the composition of morphisms: $f \longmapsto f \circ u$. It follows that $\widehat{u}_R(\widehat{id}_R) = u$ for each $u \in Hom_R(m,R)$ and $m \in I_\ell R$; therefore $e_{\mathcal{F}}$ is a right unit of $H_{\mathcal{F}}R$. The arrow $\tau_{\mathcal{F},R}: R \to H_{\mathcal{F}}R$ may be viewed as a composition of the map $R \longrightarrow Hom_R(R,R)$, which assigns to each element $x \in R$ the operator r_x of multiplication by x from the right, and the coprojection $Hom_R(R,R) \longrightarrow H_{\mathcal{F}}R$. If e' is a right unit of R, then $r_{e'} = id_R$; therefore $\tau_{\mathcal{F},R}(e') = e_{\mathcal{F}}$. $\square$

Corollary 2. For each radical filter $\mathcal{F}$ the functor $H_{\mathcal{F}}$ uniquely determines the functor ${}^uH_{\mathcal{F}}$ from R-mod into the category $H_{\mathcal{F}}R\text{-}{}^u mod$ of the unitary $H_{\mathcal{F}}R$-modules.

4. The $\mathcal{F}^1R$ -action on $\mathcal{G}^1M$. First of all, notice that there exists a commuting diagram

$$\begin{tikzcd} R\otimes M \arrow[r, "\varphi_R^{\mathcal{F}}\otimes 1_M"] \arrow[d] & \mathcal{F}^1R\otimes M \arrow[d, "\xi_M^{\mathcal{F}}"] \\ M \arrow[r, "\varphi_M^{\mathcal{F}}"] & \mathcal{F}^1M \end{tikzcd}$$

for any R-module M and topologizing set $\mathcal{F}$, where $\xi_M^{\mathcal{F}}$ is uniquely determined thanks to epimorphicy of $\varphi_R^{\mathcal{F}}\otimes 1_M$. In fact, if $x\in\mathcal{F}R$ then $x\cdot y\in\mathcal{F}M$ for any $y\in M$. Further, for a pair of topologizing sets $\mathcal{F}$ and $\mathcal{G}$ we have a commuting diagram

$$\begin{tikzcd} R\otimes M \arrow[r] \arrow[d] & \mathcal{F}^1R\otimes\mathcal{G}^1M \arrow[r] \arrow[d, dashed, "\xi_M^{\mathcal{F},\mathcal{G}}"] & H_{\mathcal{F}}R\otimes H_{\mathcal{G}}M \arrow[d, "\xi_M^{\mathcal{F},\mathcal{G}}"] \\ M \arrow[r] & (\mathcal{G}\circ\mathcal{F})^1M \arrow[r, "\xi^{\mathcal{G}\circ\mathcal{F}}"] & H_{\mathcal{G}\circ\mathcal{F}}M \end{tikzcd}$$

where $\xi_M^{\mathcal{F},\mathcal{G}}$ exists and is uniquely determined thanks to monomorphicy of $\xi^{\mathcal{G}\circ\mathcal{F}}$. As a result we have got a pair of arrows functorially depending on $\mathcal{F}$, $\mathcal{G}$ and M:

$$(\mathcal{G}\circ\mathcal{F})^1M \xleftarrow{\xi_M^{\mathcal{F},\mathcal{G}}} \mathcal{F}^1R\otimes\mathcal{G}^1M \xrightarrow{\xi^{\mathcal{F}}_{\mathcal{G}^1M}} \mathcal{F}^1\mathcal{G}^1M.$$

5. Compositions $\mathcal{F}^1\circ\mathcal{G}^1$ and $H_{\mathcal{F}}\circ H_{\mathcal{G}}$. The Morphisms occupying the last line of the above subsection are equivalent as the first of the following statements shows:

Proposition. Let $\mathcal{F}$ and $\mathcal{G}$ be topologizing sets of left ideals of R.

1) There exists an isomorphism of functors $\gamma^{\mathcal{F},\mathcal{G}}:\mathcal{F}^1\mathcal{G}^1\xrightarrow{\sim}(\mathcal{G}\circ\mathcal{F})^1$ and the diagram

$$\begin{tikzcd} \mathcal{F}^1\mathcal{G}^1M \arrow[rr, "\gamma_M^{\mathcal{F},\mathcal{G}}", "\sim"'] & & (\mathcal{G}\circ\mathcal{F})^1M \\ & \mathcal{F}^1R\otimes\mathcal{G}^1M \arrow[ul, "\xi^{\mathcal{F}}_{\mathcal{G}^1M}"] \arrow[ur, "\xi_M^{\mathcal{F},\mathcal{G}}"'] & \end{tikzcd} \qquad (1)$$

commutes.

2) There exists a canonical morphism $\psi^{\mathcal{F},\mathcal{G}}: H_{\mathcal{G}\circ\mathcal{F}} \to H_{\mathcal{F}}\circ H_{\mathcal{G}}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & H_{\mathcal{F}} \arrow[d] \arrow[dr, "H_{\mathcal{F}}\tau_{\mathcal{G}}"] & \\
Id_{R\text{-mod}} \arrow[ur, "\tau_{\mathcal{F}}"] \arrow[r, "\tau_{\mathcal{G}\circ\mathcal{F}}"] \arrow[dr, "\tau_{\mathcal{G}}"'] & H_{\mathcal{G}\circ\mathcal{F}} \arrow[r, "\psi^{\mathcal{F},\mathcal{G}}"] & H_{\mathcal{F}}\circ H_{\mathcal{G}} \\
 & H_{\mathcal{G}} \arrow[u] \arrow[ur, "\tau_{\mathcal{F}}H_{\mathcal{G}}"'] &
\end{tikzcd} \qquad (2)$$

3) If M is $\mathcal{F}$-torsion free and $\mathcal{G}$-torsion free R-module, then $\psi_M^{\mathcal{F},\mathcal{G}}: H_{\mathcal{G}\circ\mathcal{F}}M \to H_{\mathcal{F}}H_{\mathcal{G}}M$ is an isomorphism.

Sketch of the proof. 1) Let M be an R-module, $x \in \operatorname{Ker}\phi^{\mathcal{F}}_{\mathcal{G}'M}\circ\phi^{\mathcal{G}}_M$. This means that $n\cdot\phi^{\mathcal{G}}_M(x)=0$ for an ideal $n\in\mathcal{F}$ or equivalently $n\cdot x\in\mathcal{G}M$.

Clearly, $[n\cdot x\in\mathcal{G}M] \Leftrightarrow [\lambda\cdot x\in\mathcal{G}M$ for any $\lambda\in\mathcal{P}(n)] \Leftrightarrow$ $\Leftrightarrow$ [for every $\lambda\in\mathcal{P}(n)$ there exists $m_\lambda\in\mathcal{G}$ such that $m_\lambda\lambda\cdot x=0] \Leftrightarrow [\operatorname{Ann}(x)\in\mathcal{G}\circ\{n\}\subset\mathcal{G}\circ\mathcal{F}] \Rightarrow [x\in\ker\phi_M^{\mathcal{G}\circ\mathcal{F}}]$.

Conversely, if $\operatorname{Ann}(x)\in\mathcal{G}\circ\{n\}$ for some $n\in\mathcal{F}$, then, clearly, $\operatorname{Ann}(\phi^{\mathcal{G}}_M(x))\supset n$, $n\in\mathcal{F}$ and therefore $\phi^{\mathcal{F}}_{\mathcal{G}'M}(\phi^{\mathcal{G}}_M(x))=0$. Since $\phi^{\mathcal{F}}_{\mathcal{G}'M}\circ\phi^{\mathcal{G}}_M$ and $\phi_M^{\mathcal{G}\circ\mathcal{F}}$ are epimorphisms, the coincidence of their kernels implies the existence of the isomorphism $\gamma_M^{\mathcal{F},\mathcal{G}}: \mathcal{F}'\mathcal{G}'M \xrightarrow{\sim} (\mathcal{G}\circ\mathcal{F})'M$.

The verification of the commutativity of 1) is left to the reader.

2) Let, as earlier, M be an R-module; $x\in H_{\mathcal{G}\circ\mathcal{F}}M$; n an ideal of $\mathcal{G}\circ\mathcal{F}$ such that x is the image of some $x_n\in\operatorname{Hom}_R(n,M)$ with respect to the coprojection $\operatorname{Hom}_R(n,M)\longrightarrow H_{\mathcal{G}\circ\mathcal{F}}M$. By definition of R-action on $H_{\mathcal{G}\circ\mathcal{F}}M$ the morphism $n\xrightarrow{n\cdot x} H_{\mathcal{G}\circ\mathcal{F}}M$ of multiplication

of n by x factors through $\tau_{\mathcal{G}\circ\mathcal{F},M}: M \longrightarrow H_{\mathcal{G}\circ\mathcal{F}}M$. Let $n \in \mathcal{G}\circ\{m\}$, where $m \in \mathcal{F}$. Then λx is the image of a Z-submodule x_λ of finite type of the Z-module $\mathrm{Hom}_R((n:\lambda),M)$ for any $\lambda \in \mathcal{P}(m)$. Since $(n:\lambda) \in \overline{\mathcal{G}}$, there exists a coprojection $\mathrm{Hom}_R((n:\lambda),M) \longrightarrow H_{\overline{\mathcal{G}}}M = H_{\mathcal{G}}M$ which easily implies that the multiplication of x by m factors through $H_{\mathcal{G}}M \longrightarrow H_{\mathcal{G}\circ\mathcal{F}}M$:

$$\begin{tikzcd}
 & m \arrow[rr, "m\cdot x"] \arrow[drr, dashed, "x_m"] & & H_{\mathcal{G}\circ\mathcal{F}}M \\
\lambda \arrow[ur, hook] \arrow[r, "g_\lambda"'] & \mathrm{Hom}_R((n:\lambda),M) \arrow[rr] & & H_{\mathcal{G}}M \arrow[u]
\end{tikzcd} \qquad (3)$$

Denote $\hat{x}$ the image of $x_m \in \mathrm{Hom}_R(m, H_{\mathcal{G}}M)$ with respect to the coprojection $\mathrm{Hom}_R(m, H_{\mathcal{G}}M) \longrightarrow H_{\mathcal{F}}H_{\mathcal{G}}M$. It is not difficult to see that $\hat{x}$ does not depend on the arbitrariness in the choice of $n \in \mathcal{G}\circ\mathcal{F}$ and $m \in \mathcal{F}$. Therefore $\psi_M^{\mathcal{F},\mathcal{G}}: H_{\mathcal{G}\circ\mathcal{F}}M \rightarrow H_{\mathcal{F}}H_{\mathcal{G}}M$ is well defined. It remains to verify that $\psi_M^{\mathcal{F},\mathcal{G}}$ is an R-module morphism and prove the commutativity of (2). This is left to the reader.

3) Now let M be an $\mathcal{F}$ -torsion free and $\mathcal{G}$ -torsion free module or, equivalently, the canonical arrows $\tau_{\mathcal{F},M}$ and $\tau_{\mathcal{G},M}$ are injective. Then the "through map" $M \xrightarrow{\tau_{\mathcal{F},M}} H_{\mathcal{F}}M \xrightarrow{H_{\mathcal{F}}\tau_{\mathcal{G},M}} H_{\mathcal{F}}H_{\mathcal{G}}M$ is also a monoarrow, since $H_{\mathcal{F}}$ sends monomorphisms into monomorphisms.

Let $x \in H_{\mathcal{F}}H_{\mathcal{G}}M$; m be an ideal of $\mathcal{F}$ such that x is the image of some $x_m \in \mathrm{Hom}_R(m, H_{\mathcal{G}}M)$ with respect to the coprojection $\mathrm{Hom}_R(m, H_{\mathcal{G}}M) \longrightarrow H_{\mathcal{F}}H_{\mathcal{G}}M$. For any $\lambda \in \mathcal{P}(m)$ there exists $n_\lambda \in \mathcal{G}$ such that the restriction x_λ of x_m onto λ factors through $\mathrm{Hom}_R(n_\lambda, M) \longrightarrow H_{\mathcal{G}}M$.

Denote $\tau_{\mathcal{F},H_{\mathcal{G}}M} \circ \tau_{\mathcal{G},M}$ by $\tau^{(2)}$. The commutativity of

(2) implies $\tau^{(2)} = H_{\mathcal F}\tau_{\mathcal G, M} \circ \tau_{\mathcal F, M}$ and, in particular, the monomorphicy of $\tau^{(2)}$. Let Ξ_x be the full subcategory of $I_\ell R$ formed by the ideals $\mathfrak n$ such that $\mathfrak n \cdot x$ factors through $\tau^{(2)} : M \longrightarrow H_{\mathcal F} H_{\mathcal G} M$, i.e. $\mathfrak n \cdot x = \tau^{(2)} \circ q_{\mathfrak n}$.

(i) Ξ_x contains all the ideals $(\mathfrak n : \lambda)\lambda$, $\lambda \in \mathcal P(m)$. In fact, there exists a commutative diagram

$$\begin{tikzcd}
(\mathfrak n:\lambda)\otimes\lambda \arrow[r, two heads] \arrow[dr, "\hat q_\lambda"'] & (\mathfrak n:\lambda)\lambda \arrow[rr, "(\mathfrak n:\lambda)\lambda\cdot x"] \arrow[d, dashed] & & H_{\mathcal F} H_{\mathcal G} M \\
& M \arrow[urr, "\tau^{(2)}"] \arrow[rr] & & H_{\mathcal G} M \arrow[u]
\end{tikzcd} \qquad (4)$$

where $\hat q_\lambda$ is a morphism "adjoint" to the morphism g_λ of diagram (3). The dotted line exists thanks to monomorphicy of $\tau^{(2)}$.

(ii) If $\{\mathfrak n_1, \mathfrak n_2\} \subset \Xi_x$, then $\mathfrak n_1 + \mathfrak n_2 = \sup(\mathfrak n_1, \mathfrak n_2) \in \Xi_x$. This is clear from the diagram

$$\begin{tikzcd}
\mathfrak n_1 \amalg \mathfrak n_2 \arrow[r, "q_{\mathfrak n_1} \times q_{\mathfrak n_2}"] \arrow[d, two heads] & M \arrow[d, tail, "\tau^{(2)}"] \\
\mathfrak n_1 + \mathfrak n_2 \arrow[ur, dashed] \arrow[r, "(\mathfrak n_1 + \mathfrak n_2)\cdot x"'] & H_{\mathcal F} H_{\mathcal G} M
\end{tikzcd} \qquad (5)$$

which is also commutative and in which the dotted line, as in (4), exists thanks to the monomorphicy of $\tau^{(2)}$.

(iii) Clearly, together with any ascending family of ideals $\{\mathfrak n_i \mid i \in J\}$ the category Ξ_x contains its upper bound $\sup\{\mathfrak n_i \mid i \in J\}$.

It follows from (ii) and (iii) that Ξ_x possesses a final object m_x. Thanks to (i) m_x belongs to $\mathcal G \circ \{m\} \subset \mathcal G \circ \mathcal F$.

Let $m_x \cdot x = \tau^{(2)} \circ f_x$. Notice that thanks to monomorphicy of $\tau^{(2)}$ the morphism f_x is uniquely determined.

Assign to an element $x \in R$ the image $\hat{x}$ of the morphism f_x under the canonical mapping $Hom_R(m_x, \mathcal{M}) \longrightarrow$ $\longrightarrow H_{\mathcal{G}\circ\mathcal{F}}\mathcal{M}$. A direct verification shows that the map $\varphi : x \longmapsto \hat{x}$ is inverse to the map $\psi^{\mathcal{F},\mathcal{G}}_{\mathcal{M}}$ constructed in the preceding step of the proof. □

Corollary 1. *Let $\mathcal{F}$ and $\mathcal{G}$ are radical filters of left ideals of R.*

1) *There exists a unique ring structure on $\mathcal{F}^1 R$ such that $\varphi^{\mathcal{F}}_R : R \longrightarrow \mathcal{F}^1 R$ is a ring morphism; and for any R-module M there exists a unique $\mathcal{F}^1R$-module structure on $\mathcal{F}^1\mathcal{M}$ such that the diagram*

$$\begin{tikzcd}
R \otimes \mathcal{M} \arrow[r, "\varphi^{\mathcal{F}}_R \otimes \varphi^{\mathcal{F}}_{\mathcal{M}}"] \arrow[d] & \mathcal{F}^1 R \otimes \mathcal{F}^1 \mathcal{M} \arrow[d] \\
\mathcal{M} \arrow[r, "\varphi^{\mathcal{F}}_{\mathcal{M}}"] & \mathcal{F}^1 \mathcal{M}
\end{tikzcd}$$

commutes.

2) *There exists a canonical isomorphism $G_{\mathcal{G}\circ\mathcal{F}} \xrightarrow{\sim}$ $\xrightarrow{\sim} G_{\mathcal{F}} \circ G_{\mathcal{G}}$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & G_{\mathcal{F}} \arrow[dr] \arrow[drr, "G_{\mathcal{F}} j_{\mathcal{G}}"] & & \\
Id_{R\text{-mod}} \arrow[ur, "j_{\mathcal{F}}"] \arrow[dr, "j_{\mathcal{G}}"'] & & G_{\mathcal{G}\circ\mathcal{F}} \arrow[r] & G_{\mathcal{F}} \circ G_{\mathcal{G}} \\
 & G_{\mathcal{G}} \arrow[ur] \arrow[urr, "j_{\mathcal{F}} G_{\mathcal{G}}"'] & &
\end{tikzcd}$$

Moreover, $G_{\mathcal{F}} j_{\mathcal{F}}$ is an isomorphism and $G_{\mathcal{F}} j_{\mathcal{F}} = j_{\mathcal{F}} G_{\mathcal{F}}$.

Proof. The first statement follows from heading 1) of Proposition 5, the second one from headings 2),3) and from the implications

$[\tau_{\mathcal{E},M}, \tau_{\mathcal{F},M}$ are monomorphisms $] \Leftrightarrow$
$\Leftrightarrow [H_{\mathcal{F}}\tau_{\mathcal{E},M} \circ \tau_{\mathcal{F},M}$ is a monomorphism$] \Longleftrightarrow$ [the module M is $\mathcal{E} \circ \mathcal{F}$-torsion free],
which also follow from Proposition 5.

The isomorphicy of $G_{\mathcal{F}}\, j_{\mathcal{F}}$ and the equality $G_{\mathcal{F}}\, j_{\mathcal{F}} = j_{\mathcal{F}} G_{\mathcal{F}}$ follow straightforwardly from the commutativity of the diagram in heading 2) in particular case $\mathcal{F} = \mathcal{E}$. □

Corollary 2. *Let $\mathcal{F}$ be a radical filter of the left ideals of R. Then there exists a uniquely determined structure of a ring on $G_{\mathcal{F}} R$ and uniquely determined structure of $G_{\mathcal{F}} R$-module on $G_{\mathcal{F}} M$ for any R-module M such that the diagram*

$$\begin{tikzcd}
G_{\mathcal{F}} R \otimes G_{\mathcal{F}} M \arrow[d] & R \otimes M \arrow[l, "j_{\mathcal{F},R} \otimes j_{\mathcal{F},M}"'] \arrow[d] \\
G_{\mathcal{F}} M & M \arrow[l, "j_{\mathcal{F},M}"']
\end{tikzcd}$$

commutes. The ring $G_{\mathcal{F}} R$ has a unit, and the $G_{\mathcal{F}} R$-module $G_{\mathcal{F}} M$ turn out to be unitary for any R-module M. The canonical arrow $j_{\mathcal{F},R} : R \longrightarrow G_{\mathcal{F}} R$ is a ring morphism which sends the right units of R (if any) into the unit of the ring $G_{\mathcal{F}} R$.

Proof. The statement follows directly from the first heading of Corollary 1 and from Corollary 1 of Proposition 3. □

So, the map $M \longmapsto G_{\mathcal{F}} M$ extends uniquely to a functor from the category of R-modules into the category of the unitary $G_{\mathcal{F}} R$ -modules, if $\mathcal{F}$ is a radical filter (see Corollary 2 of Proposition 3). Denote this functor by ${}^{u}G_{\mathcal{F}}$.

6. <u>Localizations</u>. For an arbitrary subset $\mathcal{G}$ of $I_\ell R$ consider the full subcategory R-mod$_{\mathcal{G}}$ of R-mod formed by all the modules M such that the canonical maps $\rho_M(m): M \longrightarrow \longrightarrow \mathrm{Hom}_R(m, M)$ are isomorphisms for all $m \in \mathcal{G}$.

A $\mathcal{G}$*-localization* of R-module M is an R-module $\mathcal{G}^{-1}M$ from R-mod$_{\mathcal{G}}$ such that there exists a universal arrow $M \longrightarrow \mathcal{G}^{-1}M$; i.e. any morphism $M \longrightarrow M'$, where $M' \in Ob\,R\text{-}mod_{\mathcal{G}}$, uniquely represents as the composition $M \longrightarrow \mathcal{G}^{-1}M \longrightarrow M'$.

<u>Proposition</u>. 1) *For an arbitrary subset* $\mathcal{G}$ *of* $I_\ell R$ *and any left* R-*module* M *there exists a* $\mathcal{G}$*-localization* $\mathcal{G}^{-1}M$. *The map* $M \longmapsto \mathcal{G}^{-1}M$ *uniquely extends up to a functor* $\mathcal{G}^{-1}: R\text{-}mod \longrightarrow R\text{-}mod_{\mathcal{G}}$, *which is left adjoint to the embedding* $J_{\mathcal{G}}: R\text{-}mod_{\mathcal{G}} \hookrightarrow R\text{-}mod$.

2) Let $\mathfrak{F}$ be a topologizing set of left ideals of R. Then

A) For an R-module M the following conditions are equivalent:

(i) $\tau_{\mathfrak{F},M}: M \to H_{\mathfrak{F}}M$ is isomorphism;

(ii) $\tau^{\infty}_{\mathfrak{F},M}: M \to H^{\infty}_{\mathfrak{F}}M$ is isomorphism;

(iii) $M \in R\text{-mod}_{\mathfrak{F}}$.

B) $\mathfrak{F}^{-1}$ is isomorphic to the functor $G_{\hat{\mathfrak{F}}}|^{R\text{-mod}_{\mathfrak{F}}}$ where $\hat{\mathfrak{F}}$ is the minimal of radical filters containing $\mathfrak{F}$, and $\tau_{\hat{\mathfrak{F}},M}$ for any $M \in Ob\, R\text{-mod}$ is the universal arrow.

C) If $\mathfrak{F}$ is of finite type, then the restrictions of $H^{\infty}_{\mathfrak{F}}$ and $G_{\hat{\mathfrak{F}}}$ onto the subcategory $R\text{-mod}^{\mathfrak{F}}$ of $\mathfrak{F}$-torsion-free R-modules are isomorphic. If, besides, $H^{\infty}_{\mathfrak{F}}$ sends epimorphisms into epimorphisms, then $H^{\infty}_{\mathfrak{F}}$ is exact, possesses a right-adjoint and is isomorphic to $G_{\hat{\mathfrak{F}}}$.

Sketch of the proof. 1) Since the functors $Hom_R(m,-)$ commute with inverse limits, then $R\text{-mod}_{\mathcal{G}}$ contains the limit of any diagram $\mathcal{D}: \delta\mathcal{D} \to R\text{-mod}$ with values in $R\text{-mod}_{\mathcal{G}}$, i.e. the embedding $J_{\mathcal{G}}: R\text{-mod}_{\mathcal{G}} \hookrightarrow R\text{-mod}$ commutes with $\varprojlim$. Since $\mathcal{G}^{-1}M \simeq \varprojlim(M\backslash J_{\mathcal{G}} \xrightarrow{res} R\text{-mod}_{\mathcal{G}})$, the domain of the map $M \mapsto \mathcal{G}^{-1}M$ coincides with the set of M for which a morphism $M \longrightarrow \mathcal{N}$ with $\mathcal{N} \in R\text{-mod}_{\mathcal{G}}$ exists. Since the zero module belongs to $R\text{-mod}_{\mathcal{G}}$, then $\mathcal{G}^{-1}$ is determined on the whole $Ob\, R\text{-mod}$.
It is easy to verify that there exists a unique extension of the map $M \longmapsto \mathcal{G}^{-1}M$ up to a functor $\mathcal{G}^{-1}: R\text{-mod} \to R\text{-mod}_{\mathcal{G}}$ such that the family of universal arrows $\varkappa_{\mathcal{G}} = \{\varkappa(M): M \longrightarrow \mathcal{G}^{-1}M\}$ is a morphism $Id_{R\text{-mod}} \longrightarrow$

$\longrightarrow J_{e_J} \circ e_J^{-1}$. Clearly, there exists an isomorphism $\psi_{e_J}: e_J^{-1} \circ J_{e_J} \longrightarrow$ $\longrightarrow Id_{R\text{-mod}_{e_J}}$ and $(\varkappa_{e_J}, \psi_{e_J})$ are unit and counit of adjunction $e_J^{-1} \dashv J_{e_J}$.

2) Let $\mathcal{F}$ be a topologizing set of left ideals of R.

A) If $\mathcal{F}'$ is a uniform subset of $I_e R$ and M a $\mathcal{F}'$-torsion free R-module, then for any $m \in \mathcal{F}'$ and an arbitrary ideal n containing m the "restriction" morphism $\pi_{n,m}: Hom_R(n,M) \to Hom_R(m,M)$ is injective.

In fact, let $f: n \longrightarrow M$ be an R-module morphism such that $f|_m = 0$. Then $(m:a)f(a) = f((m:a)a) = 0$ for any $a \in n$. Since by hypothesis $\{(m:a) | a \in n\} \subset \mathcal{F}'$ and M is $\mathcal{F}'$-torsion free, then $f(a) = 0$ for all $a \in n$.

(i) $\Longrightarrow$ (ii) The $H_{\mathcal{F}}$-invariance of $R\text{-mod}^{\mathcal{F}}$ yields the following implications: $[\tau_{\mathcal{F},M}$ is monomorphism$] \Longleftrightarrow$ $\Longleftrightarrow [\tau_{\mathcal{F},H_{\mathcal{F}}^k M}$ is monomorphism for all $k \geqslant 0] \Longleftrightarrow [\tau^{\infty}_{\mathcal{F},M}$ is monomorphism$]$ $\Longleftrightarrow$ [the canonical arrow $H_{\mathcal{F}}^k M \to H_{\mathcal{F}}^{\infty} M$ (equal to $H_{\mathcal{F}}^k \tau^{\infty}_{\mathcal{F},M}$) is monomorphism for all $k \geqslant 0]$.

Clearly, $[\tau_{\mathcal{F},M}$ is isomorphism$] \Longrightarrow [\tau^{\infty}_{\mathcal{F},M}$ is isomorphism$] \Longrightarrow [\tau^{\infty}_{\mathcal{F},H_{\mathcal{F}}M}$ is epimorphism$]$. But, as has been just noted, $\tau^{\infty}_{\mathcal{F},H_{\mathcal{F}}M}$ is monomorphism. Therefore $\tau^{\infty}_{\mathcal{F},H_{\mathcal{F}}M}$ is isomorphism and, hence, so is

$$\tau_{\mathcal{F},M} = (\tau^{\infty}_{\mathcal{F},H_{\mathcal{F}}M})^{-1} \circ \tau^{\infty}_{\mathcal{F},M}.$$

(i) $\Longrightarrow$(ii). Clearly, $\tau_{\mathcal{F},M}$ is isomorphism if $M \in R\text{-mod}_{\mathcal{F}}$, since all the arrows $\rho_M(m): M \longrightarrow Hom_R(m,M)$, $m \in \mathcal{F}$, are isomorphisms. Conversely, let $\tau_{\mathcal{F},M}$ be an isomorphism. It is not difficult to see (if one looks at the factorization of $\tau_{\mathcal{F},M}$ into the composition $M \xrightarrow{\rho_M(m)} Hom_R(m,M) \longrightarrow$

$\xrightarrow{\pi_m^{\mathcal{F}}} H_{\mathcal{F}}M)$ that this implies the monomorphicy of all $\rho_M(m)$, i.e. the fact that M is $\mathcal{F}$-torsion-free, and the epimorphicy of all the co-projections $\pi_m^{\mathcal{F}}: Hom_R(m,M) \longrightarrow H_{\mathcal{F}}M$. On the other hand, the coprojections $\pi_m^{\mathcal{F}}$ are monomorphic as follows from the statement that appeared at the beginning of the proof of n. A). Therefore all $\pi_m^{\mathcal{F}}$ and hence all $\rho_M(m)$ are isomorphisms.

B) Let $\mathcal{F}$ be a topologizing set, $M \in Ob\, R\text{-}mod_{\mathcal{F}}$; Ω_M the full subcategory of the category $\mathcal{T}_\ell R$ of topologizing sets consisting of all $\mathcal{G} \in \mathcal{T}_\ell R$ such that $\mathcal{F} \subset \mathcal{G}$ and $\tau_{\mathcal{G},M}: M \to H_{\mathcal{G}}M$ is isomorphism. Proposition 5 implies that $[\mathcal{G} \in \Omega_M \ni \mathcal{G}'] \Rightarrow [\mathcal{G} \circ \mathcal{G}' \in \Omega_M]$ since $H_{\mathcal{G}\circ\mathcal{G}'}M \simeq H_{\mathcal{G}'}H_{\mathcal{G}}M$. In particular, Ω_M is an directed category, since $\mathcal{G}, \mathcal{G}' \subset \mathcal{G} \circ \mathcal{G}'$. Therefore the union of all the sets of Ω_M also belongs to Ω_M. This union (we will denote it by $\mathcal{F}_{(M)}$) is, obviously, a finite object of Ω_M and at the same time $\mathcal{F}_{(M)} \circ \mathcal{F}_{(M)} \in \Omega_M$. Therefore $\mathcal{F}_{(M)} \circ \mathcal{F}_{(M)} = \mathcal{F}_{(M)}$. Clearly, $R\text{-}mod_{\mathcal{F}} = R\text{-}mod_{\widetilde{\mathcal{F}}}$, where $\widetilde{\mathcal{F}} = \bigcap\{\mathcal{F}_{(M)} \mid M \in Ob\,R\text{-}mod_{\mathcal{F}}\}$ is a radical filter containing $\mathcal{F}$ and therefore $\widehat{\mathcal{F}}$.

Since $\widehat{\mathcal{F}}$ is a radical set, then $\tau_{\widehat{\mathcal{F}}} G_{\widehat{\mathcal{F}}} = j_{\widehat{\mathcal{F}}} G_{\widehat{\mathcal{F}}}$ is isomorphism by Corollary 5. It follows from A) that $Im\, G_{\widehat{\mathcal{F}}} \subset R\text{-}mod_{\mathcal{F}}$ and $\{j_{\widehat{\mathcal{F}},M}\}$ are universal arrows.

C) Let $\mathcal{F}$ be a topologizing set of finite type.

a) Since $H_{\mathcal{F}}$ commutes with colimits of inductive systems, all the arrows of which are monomorphisms, then, in particular, the canonical morphism $H_{\mathcal{F}}^{\infty}M = \varinjlim_k H_{\mathcal{F}}^k M \longrightarrow$ $\longrightarrow H_{\mathcal{F}}H_{\mathcal{F}}^{\infty}$ is isomorphism for any $\mathcal{F}$-torsion free R-module M. A) implies the universality of the arrow

$$\tau^{\infty}_{\mathcal{F},M} : M \longrightarrow H^{\infty}_{\mathcal{F}} M$$

b) If $H^{\infty}_{\mathcal{F}}$ transforms epimorphisms into epimorphisms, then it is exact and commutes with arbitrary colimits (see section 2, properties 5), 6)) and, in particular, the canonical arrow $H^{\infty}M = \varinjlim_k H^{\infty}_{\mathcal{F}}(H^k_{\mathcal{F}}M) \longrightarrow (H^{\infty}_{\mathcal{F}})^2 M$ is an isomorphism. It remains again to refer to A). □

Corollary. Let $\mathcal{F}$ be a topologizing set of left ideals of R.

1) $\mathcal{F}^{-1}R$ possesses a canonical ring structure, and for any R-module M there is a canonical $\mathcal{F}^{-1}R$-module structure on $\mathcal{F}^{-1}M$ such that the diagram

$$\begin{array}{ccc} R\otimes M & \longrightarrow & \mathcal{F}^{-1}R\otimes\mathcal{F}^{-1}M \\ \downarrow & & \downarrow \\ M & \longrightarrow & \mathcal{F}^{-1}M \end{array}$$

commutes. Thus, the natural embedding $R\text{-mod}_{\mathcal{F}} \hookrightarrow \mathcal{F}^{-1}R\text{-mod}$ is well-defined.

2) If $\mathcal{F}$ is a set of finite type and $H^{\infty}_{\mathcal{F}}$ sends epimorphisms into epimorphisms, then $\mathcal{F}^{-1}$ is isomorphic to $\mathcal{F}^{-1}R^{(1)} \otimes_R -$. This takes place when the subset of projective ideals is cofinal in $\mathcal{F}$.

3) The natural arrow $G_{\hat{\mathcal{F}}} \longrightarrow H^2_{\hat{\mathcal{F}}}$ is an isomorphism.

4) $G_{\hat{\mathcal{F}}}$ is left exact and $\mathcal{F}^{-1}$ is exact.

Proof. 1) follows from Corollary of Proposition 5 and the isomorphism $\mathcal{F}^{-1} \simeq G_{\hat{\mathcal{F}}}\big|^{R\text{-mod}_{\mathcal{F}}}$; 2) follows from heading C) of Proposition and the property 7) in subsection 2.

3) Let M be an arbitrary R-module. Since $H_{\hat{\mathcal{F}}}$ is left exact, it transforms the exact sequence

$$0 \longrightarrow \hat{\mathcal{F}}M \longrightarrow M \longrightarrow \hat{\mathcal{F}}^1 M \longrightarrow 0$$

into the exact sequence

$$0 \longrightarrow H_{\hat{\mathfrak{F}}}\hat{\mathfrak{F}}\mathcal{M} \longrightarrow H_{\hat{\mathfrak{F}}}\mathcal{M} \xrightarrow{H_{\hat{\mathfrak{F}}}\varphi^{\mathfrak{F}}} H_{\hat{\mathfrak{F}}}\hat{\mathfrak{F}}^1\mathcal{M} = G_{\hat{\mathfrak{F}}}\mathcal{M} \qquad (1)$$

The exactness of (1) and the equality $H_{\hat{\mathfrak{F}}}\hat{\mathfrak{F}}\mathcal{M} = 0$ imply monomorphicy of the canonical arrow $H_{\hat{\mathfrak{F}}}\varphi^{\mathfrak{F}} : H_{\hat{\mathfrak{F}}}\mathcal{M} \longrightarrow G_{\hat{\mathfrak{F}}}\mathcal{M}$. Thus, all the arrows of the commutative diagram

$$\begin{tikzcd} H_{\hat{\mathfrak{F}}}\mathcal{M} \arrow[rr] & & G_{\hat{\mathfrak{F}}}\mathcal{M} \\ & \hat{\mathfrak{F}}^1\mathcal{M} \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad (2)$$

are monomorphisms and, thanks to the same left exactness of $H_{\hat{\mathfrak{F}}}$, so are the arrows of the diagram

$$\begin{tikzcd} H^2_{\hat{\mathfrak{F}}}\mathcal{M} \arrow[rr] & & H_{\hat{\mathfrak{F}}}G_{\hat{\mathfrak{F}}}\mathcal{M} = G^2_{\hat{\mathfrak{F}}}\mathcal{M} \\ & H_{\hat{\mathfrak{F}}}\hat{\mathfrak{F}}^1\mathcal{M} = G_{\hat{\mathfrak{F}}}\mathcal{M} \arrow[ul] \arrow[ur, "G_{\hat{\mathfrak{F}}}j_{\hat{\mathfrak{F}},\mathcal{M}}"'] & \end{tikzcd} \qquad (3)$$

By Proposition 6 $G_{\hat{\mathfrak{F}}}j_{\hat{\mathfrak{F}},\mathcal{M}}$ is an isomorphism; hence the monoarrow $H_{\hat{\mathfrak{F}}}\mathcal{M} \longrightarrow G^2_{\hat{\mathfrak{F}}}\mathcal{M}$ is an epimorphism and, therefore, an isomorphism. The isomorphicy of two arrows in (3) implies that of third one -- $G_{\hat{\mathfrak{F}}}\mathcal{M} \longrightarrow H^2_{\hat{\mathfrak{F}}}\mathcal{M}$.

4) The left exactness of $H_{\hat{\mathfrak{F}}}$ and the just established isomorphism $G_{\hat{\mathfrak{F}}} \simeq H^2_{\hat{\mathfrak{F}}}$ imply the left exactness of $G_{\hat{\mathfrak{F}}}$ and, hence, of $\mathfrak{F}^{-1} = G_{\hat{\mathfrak{F}}}\big|^{R\text{-mod}_{\mathfrak{F}}}$. On the other hand, $\mathfrak{F}^{-1}$ possesses a right adjoint functor; in particular, $\mathfrak{F}^{-1}$ is right exact. □

<u>Remark.</u> If the conditions of heading 2) of Corollary are satisfied and R is the ring with a right unit e, then the restriction of $\mathfrak{F}^{-1}$ onto the full subcategory R-^{u}mod of R-mod formed by unitary modules (i.e. the modules on which e acts identically; it is easy to see that the unitarity of a module does not depend of the choice of right unit) is isomorphic to the restriction of $\mathfrak{F}^{-1}R \otimes_R -$ onto R-^{u}mod. This implies the

equivalence of $R\text{-}^{u}\mathrm{mod}_{\mathfrak{F}} = R\text{-}\mathrm{mod}_{\mathfrak{F}}$ and the category of unitary $\mathfrak{F}^{-1}R$-modules. $\square$

7. Ideals of $\mathfrak{F}^{-1}R$. Let, as earlier, $\mathfrak{F}$ be a topologizing set of ideals of R, $\hat{\mathfrak{F}}$ the minimal radical filter "spanned" by $\mathfrak{F}$. Since $G_{\hat{\mathfrak{F}}}$ is left exact, we can (and will) identify the module $G_{\mathfrak{F}}m$ with an ideal of $G_{\mathfrak{F}}R$ for any left ideal m of R.

Proposition. *For any left ideal* n *of* R *and a topologizing filter* $\mathfrak{F}$ *there are implications:*

$$[n \in \hat{\mathfrak{F}}] \Longleftrightarrow [G_{\hat{\mathfrak{F}}}n = G_{\hat{\mathfrak{F}}}R].$$

Proof. Let $n \in I_{\ell}R$, $x \in R$ and $j_{\hat{\mathfrak{F}},R}(x) \in G_{\mathfrak{F}}n$. Then $j_{\hat{\mathfrak{F}},R}(m\cdot x) \subset \hat{\mathfrak{F}}^{1}n$ for some $m \in \hat{\mathfrak{F}}$. This, in turn, means that for any $\lambda \in \mathcal{P}(m)$ there exists $m_{\lambda} \in \hat{\mathfrak{F}}$ such that $m_{\lambda}\lambda x \subset n$; therefore $(n:x) \in \hat{\mathfrak{F}} \circ \{m\} \subset \hat{\mathfrak{F}}$. Thus, if $G_{\hat{\mathfrak{F}}}n = G_{\hat{\mathfrak{F}}}R$, then $n \in \hat{\mathfrak{F}} \circ \{R\} = \hat{\mathfrak{F}}$.

Conversely, let $n \in \hat{\mathfrak{F}}$. Then $(n:x)j_{\hat{\mathfrak{F}},R}(x) = j_{\hat{\mathfrak{F}},R}((n:x)x) \subset G_{\hat{\mathfrak{F}}}n$ for any $x \in R$. Since $(n:x) \in \hat{\mathfrak{F}}$, this implies that $j_{\hat{\mathfrak{F}},R}(x) \in G_{\hat{\mathfrak{F}}}n$. Therefore $j_{\hat{\mathfrak{F}},R}(R) \subset G_{\hat{\mathfrak{F}}}n$. Since $G_{\hat{\mathfrak{F}}}$ is left exact, then the following condition holds:

if $\mathcal{N}$ *is an* R-*submodule of* $G_{\hat{\mathfrak{F}}}\mathcal{M}$, *then* $G_{\hat{\mathfrak{F}}}(j^{-1}_{\hat{\mathfrak{F}},\mathcal{M}}\mathcal{N})$ *is canonically identified with* $G_{\hat{\mathfrak{F}}}\mathcal{N}$.

Indeed, by definition, the square

$$\begin{tikzcd}
\mathcal{N} \arrow[r] & G_{\hat{\mathfrak{F}}}\mathcal{M} \\
j^{-1}_{\mathfrak{F},\mathcal{M}}\mathcal{N} \arrow[r] \arrow[u] & \mathcal{M} \arrow[u, "j_{\hat{\mathfrak{F}},\mathcal{M}}"']
\end{tikzcd}$$

is Cartesian and $G_{\hat{\mathfrak{F}}}$ transforms Cartesian squares into Cartesian squares. Therefore the fact that the image of R belongs to $G_{\hat{\mathfrak{F}}}n$ implies the equality $G_{\hat{\mathfrak{F}}}n = G_{\hat{\mathfrak{F}}}R$. $\square$

Corollary 1. Let $\mathcal{F}$ be a topologising set of ideals of the ring R. If R-mod$_{\mathcal{F}} \subset$ R mod$_{\mathcal{F}'}$ for some radical filter $\mathcal{F}'$, then $\mathcal{F}' \subset \hat{\mathcal{F}}$.

Proof. From the universality of localisations it follows that $[R\text{-mod}_{\mathcal{F}} \subset R\text{-mod}_{\mathcal{F}'}] \Rightarrow [G_{\hat{\mathcal{F}}} \simeq G_{\hat{\mathcal{F}}} \circ G_{\mathcal{F}'}]$. If $n \in I_e R \setminus \mathcal{F}$ then $G_{\hat{\mathcal{F}}} n$ is a proper ideal of the ring $G_{\hat{\mathcal{F}}} n$, as it was just shown. At the same time $G_{\hat{\mathcal{F}}} n \simeq G_{\hat{\mathcal{F}}}(G_{\mathcal{F}'} n)$ and, consequently, $G_{\mathcal{F}'} n$ is a proper ideal of $G_{\mathcal{F}'} R$; it follows that $n \notin \mathcal{F}'$. $\square$

A full subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is called a Giraud category if the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a faithfull left-adjoint functor.

Corollary 2. The maps $\mathcal{F} \mapsto R\text{-mod}_{\mathcal{F}}$ and

$$\mathcal{C} \mapsto \mathcal{F}^{\mathcal{C}} \overset{\text{def}}{=} \{ m \in I_e R \mid \text{the canonical arrow } M \to Hom_R(m, M) \text{ is a bijection for any } M \in Ob\,\mathcal{C} \}$$

induce inverse to each other isomorphisms of the category $\mathcal{T}I_e R^{op}$ (opposite to the category of radical filters) and the category $\mathcal{G}(R\text{-mod})$ of all Giraud sub-catgories of the category of R-modules (with inclusions as morphisms).

Proof. The fact that R-mod$_{\mathcal{F}} \in \mathcal{G}$(R-mod) for every radical filter $\mathcal{F}$ (the exactness of the localisation functor $\mathcal{F}^{-1}$) is stated in Corollary 6 of Proposition. The injectivity of the map $\mathcal{F} \mapsto R\text{-mod}_{\mathcal{F}}$ follows from the previous corollary. Thus it remains to show that for every Giraud subcategory $\mathcal{C}$ of the category of R -modules the set $\mathcal{F}^{\mathcal{C}}$ is a radical filter and $\mathcal{C} =$ $= R\text{-mod}_{\mathcal{F}^{\mathcal{C}}}$. Let $T_{\mathcal{C}}$ be the functor $R\text{-mod} \to \mathcal{C}$, left adjoint to the embedding $\mathcal{C} \hookrightarrow R\text{-mod}$; the kernel of $T_{\mathcal{C}}$ Ker $T_{\mathcal{C}}$, is the full subcategory of R-mod, formed by all the modules N such that $T_{\mathcal{C}} N = 0$. Let $\mathcal{F}_{\mathcal{C}}$ be the set of all the left ideals of R which annihilate some elements of modules from $\operatorname{Ker} T_{\mathcal{C}}$. It is known (see [2], Chapter 16, or [7], Chapter 6, §5) that $\mathcal{F}_{\mathcal{C}}$ is a radical filter and the subcatgory C consists exactly of the modules M for which the canonical map $j_{\mathcal{F}_C, M} : M \to G_{\mathcal{F}_C} M$ is an isomorphism. This and Proposition 6 impl that $\mathcal{F}_C \subset \mathcal{F}^{\mathcal{C}}$. On the other hand, since $M \to \operatorname{Hom}_R(\mathfrak{n}, M),\ y \mapsto (x \xrightarrow{r_y} x \cdot y)$, is injective for any M from $\mathcal{C}$ and $\mathfrak{n} \in \mathcal{F}^{\mathcal{C}}$, it follows that the submodule $\mathcal{F}^{\mathcal{C}} \mathcal{N}$ of $\mathcal{F}^{\mathcal{C}}$-torsion of $\mathcal{N}$ belongs to $\operatorname{Ker} j_{\mathcal{F}_e, \mathcal{N}}$ for every module N. Since $\operatorname{Ker} j_{\mathcal{F}_C, \mathcal{N}} \in \operatorname{Ker} T_{\mathcal{C}}$ $(= \operatorname{Ker} G_{\mathcal{F}_C})$ for all the modules N, the definition of $\mathcal{F}_C$ implies the inverse inclusion $\mathcal{F}^{\mathcal{C}} \subset \mathcal{F}_C$. Thus, $\mathcal{F}_C = \mathcal{F}^{\mathcal{C}}$. □

Corollary 3. *Let $\mathcal{F}$ be a radical filter of finite type. Then every ideal from $I_\ell R \setminus \mathcal{F}$ is contained in some ideal from $\operatorname{Max}(I_\ell R \setminus \mathcal{F})$.*

Proof. Let $\{\mathfrak{n}_i \mid i \in J\}$ be an increasing (with respect to inclusion) family of ideals from $I_\ell R \setminus \mathcal{F}$. Then $\{G_{\mathcal{F}} \mathfrak{n}_i \mid i \in J\}$ is an increasing family of proper left ideals of $G_{\mathcal{F}} R$. The supremum of any increasing family of the proper left ideals of $G_{\mathcal{F}} R$ is a proper ideal, since $G_{\mathcal{F}} R$ is a unitary ring. In particular, $\widetilde{\mathfrak{n}}_J = \sup_{i \in J} G_{\mathcal{F}} \mathfrak{n}_i$ is proper. Since $\mathcal{F}$ is of finite type, the functor $G_{\mathcal{F}}$ commutes with the colimits of the inductive systems $\{M_i \xrightarrow{\lambda_{ij}} M_j\}$ in which all the arrows λ_{ij} are monomorphisms (see 2.5). Therefore, $G_{\mathcal{F}} \widetilde{\mathfrak{n}}_J = G_{\mathcal{F}} \sup_i G_{\mathcal{F}} \mathfrak{n}_i = \sup_i G_{\mathcal{F}} \mathfrak{n}_i = \widetilde{\mathfrak{n}}_J$; hence $G_{\mathcal{F}} \widetilde{\mathfrak{n}}_J$ is proper. According to Proposition 7 the properness of $G_{\mathcal{F}} \widetilde{\mathfrak{n}}_J$ means that $j_{\mathcal{F}}^{-1} \widetilde{\mathfrak{n}}_J \in I_\ell R \setminus \mathcal{F}$ (we use the equality $G_{\mathcal{F}} \widetilde{\mathfrak{n}}_J = G_{\mathcal{F}}(j_{\mathcal{F}}^{-1} \widetilde{\mathfrak{n}}_J)$; see the proof of Proposition 7) and, consequently, $\sup_{i \in J} \mathfrak{n}_i \in I_\ell R \setminus \mathcal{F}$, since $\sup_{i \in J} \mathfrak{n}_i \subset j_{\mathcal{F}}^{-1} \widetilde{\mathfrak{n}}_J$. It remains to apply the Zorn's lemma. □

Corollary 4. *Suppose that* R *is left noetherian. Then for every* $\mathfrak{p}\in \mathrm{Spec}_\ell R$ *there exists a maximal (with respect to the inclusion) ideal* $\mathcal{M}_o$ *among the left ideals* $\mathcal{M}$ *such that* $\mathfrak{p}\subset\mathcal{M}$ *and* $\mathfrak{p}\simeq\mathcal{M}$ *in the category* $\mathrm{I}_\ell^{\succ} R$.

Proof. Recall that $\mathcal{O}b\,\mathrm{I}_\ell^{\succ} R = \mathcal{O}b\,\mathrm{I}_\ell R$ and the arrows (or the order) are defined as follows: $m \longrightarrow n$, if $m\in\mathfrak{F}_n$ (see an equivalent definition in Introduction). This makes it clear, that any ideal from $Max(\mathrm{I}_\ell R\setminus\mathfrak{F}_\mathfrak{p})$, that contains $\mathfrak{p}$, can be taken as $\mathcal{M}_o$. Its existence is guaranteed by Corollary 3. $\square$

8. Filters of bifinite type and the prime spectrum.

We say that a set of left ideals $\mathfrak{F}$ is of *bifinite type* if $\mathfrak{F}\cap\mathrm{I}R$ has a cofinal subset of finitely generated twosided ideals. Obviously, any filter of finite type is a filter of bifinite type.

Proposition. *Let* $\mathcal{T}$ *be a family of filters of left ideals of* R, *set* $\Sigma\mathcal{T} = \bigcup\{\mathfrak{F}\mid\mathfrak{F}\in\mathcal{T}\}$.

1). *If the set* $\Sigma\mathcal{T}\cap\mathrm{I}R$ *is closed under the multiplication of ideals, then the set* $Max(\mathrm{I}R\setminus\Sigma\mathcal{T})$ *of maximal (with respect to inclusion elements of* $\mathrm{I}R\setminus\Sigma\mathcal{T}$ *belongs to the prime spectrum*

2). *If the filter* $\mathfrak{F}$ *is of bifinite type, then every ideal from* $\mathrm{I}R\setminus\Sigma\mathcal{T}$ *is contained in an ideal from* $Max(\mathrm{I}R\setminus\Sigma\mathcal{T})$.

Proof. 1) Let $\mu\in Max(\mathrm{I}R\setminus\Sigma\mathcal{T})$ and α, β be twosided ideals such, that $\alpha\beta\subset\mu$. Suppose that neither $\alpha\not\subset\mu$ nor $\beta\not\subset\mu$. The maximality of μ implies the inclu-
$, \mu+\beta\}\subset\Sigma\mathcal{T}$. . By the multiplicativity condition on $\Sigma\mathcal{T}\cap\mathrm{I}R$ we have

$(\mu+\alpha)(\mu+\beta)\in\Sigma\mathfrak{T}$. But then $\mu\in\Sigma\mathfrak{T}$, since $\Sigma\mathfrak{T}$ is a filter and $(\mu+\alpha)(\mu+\beta)\subset\mu$. Contradiction.

2) Let $\{n_i \mid i\in J\}$ be an increasing family of twosided ideals. If $\sup\{n_i \mid i\in J\}\in\Sigma\mathfrak{T}$, then by the hypothesis (on bifiniteness) there exists a finitely generated twosided ideal $\nu\in\Sigma\mathfrak{T}$ which belongs to $\sup\{n_i \mid i\in J\}$; hence $\nu\subset n_i$ for some $i\in J$. Therefore, we have $[\{n_i \mid i\in J\}\subset \mathrm{I}R\setminus\Sigma\mathfrak{T}]\Rightarrow[\sup_{i\in J} n_i\notin\Sigma\mathfrak{T}]$. The proof terminates by one more standard application of Zorn's lemma. □

Corollary. *Let* $\mathfrak{T}$ *be a family of filters of the left ideals of the* R, *such that for any* $\{\mathcal{F},\mathcal{G}\}\subset\mathfrak{T}$ *the filter* $\mathcal{F}\circ\mathcal{G}$ *belongs to* $\Sigma\mathfrak{T}$. *Then* $\mathrm{Max}(\mathrm{I}R\setminus\Sigma\mathfrak{T})\subset \mathrm{Spec}\,R$.

In particular, $\mathrm{Max}(\mathrm{I}R\setminus\mathcal{F})\subset\mathrm{Spec}\,R$ *for every radical filter* $\mathcal{F}$.

Proof. If $\mathcal{F}$ and $\mathcal{G}$ are subsets of $\mathrm{I}_l R$, $n\in\mathcal{F}$ and $m\in\mathcal{G}$, then $nm\in\mathcal{F}\circ\{m\}\subset\mathcal{F}\circ\mathcal{G}$. Therefore the set of ideals $\Sigma\mathfrak{T}$ (and, consequently, its subset $\mathrm{I}R\cap\Sigma\mathfrak{T}$) are multiplicative. □

2.9. Localizations, categories $\mathrm{I}_l^{\succ}$... , the left spectrum. Let R and B be associative rings. Fix an (R,B)-bimodule M and denote by $\mathrm{I}_l^{\succ}M$ the preorder category, whose objects are all the R-submodules of M. The arrows are defined as follows: $N\rightarrow N'$ if either $N\subset N'$ or for some $b\in\mathcal{P}(B)$ the R-submodule $(N:b)\overset{\mathrm{def}}{=}\{\xi\in M \mid \xi b\subset N\}$ belongs to N'. The notation $\mathrm{I}_l M$ will stand for the preorder of R-submodules of M with respect to inclusion; it is a subcategory of the category $\mathrm{I}_l^{\succ}M$.

If M is the (R,R) bimodule R, the category $\mathrm{I}_l^{\succ}M$ coincides with the category $I_\ell^{\succ} R$.

Any functor $F: R\text{-mod} \longrightarrow R'\text{-mod}$ uniquely defines the functor ${}^{B}F$ from the category (R,B)-bi of (R,B)-bimodules into (R', B)-bi. In particular, to the functor $G_{\mathcal{F}}$ the functor ${}^{B}G_{\mathcal{F}}: (R,B)\text{-bi} \longrightarrow (R,B)\text{-bi}$ for $\mathcal{F} \subset \mathcal{T}_\ell R$ corresponds.

Proposition. Let $\mathcal{F}$ be a radical filter of left ideals of R, M an (R,B)-bimodule, $N \in \mathcal{O}b\, I_\ell^{\succ} M$.

1) $G_{\mathcal{F}}(N : \tilde{b}) = (G_{\mathcal{F}} N : \tilde{b})$ for any $\tilde{b} \in \mathcal{P}(B)$.

2) ${}^{B}G_{\mathcal{F}}$ determines a functor $I_\ell^{\succ} M \longrightarrow I_\ell^{\succ}\, {}^{B}G_{\mathcal{F}} M$.

Proof. 1). let $b \in B$ be an arbitary element of B, r_b the b-action on M. Since the square

$$\begin{array}{ccc} N & \hookrightarrow & M \\ \uparrow & & \uparrow r_b \\ (N:b) & \hookrightarrow & M \end{array}$$

is Cartesian and $G_{\mathcal{F}}$ is left-exact, then $G_{\mathcal{F}}(N:b) = (G_{\mathcal{F}} N : b)$. Now let $\tilde{b} \in \mathcal{P}(B)$ and $\{b_i\}_{i \in I}$ be a finite set of generators of Z-module b. Because of the same left-faithfulness of $G_{\mathcal{F}}$ we have

$$G_{\mathcal{F}}(N:\tilde{b}) = G_{\mathcal{F}}\Big(\bigcap_{i\in I}(N:b_i)\Big) = \bigcap_{i \in I} G_{\mathcal{F}}(N:b_i) =$$
$$= \bigcap_{i\in I}(G_{\mathcal{F}}N : b_i) = (G_{\mathcal{F}} N : \tilde{b}).$$

2) The first statement and from the definition of the arrows in $I_\ell^{\succ} M$ make it is clear that $[N \to N'] \Rightarrow [G_{\mathcal{F}} N \to G_{\mathcal{F}} N']$ for any pair of R-submodules N, N' of (R,B)-bimodule M. $\square$

Remark. Let $\hat{G}_{\mathcal{F}}$ be a functor from R-mod into the category $G_{\mathcal{F}}R\text{-}{}^{u}\mathrm{mod}$ of the unitary $G_{\mathcal{F}}R$-modules corresponding to a radical filter $\mathcal{F}$ (see the concluding lines of subsect 5) -- the composition of the localization functor $\mathcal{F}^{-1}$ and of the embedding $R\text{-}mod_{\mathcal{F}} \hookrightarrow G_{\mathcal{F}}R\text{-}{}^{u}mod$. Obviously, for any (R,B)-bimodule M the functor $\widehat{G}_{\mathcal{F}}$ induces the functor $I_\ell^{\succ} M \longrightarrow I_\ell^{\succ\, B}\widehat{G}_{\mathcal{F}}M$.

Corollary 1. (a) *For every radical filter* $\mathcal{F}$ *of the left ideals of* R *the map* $m \longmapsto G_{\mathcal{F}}m$, $m \in I_\ell G_{\mathcal{F}}R$, *is a functor from* $I_\ell^{\succ}G_{\mathcal{F}}R$ *into* $I_\ell^{\succ}G_{\mathcal{F}}R$ (*here* $G_{\mathcal{F}}m$ *is identified with an ideal of* $G_{\mathcal{F}}R$, *as it was done earlier in similar cases*).

(b) *The map* $n \longmapsto G_{\mathcal{F}}n$, $n \in I_\ell R$, *determines a functor*

$$I_\ell^{\succ}R \longrightarrow I_\ell^{\succ}G_{\mathcal{F}}R.$$

Corollary 2. *Let* $\mathcal{F}$ *be a radical filter in* $I_\ell R$. *The map* $n \mapsto G_{\mathcal{F}}n$, $n \in I_\ell R$, *sends the ideals from* $Spec_\ell R \setminus \mathcal{F}$ ($\widehat{Spec}_\ell R \setminus \mathcal{F}$) *into the ideals from* $Spec_\ell G_{\mathcal{F}}R$ ($\widehat{Spec}_\ell G_{\mathcal{F}}R$ *respectively*).

Proof. Let $\mu \in Spec_\ell R \setminus \mathcal{F}$, $n \in I_\ell G_{\mathcal{F}}R$, and $(G_{\mathcal{F}}\mu : x) \not\subset G_{\mathcal{F}}\mu$ for any $x \in \mathcal{P}(n)$. In particular, according to Proposition 9, $(\mu : y) \not\subset \mu$ for any $y \in \mathcal{P}(j_{\mathcal{F}}^{-1}n)$. Since $\mu \in Spec_\ell R$, there exists an arrow $j_{\mathcal{F}}^{-1}n \longrightarrow \mu$. The equality $G_{\mathcal{F}}n = G_{\mathcal{F}}(j_{\mathcal{F}}^{-1}n)$ and Corollary 1 imply that $n \subset G_{\mathcal{F}}n \to \to G_{\mathcal{F}}\mu$. Therefore $G_{\mathcal{F}}\mu \in Spec_\ell G_{\mathcal{F}}R$.

If $\mu \in \widehat{Spec}_\ell R \setminus \mathcal{F}$, then the following implications hold: $[(\mu : y) \not\subset \mu$ for every $y \in \mathcal{P}(j_{\mathcal{F}}^{-1}n)] \Rightarrow [j_{\mathcal{F}}^{-1}n \subset \mu] \Rightarrow \Rightarrow [n \subset G_{\mathcal{F}}n \subset G_{\mathcal{F}}\mu]$. $\square$

The following lemma provides us with one more trait on $Spec_\ell G_{\mathcal{F}}R$.

Lemma. *Let* $\widetilde{p} \in \widehat{Spec}_\ell G_{\mathcal{F}}R$ *and* $p \overset{def}{=} j_{\mathcal{F}}^{-1}\widetilde{p} \notin \mathcal{F}$. *Then* $\widetilde{p} = G_{\mathcal{F}}p$.

Proof. Suppose that $(\widetilde{p} : x) \subset \widetilde{p}$ for some $x \in \mathcal{P}(G_{\mathcal{F}}p)$. Then there exists an ideal $m \in \mathcal{F}$ such that $j_{\mathcal{F}}(m) \cdot x \subset j_{\mathcal{F}}(p) \subset \subset \widetilde{p}$. Since $p \notin \mathcal{F}$, the set $j_{\mathcal{F}}(m) \setminus \widetilde{p}$ is non-empty;

and for any $y \in j_{\mathcal{F}}(m) \setminus \tilde{p}$ the ideals $\tilde{p}$ and $(\tilde{p}:y)$ are isomorphic, since $\tilde{p} \in \widehat{Spec}_\ell G_{\mathcal{F}}R$. In particular, $G_{\mathcal{F}}(\tilde{p}:y) \neq G_{\mathcal{F}}R$ and, therefore, $G_{\mathcal{F}}(\tilde{p}:yx) \neq G_{\mathcal{F}}R$ thanks to the inclusion $(\tilde{p}:yx) \subset (\tilde{p}:y)$. But $yx \subset \tilde{p}$ and, hence, $(\tilde{p}:yx)$ coincides with $G_{\mathcal{F}}R$. We have come to a contradiction with the initial supposition. So, $(\tilde{p}:x) \not\subset \tilde{p}$ for any $x \in \mathcal{P}(G_{\mathcal{F}}p)$. Since $\tilde{p} \in \widehat{Spec}_\ell G_{\mathcal{F}}R$ and $\tilde{p} \subset G_{\mathcal{F}}p$, this means that the ideals $\tilde{p}$ and $G_{\mathcal{F}}p$ coincide. $\square$

<u>Example</u>. Let $\mathcal{F}$ be a radical filter of left ideals of R, p a left ideal of R such that $G_{\mathcal{F}}p$ is a completely prime left ideal of $G_{\mathcal{F}}R$. Then the ideal $p_{\mathcal{F}} \overset{def}{=} j_{\mathcal{F}}^{-1}(G_{\mathcal{F}}p)$ turn out to be a completely prime ideal of R.

Indeed, if $y \in R \setminus p_{\mathcal{F}}$ and $x \in R$, then

$$[yx \in p_{\mathcal{F}}] \Leftrightarrow [j_{\mathcal{F}}(y) \cdot j_{\mathcal{F}}(x) \in G_{\mathcal{F}}p] \Leftrightarrow [\, j_{\mathcal{F}}(x) \in G_{\mathcal{F}}p$$

(since $G_{\mathcal{F}}p \in \widehat{\widehat{Spec}}_\ell G_{\mathcal{F}}R$ and $j_{\mathcal{F}}(y) \notin G_{\mathcal{F}}p$)$] \Rightarrow [x \in p_{\mathcal{F}}]$.

In particular, if the ring R is commutative, then for any radical filter $\mathcal{F}$ the map $\mu \longmapsto G_{\mathcal{F}}\mu$ determines a bijection of the set $Spec\, R \setminus \mathcal{F}$ onto the set of the prime ideals $\tilde{p}$ of $G_{\mathcal{F}}R$ such that $j_{\mathcal{F}}^{-1}\tilde{p} \notin \mathcal{F}$. $\square$

In general situation we cannot maintain that

- $G_{\mathcal{F}}p$ is completely prime for any completely prime ideal p of R;

- the ideal $p_{\mathcal{F}} = j_{\mathcal{F}}^{-1}(G_{\mathcal{F}}p)$ belongs to $\widehat{Spec}_\ell R$, if $G_{\mathcal{F}}p \in \widehat{Spec}_\ell G_{\mathcal{F}}R$.

However, the last statement is true under condition:

" p is a maximal element (in $I_\ell^{\succ}R$) of the set $\{p, (p:x) \mid x \in R, (p:x) \notin \mathcal{F}\}$.

In fact, by hypothesis, $(p:x) \in \mathcal{F}$ for any $x \in R$ such that $(p:x) \not\to p$. This means that $p \in I_\ell^{*}R$ (see 1.6);

hence $\hat{p} \overset{def}{=} \{\lambda \in R \mid (p:\lambda) \not\to p\} \in \widehat{Spec}_\ell R$ (Proposition 1.6). On the other hand, $\hat{p} = \{z \in R \mid (p:z) \in \mathcal{F}\}$, and the last set, as the reader can easily verify, coincides with $p_{\mathcal{F}} = j_{\mathcal{F}}^{-1}(G_{\mathcal{F}} p)$.

10. q-proper and q-improper ideals. A left ideal m of R will be called q-improper, if the canonical map $R \longrightarrow \operatorname{Hom}_R(R,R)$ is a bijection, and q-proper otherwise.

Proposition. 1) *The following conditions are equivalent:*

(a) R *is a ring with right unit*

$[y \in R,\ Ry = 0] \Longrightarrow [y = 0]$;

(b) R *is a* q-*improper ideal of* R;

(c) R *is a ring with unit.*

2) *Let* $\mathcal{F}$ *be a radical filter of left ideals of* R. *If* m *is a* q-*proper ideal of* $G_{\mathcal{F}} R$, *then so is* $G_{\mathcal{F}} m$, *and* $j_{\mathcal{F}}^{-1} m \notin \mathcal{F}$.

Proof. 1) (a) $\Longrightarrow$ (b). If R is a ring with right unit e, then for every R-module M the canonical map $\rho_M : M \longrightarrow \operatorname{Hom}_R(R, M)$ is surjective, since every morphism of R-modules f:R$\longrightarrow$M coincides with the "right multiplication" by the image f(e) of the right unit. Clearly, ρ_M is injective if and only if $\operatorname{Ann} \xi \neq R$ for any $\xi \in M \setminus \{0\}$.

(b) $\Longrightarrow$ (c). The map ρ_R is a ring morphism of R into the unitary ring $R^{\#} = \operatorname{Hom}_R(R,R)^{o}$, opposite to the endomorphism ring of the left R-module R. Therefore the bijectivity of ρ_R means that R is isomorphic to a ring with unit.

(c)$\Rightarrow$(a) is obvious.

2) Now let $\mathcal{F}$ be a radical filter of left ideals of R. For every left ideal m of the ring $G_{\mathcal{F}}R$ and an arbitrary R-module M there is a commutative diagram

$$\begin{array}{ccc} Hom_R(m, G_{\mathcal{F}}M) & \xleftarrow{\eta'} & Hom_R(G_{\mathcal{F}}m, G_{\mathcal{F}}M) \\ \uparrow{\scriptstyle \xi'} & & \uparrow{\scriptstyle \xi} \\ Hom_{G_{\mathcal{F}}R}(m, G_{\mathcal{F}}M) & \xleftarrow{\eta} & Hom_{G_{\mathcal{F}}R}(G_{\mathcal{F}}m, G_{\mathcal{F}}M) \end{array} \quad (1),$$

in which the arrow $\eta' = Hom_R(j_{\mathcal{F},m}, 1_{G_{\mathcal{F}}M})$ is bijective due to the universality of $j_{\mathcal{F},m}$; the arrow ξ' (the natural ambedding) is bijective, since any morphism of the category R-mod$_{\mathcal{F}}$ is a morphism of $G_{\mathcal{F}}R$-modules. Since $\eta' \circ \xi$ is surjective, then so (and hence the bijective) is the embedding $\xi' : Hom_{G_{\mathcal{F}}R}(m, G_{\mathcal{F}}M) \to$ $\longrightarrow Hom_R(m, G_{\mathcal{F}}M)$. Finally, bijectivity of three arrows implies the bijectivity of the fourth one: – $\eta_M = Hom_{G_{\mathcal{F}}R}(j_{\mathcal{F},m}, 1_{\sim})$.

2) follows from the bijectivity of the map η_R^{-1} and Proposition 7, since $G_{\mathcal{F}}m = G_{\mathcal{F}}(j_{\mathcal{F}}^{-1}m)$. $\square$

11. Flat localisations.

Proposition. *The following properties of the radical filter $\mathcal{F}$ of the left ideals of R are equivalent:*

(a) *for every proper ideal m of the ring $G_{\mathcal{F}}R$ the ideal $G_{\mathcal{F}}m$ is also proper;*

(b) $m = G_{\mathcal{F}}m$ *for every left ideal m of* $G_{\mathcal{F}}R$;

(c) *if* $\mu \in Max_{\ell} G_{\mathcal{F}}R$, *then* $\mu = G_{\mathcal{F}}\mu$.

(d) *The functor* $G_{\mathcal{F}}$ *is exact* .

Proof. The implications (a)$\Leftarrow$ (b)$\Rightarrow$(c) are trivial. The implication (d)$\Rightarrow$(b) follows from the implication [the functor $G_{\mathfrak{F}}$] is **exact**]$\Rightarrow$[$G_{\mathfrak{F}} \simeq G_{\mathfrak{F}} R^{(1)} \otimes_R \sim$] (see sect. 2, property 7).

(c) $\Longrightarrow$ (a). Every proper ideal $m \in I_\ell G_{\mathfrak{F}} R$ is contained inan ideal $\mu \in Max_\ell G_{\mathfrak{F}} R$, since $G_{\mathfrak{F}} R$ is a ring with unit. The inclusion $G_{\mathfrak{F}} m \subset G_{\mathfrak{F}} \mu$ and the equality $\mu = G_{\mathfrak{F}} \mu$ imply that $G_{\mathfrak{F}} m$ is a proper ideal.

(a) $\Longrightarrow$ (d). Let $C_{\mathfrak{F}}$ be the image of the functor $\hat{G}_{\mathfrak{F}}$: R-mod $\to G_{\mathfrak{F}} R$-^{u}mod, and $\tilde{\mathfrak{F}}$ the collection of all the left ideals ν of $G_{\mathfrak{F}} R$ such that for every module M from $C_{\mathfrak{F}}$ the natural map $M \to Hom_{G_{\mathfrak{F}} R}(\nu, M)$ is bijective. From the second step of the proof of Proposition 10 (concerning the bijectivity of arrows in diagram (I)) and Corollary 2 of Proposition 7 it follows that

[the natural map $M \to Hom_R(j_{\mathfrak{F}}^{-1}\nu, M)$ is bijective for every M from R-mod$_{\mathfrak{F}}$] $\Leftrightarrow$ [$j_{\mathfrak{F}}^{-1}\nu \in \mathfrak{F}$]$\Leftrightarrow$ [$G_{\mathfrak{F}}\nu = G_{\mathfrak{F}}(j_{\mathfrak{F}}^{-1}\nu) = G_{\mathfrak{F}} R$]. Hence, if (a) is satisfied, then $\tilde{\mathfrak{F}}$ consists of the one ideal $G_{\mathfrak{F}} R$. We leave to the reader verification of the implications

[$\tilde{\mathfrak{F}} = \{G_{\mathfrak{F}} R\}$]$\Leftrightarrow$[$C_{\mathfrak{F}} = G_{\mathfrak{F}} R$-^{u}mod]$\Leftrightarrow$

$\Leftrightarrow$ [the functor $G_{\mathfrak{F}}$ is exact] . $\square$

Corollary Let $\mathfrak{F}$ be a radical filter of left ideals of R. If the functor $G_{\mathfrak{F}}$ induces the surjection of the set $Spec_\ell R \setminus \mathfrak{F}$ onto $Spec_\ell G_{\mathfrak{F}} R$ (see Corollary 2 of Proposition 9 and the subsequent Remark..) or if the map $\nu \mapsto j_{\mathfrak{F}}^{-1}$ sends the ideals from $\hat{Spec}_\ell G_{\mathfrak{F}} R$ into the ideals from $Spec_\ell R \setminus \mathfrak{F}$, or at least into the ideals from $\mathcal{I}_\ell R - \mathcal{F}$, then $G_\mathcal{F}$ is exact .

Proof. Since $G_\mathcal{F} R$ is a ring with unit, all its maximal left ideals are points of the left spectrum. Hence, if any of the conditions of the corollary is satisfied, then $\mu = G_\mathcal{F}\mu$ for every $\mu \in Max_\ell G_\mathcal{F} R$ The statement follows, therefore, from the equivalence of the conditions (c) and (d) in Proposition II. □

Note that in general there are very few radical filters $\mathcal{F}$ such that $G_\mathcal{F}$ is exact .

12. The inductive limits of localisations. Let $\mathcal{F}_J = \{\mathcal{F}_i \mid i \in J\}$ be a directed with respect to inclusion set of radical filters, such that $\mathcal{F} = \bigcup_{i \in J} \mathcal{F}_i$ is a radical filter. Then there are canonical morphisms

$$\psi_J : \varinjlim_i \mathcal{F}_i^1 \to \mathcal{F}^1, \quad \varphi_J : \varinjlim_i H_{\mathcal{F}_i} \to H_\mathcal{F}, \quad \phi_J : \varinjlim_i G_{\mathcal{F}_i} \to G_\mathcal{F}$$

Proposition 1) $\psi_J : \varinjlim \mathcal{F}_i^1 \to \mathcal{F}^1$ and $\varphi_J : \varinjlim H_{\mathcal{F}_i} \to H_\mathcal{F}$ are isomorphisms.

2) $\phi_J : \varinjlim G_{\mathcal{F}_i} \to G_\mathcal{F}$ is a monomorphism; the arrow $\phi_J(M) : \varinjlim G_{\mathcal{F}_i} M \to G_\mathcal{F} M$ is an isomorphism, if the natural arrow $H_\mathcal{F} \mathcal{F}_{i_0}^1 M \longrightarrow G_\mathcal{F} M$ is epimorphism for some $i_0 \in J$. In particular, $\phi_J(M)$ is isomorphism, if the directed with respect to inclusion set of torsion submodules $\{\mathcal{F}_i M \mid i \in J\}$ stabilizes.

3). The following properties of $n \in \mathcal{I}_\ell R$ are equivalent:

(i) $n \notin \mathcal{F}$;

(ii) $\varinjlim_i G_{\mathcal{F}_i} n$ is a proper ideal of the ring $\varinjlim_i G_{\mathcal{F}_i} R$.

Proof. 1) (a) The formulas for $H_{\mathcal{F}}$ immediately imply that $\varinjlim H_{\mathcal{F}_i} \to H_{\mathcal{F}}$ is an isomorphism.

(b) Clearly, $\varinjlim \mathcal{F}_i^1 \longrightarrow \mathcal{F}^1$ is an epimorphism, since so are $\mathcal{F}_i^1 \to \mathcal{F}^1$ and the diagrams

$$\begin{tikzcd} & \varinjlim_i \mathcal{F}_i^1 \arrow[dr, "\psi_J"] & \\ \mathcal{F}_i^1 \arrow[ur] \arrow[rr, two heads] & & \mathcal{F}^1 \end{tikzcd}$$

commutate by definition of ψ_J and since g is an epimorphism if so is $g \circ f$ for some f.

On the other hand, since in the commutative diagram

$$\begin{tikzcd} \varinjlim_i H_{\mathcal{F}_i} \arrow[r, "\sim"] & H_{\mathcal{F}} \\ \varinjlim_i \mathcal{F}_i^1 \arrow[u] \arrow[r, "\psi_J"] & \mathcal{F}^1 \arrow[u] \end{tikzcd}$$

the arrows $\uparrow \longrightarrow$ are the monomorphisms, then ψ_J is a monomorphism (since g is a monomorphism, if so is $f \circ g$ for some f). Therefore, ψ_J is an isomorphism.

2) (c) The functor $\varinjlim_i G_{\mathcal{F}_i}$: R-mod $\longrightarrow$ R-mod, being the inductive limit of left-exact functors, is left-exact. In particular, it assigns to an exact sequence

$$0 \to \mathcal{F}M \longrightarrow M \longrightarrow \mathcal{F}^1 M \longrightarrow 0$$

The exact sequence

$$0 \to \varinjlim G_{\mathcal{F}_i} \mathcal{F}M \longrightarrow \varinjlim G_{\mathcal{F}_i} M \xrightarrow{\phi_J(M)} \varinjlim G_{\mathcal{F}_i} \mathcal{F}^1 M = G_{\mathcal{F}} M$$

It is subject to direct verification that $\varinjlim G_{\mathcal{F}_i} \mathcal{F}M = 0$; consequently, $\phi_J(M)$: $\varinjlim G_{\mathcal{F}_i} M \to G_{\mathcal{F}} M$ is a monomorphism.

(d) Let the canonical map $H_{\mathcal{F}} \mathcal{F}_{i_0}^1 M \longrightarrow G_{\mathcal{F}} M$ be an epimorphism for some $i_0 \in J$. Then in the commutative diagram

$$\begin{tikzcd} \varinjlim_i H_{\mathcal{F}_i} \mathcal{F}_i^1 M = \varinjlim G_{\mathcal{F}_i} M \arrow[d, tail, "\varphi_J(M)"] \\ H_{\mathcal{F}} \mathcal{F}^1_{i_0} M \arrow[u] \arrow[r, two heads] & G_{\mathcal{F}} M \end{tikzcd}$$

The arrow $\varphi_J(M)$ is an epimorphism. On the other hand, as it was just established, $\varphi_J(M)$ is a monomorphism for all the modules M. Therefore, $\varphi_J(M)$ is an isomorphism.

(e) The stabilization of the torsion submodules $\{\mathcal{F}_i M \mid i \in J\}$ is equivalent to that of the quotient-modules $\mathcal{F}_i^1 M \stackrel{\text{def}}{=} M/\mathcal{F}_i M, i \in J$; this means that $\mathcal{F}^1_{i_0} M \longrightarrow \mathcal{F}^1 M$ is an isomorphism for some $i_0 \in J$; but then $H_{\mathcal{F}} \mathcal{F}^1_{i_0} M \simeq G_{\mathcal{F}} M$.

3) (i) $\Longrightarrow$ (ii). If $n \in I_\ell R \setminus \mathcal{F}$, then, according to Proposition 7, $G_{\mathcal{F}} n$ is a proper ideal of $G_{\mathcal{F}} R$. Since $\varphi_J(R) : \varinjlim_i G_{\mathcal{F}_i} R \to G_{\mathcal{F}} R$ is a (mono)morphism of unitary rings, then it is clear from the commutative diagram

$$\begin{tikzcd} \varinjlim G_{\mathcal{F}_i} R \arrow[r] & G_{\mathcal{F}} R \\ \varinjlim G_{\mathcal{F}_i} n \arrow[u] \arrow[r] & G_{\mathcal{F}} n \arrow[u] \end{tikzcd}$$

that $\varinjlim G_{\mathcal{F}_i} n$ is also a proper ideal.

(ii) $\Rightarrow$ (i). Let $n \in I_\ell R$ and $\varinjlim G_{\mathcal{F}_i} n$ be a proper ideal of the ring $\varinjlim G_{\mathcal{F}_i} R$. Then $G_{\mathcal{F}_i} n$ is a proper ideal of the ring $G_{\mathcal{F}_i} R$ for every $i \in J$. By Proposition 7 $n \notin \mathcal{F}_i$ for any $i \in J$, hence $n \notin \bigcup_{i \in J} \mathcal{F}_i = \mathcal{F}$. □

Corollary 1. Let M be either a noetherian R-module or a $\mathcal{F}$-torsion free R-module. Then $\varinjlim G_{\mathcal{F}_i} M \simeq G_{\mathcal{F}} M$.

Proof. In the first case the directed set of submodules $\{\mathcal{F}_i M \mid i \in J\}$ stabilises. In the second case $G_{\mathcal{F}} M = H_{\mathcal{F}} M$ and $G_{\mathcal{F}_i} M = H_{\mathcal{F}_i} M$ for all $i \in J$. $\square$.

Corollary 2. Let R be a left noetherian ring. Then $\Phi_J : \varinjlim G_{\mathcal{F}_i} \to G_{\mathcal{F}}$ is isomorphism.

Proof. 1) $\Phi_J(M) : \varinjlim G_{\mathcal{F}_i} M \to G_{\mathcal{F}} M$ is isomorphism, if M is of finite type.

Indeed, every module of finite type over a left noetherian ring is noetherian. Hence Proposition follows from Corollary 1.

2) Any R-module M is a colimit (a union) of the directed set $\{M_\alpha \mid \alpha \in \mathfrak{A}\}$ of its finitely generated submodules. Since all the filters $\{\mathcal{F}, \mathcal{F}_i \mid i \in J\}$ are of finite type, then

$$\varinjlim_i G_{\mathcal{F}_i} M = \varinjlim_i G_{\mathcal{F}_i} \varinjlim_\alpha M_\alpha \simeq \varinjlim_i \varinjlim_\alpha G_{\mathcal{F}_i} M_\alpha \simeq$$
$$\simeq \varinjlim_\alpha \varinjlim_i G_{\mathcal{F}_i} M_\alpha \simeq \varinjlim_\alpha G_{\mathcal{F}} M_\alpha \simeq G_{\mathcal{F}} \varinjlim_\alpha M_\alpha = G_{\mathcal{F}} M. \square$$

Corollary 3. The square of the functor $\varinjlim G_{\mathcal{F}_i}$ is isomorphic to the functor $G_{\mathcal{F}}$.

Proof. Since by Proposition 12 $\Phi_J(M)$ is a monomorphism for any R-module M, then $\varinjlim G_{\mathcal{F}_i} M$ is an $\mathcal{F}$-torsion-free module. Therefore $G_{\mathcal{F}} \varinjlim G_{\mathcal{F}_i} M = H_{\mathcal{F}} \varinjlim G_{\mathcal{F}_i} = \simeq (\varinjlim G_{\mathcal{F}_i})^2 M$.
The injectivity of $\Phi_J(M)$ implies injectivity of $G_{\mathcal{F}} \Phi_J(M)$. On theother hand, the bijectivity of $G_{\mathcal{F}} j_{\mathcal{F},M}$ implies the surjectivity of $G_{\mathcal{F}} \Phi_J(M)$ as is clear from the commutative diagram

$$\begin{tikzcd}
G_{\mathcal{F}} \varinjlim G_{\mathcal{F}_i} M \arrow[rr, "G_{\mathcal{F}} \Phi_J(M)"] & & G^2_{\mathcal{F}} M \\
& G_{\mathcal{F}} M \arrow[ul, "G_{\mathcal{F}} j_M"] \arrow[ur, "\sim"', "G_{\mathcal{F}} j_{\mathcal{F},M}"] &
\end{tikzcd}$$

. $\square$

Corollary 4. 1) For every R-submodule N of $\varinjlim_i G_{\mathcal{F}_i} M$, where M is an arbitrary R-module, the inclusion $N \subset \varinjlim_i G_{\mathcal{F}_i}(\underline{j}_M^{-1} N)$ holds. More exactly, $N \subset \varinjlim G_{\mathcal{F}_i} N_i = \varinjlim G_{\mathcal{F}_i}(\underline{j}_M^{-1} N) = G_{\mathcal{F}} N \cap \varinjlim G_{\mathcal{F}_i} M$. Here $\underline{j}_M$ is the canonical arrow $M \to \varinjlim G_{\mathcal{F}_i} M$; N_j the preimage of N with respect to the coprojection $G_{\mathcal{F}_j} M \to \varinjlim_i G_{\mathcal{F}_i}$.

2) Let B be a ring, M an (R,B) -bimodule, $N \subset I_\ell M$. For any $\tilde{\mathcal{E}} \in \mathcal{P}(B)$ we have (see subsection 9):

$$\varinjlim G_{\mathcal{F}_i}(N : \tilde{\mathcal{E}}) = (\varinjlim G_{\mathcal{F}_i} N : \tilde{\mathcal{E}}).$$

3) The functor $\varinjlim G_{\mathcal{F}_i}$ sends $Spec_\ell R \setminus \mathcal{F}$ and $\widehat{Spec}_\ell R \setminus \mathcal{F}$ into $Spec_\ell \varinjlim G_{\mathcal{F}_i} R$ and $\widehat{Spec}_\ell \varinjlim G_{\mathcal{F}_i} R$ respectively.

Proof. 1) The statement follows directly from the equalities $G_{\mathcal{F}_i} \underline{j}_M^{-1}(N) = G_{\mathcal{F}_i} N_i$ (see the end of the proof of Proposition 7)

2) Proposition 9 implies

$$\varinjlim G_{\mathcal{F}_i}(N : \tilde{\mathcal{E}}) = \varinjlim (G_{\mathcal{F}_i} N : \tilde{\mathcal{E}}) = (\varinjlim G_{\mathcal{F}_i} N : \tilde{\mathcal{E}}).$$

3) Let $p \in Spec_\ell R$ and n be an arbitrary left ideal of the ring $\varinjlim G_{\mathcal{F}_i} R$.

Consider the possible alternatives:

(a) $\underline{j}_R^{-1}(n) \not\to p$. Then $(p : x) \subset p$ for some $x \in \mathcal{P}(\underline{j}_R^{-1}(n))$. From the second statement (applied to the (R,R) -bimodule R) we obtain

$$(\varinjlim G_{\mathcal{F}_i} p : \underline{j}_R(x)) = \varinjlim G_{\mathcal{F}_i}(p : x) \subset \varinjlim G_{\mathcal{F}_i} p .$$

(b) $\underline{j}_R^{-1}(n) \to p$. Then either $\underline{j}_R^{-1}(n) \subset p$ or $(\underline{j}_R^{-1}(n) : x) \subset p$ for some $x \in \mathcal{P}(R)$.

In the first case according to the first statement of the corollary $n \subset \varinjlim G_{\mathcal{F}_i}(\underline{j}_R^{-1}(n)) \subset \varinjlim G_{\mathcal{F}_i} p$.

in the second case

$$(n : \underline{j}_R(x)) \subset \varinjlim (G_{\mathcal{F}_i}(\underline{j}_R^{-1}(n)) : \underline{j}_R(x)) = \varinjlim G_{\mathcal{F}_i}(\underline{j}_R^{-1}(n) : x) \subset \varinjlim G_{\mathcal{F}_i} p.$$

Similarly, if $p \in \widehat{\operatorname{Spec}}_e R$, $n \in I_e \varinjlim G_{\mathcal{F}_i} R$ and $\underline{j}_R^{-1}(n) \not\subset p$, then $(p : x) \subset p$ for some $x \in \mathcal{P}(\underline{j}_R^{-1}(n))$, and, consequently, $(\varinjlim G_{\mathcal{F}_i} p : \underline{j}_R(x)) \subset \varinjlim G_{\mathcal{F}_i} p$; whereas if $\underline{j}_R^{-1}(n) \subset p$, then

$$n \subset \varinjlim_i G_{\mathcal{F}_i}(\underline{j}_R^{-1}(n)) \subset \varinjlim_i G_{\mathcal{F}_i} p . \ \square$$

3. PRECOSITI AND ω- SHEAVES

1. Definitions. For an arbitrary category A denote by $\downarrow A$ the collection of all the sets of arrows from $Hom A$ with the common origin and by $\uparrow A$ the collection of the sets of arrows from $Hom A$ with the common end.

A pair $(A, \bar{A})$, where A is a category, $\bar{A}$ a subset of $\downarrow A$, will be called a precositus, satisfying the following conditions

(1) $(x \xrightarrow{id} x) \in \bar{A}$ for every $x \in Ob\, A$,

(2) if $\{x \to x_i\}_{i \in J} \in \bar{A}$, then for any $(i, j) \in J \times J$ there exists a fiber coproduct $x_i \coprod_x x_j$,

A precositus $(A, \bar{A})$ is called a cositus, if (2) is replaced by "invariance under the base change":

(2') if $\{x \to x_i\}_{i \in J} \in \bar{A}$, then for every arrow $x \to y$ there exist fiber coproducts $x_i \coprod_x y$ and the set of coprojections $\{y \to x_i \coprod_x y\}_{i \in J}$ belongs to $\bar{A}$; and, besides, the "composition" property holds:

(3) if $\{x \xrightarrow{\xi_i} x_i\}_{i \in J} \in \bar{A}$ and $\{x_i \xrightarrow{\xi_{ij}} x_{ij}\}_{j \in J_i} \in \bar{A}$ for each $i \in J$, then $\{x \xrightarrow{\xi_{ij} \circ \xi_i} x_{ij} \mid i \in J, j \in J_i\} \in \bar{A}$.

Under the dualisation (the arrows change directions and the fiber coproducts become the fiber products) the precositi turn into the formations that will be called presiti. It is easy to see that the dualisation of the cositi are siti (alternatively called the Grothendieck topologies).

A morphism of a precositus $\underline{A} = (A, \bar{A})$ into a precositus $\underline{B} = (B, \bar{B})$ is a triple $(\underline{A}, F, \underline{B})$, where F is a fiber coproducts-preserving functor $B \to A$ such that for every set $\{x \xrightarrow{\xi_i} x_i\}_{i \in J}$ from B its image $\{Fx \xrightarrow{F\xi_i} Fx_i\}_{i \in J}$ belongs to A. The composition is naturally defined :

$(\underset{\sim}{B}, G, \underset{\sim}{C}) \circ (\underset{\sim}{A}, F, \underset{\sim}{B}) = (\underset{\sim}{A}, F \circ G, \underset{\sim}{C})$. Therefore the collection of the precositi $(A, \bar{A})$ with $\mathrm{Hom} A$ and $\bar{A}$ belonging to some fixed universum form a category, which will be denoted $\widehat{Cov}$. In a dual way the presiti category Cov is defined. Denote by $G\widehat{Top}$ and $GTop$ the full subcategories of $\widehat{Cov}$ and Cov formed by the cositi and by the siti respectively.

2. Examples.

2.1. Let X be a topological space, $\mathcal{cl}X$ the category of the closed subsets of X (the inclusions are morphisms); $\overline{\mathcal{cl}X}$ consists of all the sets $\{V \hookrightarrow V_i \mid i \in J\}$ such that $V = \bigcap\{V_i \mid i \in J\}$. Clearly, $\underline{X} = (\mathcal{cl}X, \overline{\mathcal{cl}X})$ is a cositus. Its dual situs is identified naturally with the situs of the open sets of the space X.

A continuous map of the topological spaces $f : X \to Y$ gives a functor ${}^a f : \mathcal{cl}Y \to \mathcal{cl}X$, which obviously defines a morphism ${}^a\underline{f} : \underline{X} \to \underline{Y}$ of the corresponding cositi. Clearly, $(X \xrightarrow{f} Y) \mapsto {}^a\underline{f} = (\underline{X}, {}^a f, \underline{Y})$ is a functor from Top into $G\widehat{Top}$. Note that this functor is not full; i.e., there exist a morphisms (even isomorphisms) of the cositi of the closed sets, that does not originate from any continuous map. Let, for instance $X =$ Spec R, where R is a commutative ring with unit, $|X| =$ Max R the subspace of X consisting of the maximal ideals (with the induced topology). If R is a finitely generated algebra over a field or $\mathbb{Z}$, then by Hibert's Nullstellensatz for every closed subset W of X the subset $|X| \cap W$ is dense in W. This means that the inclusion $|X| \hookrightarrow X$ induces an isomorphism of the cositi $\underline{|X|} \xrightarrow{\sim} \underline{X}$. The inverse isomorphism only in exceptional situations.

turns out to be the image of a continuous map.

2.2. Let R be an arbitrary associative ring, $\mathcal{T}$ a family of radical filters of the left ideals of R such that for any pair $\{\mathcal{F}, \mathcal{G}\} \subset \mathcal{T}$ the intersection of all the filters from $\mathcal{T}$ containing $\mathcal{F} \cup \mathcal{G}$, belongs to $\mathcal{T}$. This condition is equivalent to the following one: the category $\mathcal{T}$ (the inclusions are morphisms) is closed under the finite coproducts. As $\overline{\mathcal{T}}$ take the collection of all the sets $\{\mathcal{G} \hookrightarrow \mathcal{G}_i\}_{i \in J}$ such that $\bigcap\{\mathcal{G}_i \mid i \in J\} = \mathcal{G}$. We see that the precositus $\underline{\mathcal{T}} = (\mathcal{T}, \overline{\mathcal{T}})$ has not a few common features with the cositi of the closed sets of the topological spaces. For instance, the category $\mathcal{T}$ is also a preorder (any pair of objects is connected by no more than one arrow) and the "composition" property is satisfied. The most essential difference is the following one:

as a rule, $\underline{\mathcal{T}}$ is not a cositus; i.e., the inclusion $(\bigcap_{i \in J} \mathcal{G}_i) \amalg \mathcal{F} \hookrightarrow \bigcap_{i \in J} (\mathcal{G}_i \amalg \mathcal{F})$, where $\{\bigcap_{i \in J} \mathcal{G}_i, \mathcal{F}, \mathcal{G}_j \mid j \in J\} \subset \mathcal{T}$, usually is a proper one . □

Remark. The example 2.2. is principal for this paper. It was this example that had caused the appearance here of pre(co)siti and is responsible for the preference paid to the (pre)cositi as compared with the (pre)siti, contrary to the existing tradition. Note, that the precositi appear in numerous context, for instance, in constructing a geometry, connected with the recursive functions [9] . □

The following two examples are not so directly related to the events that will happen on our stage, but hopefully they may turn to be useful to the reader. The first of them is the "classical" example of a nontrivial Grothendieck topology.

2.3. Let G be a group; $G\text{-}\mathcal{E}ns$ the category whose objects are the left G-sets $(X, G\times X \xrightarrow{\xi} X)$, morphisms $(X, \xi) \to (X', \xi')$ the maps $f : X \to X'$ such that the diagram

$$\begin{tikzcd} G\times X \arrow[r, "id\times f"] \arrow[d, "\xi"'] & G\times X' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

commutates. For "coverings" take all the families $\{(X_i, \xi_i) \xrightarrow{f_i} (X, \xi) \mid i \in J\}$ of morphisms of G-sets, for which $\bigcup_{i\in J} f_i(X_i) = X$. It is easy to verify that the situs $\underline{G\text{-}\mathcal{E}ns} = (G\text{-}\mathcal{E}ns, \overline{G\text{-}\mathcal{E}ns})$ is thus defined. The category of the "open sets" $G\text{-}\mathcal{E}ns$ is not a preorder anymore. Instead it has a finite object (the one-point G-set), which may be considered as an analogue of a topological space.

To a group morphism $\varphi : G \to G'$ the functor of the "base change" $\varphi_*: G'\text{-}\mathcal{E}ns \to G\text{-}\mathcal{E}ns$ corresponds which clearly defines a morphism of siti $\underline{G\text{-}\mathcal{E}ns} \to \underline{G'\text{-}\mathcal{E}ns}$.

2.4. Let R be an associative ring, $\overline{I_\ell R}$ the collection of all the sets $\{m_i \hookrightarrow m \mid i \in J\}$ of morphisms from the category $I_\ell R$ (the inclusions of ideals) such that $m = \sup\{m_i \mid i \in J\}$. The category $I_\ell R$ is a preorder with products (as is readily seen, the product of a family of ideals coincides with their intersection) and therefore $\underline{I_\ell R} = (I_\ell R, \overline{I_\ell R})$ is a presitus. With a ring morphism $\varphi : R \to R'$ the functor $\varphi^\# : I_\ell R \to I_\ell R'$, $m \mapsto (R'\varphi(m))$ is associated which obviously defines a presiti morphism $\underline{I_\ell R'} \to \underline{I_\ell R}$. Note as an aside that the functor $\varphi^\#$ is left-adjoint to the functor $\varphi_\# : n \mapsto \varphi^{-1} n$ which, in general, is not a presiti morphism.

The presitus structure on $I_\ell R$ induces the structure of the presitus $\underline{IR} = (IR, \overline{IR})$ on the category IR of the two-sided ideals R. The map that assigns to an ideal $\alpha \in IR$ the open set $U(\alpha) \overset{def}{=} \{p \in \mathrm{Spec}\,R \mid \alpha \not\subset p\}$ of the prime spectrum Spec R is a surjective functor from IR into the category $\mathcal{O}p$ Spec R of the open sets of Spec R. The equality $U(\alpha \cdot \beta) = U(\alpha) \cap U(\beta)$ and the inclusion $\alpha \cdot \beta \subset \alpha \cap \beta$ for any $\{\alpha, \beta\} \subset IR$ imply $U(\alpha \cap \beta) = U(\alpha) \cap U(\beta)$; i.e., the functor U commutes with finite products. The equality $U(\sup_{i \in J} \alpha_i) = \bigcup_{i \in J} U(\alpha_i)$ true for an arbitrary family $\{\alpha_i \mid i \in J\} \subset IR$ implies that U defines a morphism from the situs $\mathcal{T}\mathrm{Spec}\,R$ of the open sets of the Spec R into the presitus $\underline{IR}$.

3. Presheaves and sheaves. The presheaves on a precositus $\underline{A} = (A, \overline{A})$ with the values in a category C are arbitrary functors from A into C; by definition the category of the presheaves $F_o(\underline{A}, C)$ coincides with the category of the functors from A into C – $F_o(\underline{A}, C) = C^A$.

A presheaf $G : A \to C$ will be called a sheaf (respectively an ω-sheaf), if for every $\{x \xrightarrow{\xi_i} x_i \mid i \in J\} \in \overline{A}$ (where J is a finite set) the cone

$$\begin{tikzcd} & Fx_i \arrow[dr] & \\ Fx \arrow[ur, "F\xi_i"] \arrow[dr, "F\xi_j"'] & & F(x_i \amalg_x x_j) \\ & Fx_j \arrow[ur] & \end{tikzcd} \qquad (i, j) \in J \times J \tag{1}$$

is initial. If C is a category with products, then to a cone (1) the diagram

$$Fx \longrightarrow \prod_{i \in J} Fx_i \rightrightarrows \prod_{(i,j) \in J \times J} F(x_i \amalg_x x_j) \tag{2}$$

is naturally assigned, and (1) is initial if and only if (2) is exact.

The full subcategory of the category of presheaves $F_o(\underline{A}, \mathcal{C})$ formed by (ω-)sheaves will be denoted by $F(A, C)$ (respectively, by $F_\omega(\underline{A}, \mathcal{C})$).

Remarks. 1) Obviously, the category of (pre)sheaves on a cositus with values in a category C coincides with the category of (pre)sheaves on the dual situs understood in usual way.

2) As for the topological spaces, many problems of sheaves on precositi with values in an arbitrary category C can be reduced to the corresponding problems of the sheaves of sets thanks to the following criterion that is directly veryfied:

A presheaf $F : A \longrightarrow C$ on the precositus $\underline{A} = (A, \bar{A})$ is a sheaf if and only if the presheaf of sets $C(y, F-)$ is a sheaf for every $y \in \operatorname{Ob} C$.

Let $\underline{A} = (A, \bar{A})$ be a precositus. Denote by $\bar{A}_\omega$ the collection of all the $\{x \to x_i \mid i \in I\} \in \bar{A}$ such that

for each finite subset $J_o \subset I$ there exists a finite subset $J_1 \subset I$ such that

(a) $\{x \to x_j \mid j \in J_1\} \in \bar{A}$;

(b) if $i \in J_o \setminus J_1$, then a set of the coprojections $\{x_i \longrightarrow x_i \amalg_x x_j \mid j \in J_1\}$ belongs to A.

The following properties of this construction are obvious, more or less:

1) $\underline{A}_\omega = (A, \bar{A}_\omega)$ is a precositus.

2) Let $\{x \to x_i \mid i \in J\} \in \bar{A}$ and any finite subset $J_o \subset J$ be contained in a finite subset $J_1 \subset J$ such that $\{x \to x_j \mid j \in J_1\} \in \bar{A}$. Then $\{x \to x_i \mid i \in J\} \in \bar{A}_\omega$.

3) Suppose that if $\{x \to x_i \mid i \in J\} \in \bar{A}$ and $Card(J) < \infty$, then for any arrow $x \to y$ the family $\{x \to y, x \to x_i\}_{i \in J}$ belongs to $\bar{A}$. Then, as follows from 2), $\bar{A}_\omega$ consists of all $\{x \to x_i \mid i \in I\} \in \bar{A}$ such that $\{x \to x_i \mid i \in J_1\} \in \bar{A}$ for a finite subset $J_1 \subset J$.

4) If $\underline{A}$ is the cositus of the closed subsets of a topological space or the precosite of the radical filters of the left ideals of a ring, then $\underline{A}$ has a property stronger than 3): the collection of arrows $\{x \to x_j \mid j \in J\}$ belongs to $\bar{A}_\omega$ if and only if $\{x \to x_i \mid i \in J_1\} \subset \bar{A}$ for some $J_1 \subset J$.

5) If $\underline{A}$ is a cositus, then $\bar{A}_\omega$ also consists if all $\{x \to x_i \mid i \in J\}$ such that $\{x \to x_j \mid j \in J_0\} \in \bar{A}$ for a finite subset $J_0 \subset J$.

Proposition. *For every predcositus* $\underline{A} = (A, \bar{A})$ *and for an arbitrary category* C *the category of* ω*-sheaves* $F(\underline{A}, \mathcal{C})$ *coincides with the category* $F(\underline{A}_\omega, \mathcal{C})$ *of sheaves on* $\underline{A}_\omega = (A, \bar{A}_\omega)$.

Sketch of the proof. (i) Clearly, $F(\underline{A}_\omega, \mathcal{C}) \subset F_\omega(\underline{A}, \mathcal{C})$. It suffices to prove the opposite inclusion for $C = \mathcal{E}ns$ (see Remark 2)). Therefore hereafter we will deal with the ω-sheaves of sets.

(ii) To every presheaf $G : A \to \mathcal{E}ns$ we assign the function $\hat{G}$ on the collection $A_{\amalg}$ of all the sets of arrows of the form $\{x \to x_i \mid i \in J\}$ for which the fibre coproducts $x_i \coprod_x x_j$ exist, whose value in $\bar{x} = \{x \to x_i \mid i \in J\}$ is a kernel of the canonical pair $\prod_{i \in J} G x_i \rightrightarrows \prod_{(i,j) \in J \times J} G(x_i \coprod_x x_j)$. For every $\bar{x} = \{x \to x_i \mid i \in I\}$ there exists a canonical arrow (map) $\rho_{\bar{x}} : G x \to \hat{G} \bar{x}$, and for any $J_0 \subset I$ the morphism $\hat{G} \bar{x} \to \prod_{i \in J_0} G x_i$ (the composition of the embedding $k_{\bar{x}}: \widehat{G}\bar{x} \to \prod_{j\in I} Gx_j$ and of the projection) uniquely factors through $k_{\bar{x}_{J_0}}: \widehat{G}\bar{x}_{J_0} \rightarrowtail \prod_{i\in J_0} Gx_i$, where $\bar{x}_{J_0} \overset{def}{=} \{x \to x_i \mid i \in J_0\}$. In particular, if $\rho_{\bar{x}_{J_0}}: Gx \longrightarrow \widehat{G}\bar{x}_{J_0}$ is an isomorphism, then there exists a coretraction $\psi_{J_0}: \widehat{G}\bar{x} \to Gx$, which is uniquely defined from the commutativity of the diagram

$$\begin{tikzcd} \widehat{G}\bar{x} \arrow[r, "\psi_{J_0}"] \arrow[dr] & Gx \arrow[d] \\ & \prod_{i\in J_0} Gx_i \end{tikzcd}$$

we are interested in

An example: G is an ω-sheaf on $\underline{A}$, $\mathrm{card}(J_0) < \infty$ and $\bar{x}_{J_0} \in \widetilde{A}$.

(iii) Now let G be an ω-sheaf on $\underline{A}$; $\bar{x} = \{x \to x_i \mid i \in I\}$ a family from $\widetilde{A}_\omega$; $J_0 \subset I$ an arbitrary finite subset, $J_1 \subset I$ a finite subset, such that $\bar{x}_{J_1}$ and $\bar{x}^{(i)}_{J_1} \overset{def}{=} \{x_i \to x_i \amalg_x x_j \mid j \in J_1\}$ belong to $\widetilde{A}$ for all $i \in J_0 \setminus J_1$. Denote $J_1 \cup J_0$ by J, fix $i \in J_0 \setminus J_1$, and consider the diagram

$$\begin{tikzcd} \widehat{G}\bar{x}_I \arrow[r, "p_i"] \arrow[dd, "\psi_{J_1}"'] & Gx_i \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "1"] & \prod_{\ell\in J_1} G(x_i \amalg_x x_\ell) \arrow[r, shift left] \arrow[r, shift right] & \prod \sim \\ & \widehat{G}x^{(i)}_{J_1} & & \\ Gx \arrow[uur, dashed, "G\xi_i"] \arrow[ur, dashed, "\eta"] \arrow[rr, "j_{\bar{x}}"'] & & \prod_{\ell\in J_1} Gx_\ell \arrow[uu, "\alpha"'] \arrow[r, shift left] \arrow[r, shift right] & \prod \approx \arrow[uu] \end{tikzcd} \tag{3}$$

Here η is an arrow, uniquely defined by the commutativity of the subdiagram (1); ψ_{J_1} is a coretraction mentioned in (ii), the other arrows are obvious. The subdiagram distinguished by the solid arrows is commutative. Hence so is (3), since

$$j_{\bar{x}^{(i)}} \circ p_i = \alpha \circ j_{\bar{x}} \circ \psi_{J_1} = j_{\bar{x}^{(i)}} \circ (G\xi_i \circ \psi_{J_1})$$

and, therefore, $p_i = G\xi_i \circ \psi_{J_1}$, since $j_{x^{(i)}}$ is a monomorphism. Of the above, it is important that the projection $p_i: \widehat{G}\bar{x}_i \to Gx_i$ is the composition of ψ_{J_1} and of the natural map $G_I x \to G_I x_i$. From this and from the arbitrariness of $i \in I \setminus J_1$ the injectiveness of Ψ_{J_1} follows. Hence Ψ_{J_1} is an isomorphism.

(iv) It is easy to see, that $\widehat{G}_I(\bar{x})$ is isomorphic to the inverse limit of $\widehat{G}_I(\bar{x}_{I'})$, where I' runs the finite subsets of J. The result of the previous subsection may be presented as follows: for any finite subset $J_0 \subset J$ there exists a finite subset $I \subset J$ such that $J_0 \subset I$ and the canonical arrow $G_I x \to \widehat{G}_I \bar{x}_I$ is an isomorphism. This implies that $G_I x \to \widehat{G}_I \bar{x} = \varprojlim_{Card(I)<\infty} \widehat{G}_I \bar{x}_I$ is an isomorphism. □

Example (1). Let R be a ring, $\mathcal{T}$ a family of the radical filters, satisfying the conditions of Example 2.2.; $\underline{\mathcal{T}} = (\mathcal{T}, \overline{\mathcal{T}})$ an associated precositus. For every R-module M the map $\mathcal{F} \mapsto G_{\mathcal{F}} M$ extends to the presheaf $M_{\mathcal{T}}$ on $\underline{\mathcal{T}}$ – the "structural" presheaf of the module M. In the next section one of the key results of the present paper will be proved:

The presheaf $M_{\mathcal{T}}$ turns out to be an $\underline{\omega}$-sheaf iff the following condition is satisfied:

(b) for every $\{\mathcal{F}, \mathcal{G}\} \subset \mathcal{T}$ the canonical morphism $\mathcal{F}^1 \mathcal{G}^1 M \to (\mathcal{F} \amalg \mathcal{G})^1 M$ is an isomorphism.

According to Proposition 3 every module M, satisfying the condition (b), defines a sheaf on $\underline{\mathcal{T}}_\omega = (\mathcal{T}, \overline{\mathcal{T}}_\omega)$. As was already pointed out (the statements 3) and 4) concerning $\underline{A}_\omega$), $\overline{\mathcal{T}}_\omega$ consists of all $\{\mathcal{F} \hookrightarrow \mathcal{F}_i \mid i \in I\}$ such that $\{\mathcal{F} \hookrightarrow \mathcal{F}_j \mid j \in J\} \in \overline{\mathcal{T}}$ for a finite subset $J \subset I$. Therefore, if for every family $\{\mathcal{F}, \mathcal{F}_i \mid i \in I\}$ of the radical filters from $\mathcal{T}$, such that $\mathcal{F} = \bigcap\{\mathcal{F}_i \mid i \in I\}$ there exist a finite subset $J \subset I$ for which $\mathcal{F} = \bigcap \{\mathcal{F}_j \mid j \in J\}$, then the following properties of M are equivalent:

(i) the structural presheaf $\mathcal{M}_{\mathcal{T}}$ is a sheaf;

(ii) M satisfies the condition (b) . □

Remark. All the constructions appearing here as well as the statements about them are readily dualised. In particular, the notation $\underline{A}_{\omega} = (A, \overline{A}_{\omega})$ does not require any explanation when $\underline{A} = (A, \overline{A})$ is a presitus and referring to Proposition 3 we mean, when needed, its dual formulation. □

Example (2). Consider the presitus $\underline{I_\ell R} = (I_\ell R, \overline{I_\ell R})$ of Example 2.4. Obviously, $\overline{I_\ell R}$ enjoyes the dual of the property, mentioned in the statement 4) concerning $\overline{A}_{\omega}$. Therefore $\overline{I_\ell R}_{\omega}$ consists of all the "coverings" $\{m_i \hookrightarrow m\}_{i \in I}$ such that $\{m_i \hookrightarrow m\}_{i \in J}$ is a covering for some finite subset $J \subset I$. In particular, if m is a finitely generated left ideal then any covering $\{m_i \hookrightarrow m\}_{i \in I}$ belongs to $\overline{I_\ell R}_{\omega}$. Therefore, Proposition 3 implies that if R is a left-noetherian ring, the category of the ω-sheaves on $I_\ell R$ coincides with the category of the sheaves. □

Example(3). The state of affairs is similarly in the case of the presite IR (see the second half of Example 2.4). E.g., if an ideal $m \in IR$ is finitely generated (as a two-sided ideal), then every its covering belongs to $\overline{IR}_{\omega}$. In particular, if the ring R is (symmetrically) noetherian then $\underline{IR} = \underline{IR}_{\omega}$ and therefore every ω-sheaf on IR is a sheaf. □

4. The "direct image" functors. If $\underline{F} = (\underline{A}, F, \underline{A}')$ is a morphism of the pre(co)siti, then, as is easy to verify, the

functor $F_{\#} = \mathcal{C}^{F}: F_o(\underline{A}, \mathcal{C}) \longrightarrow F_o(\underline{A}', \mathcal{C})$, $G \mapsto G \circ F$, sends (ω-) sheaves into (ω-) sheaves; i.e., it induces the functors of the "direct image"

$\underline{F}_{*}: F(\underline{A}, \mathcal{C}) \to F(\underline{A}', \mathcal{C})$ and $F_{*\omega}: F_\omega(\underline{A}, \mathcal{C}) \to F_\omega(\underline{A}', \mathcal{C})$.

Examples. 1) The direct image functor corresponding to the morphism $\mathcal{U}: \mathcal{T}\,\mathrm{Spec}\ R \to \mathbb{I}\mathrm{R}$ of Example 2.4, realizes the complete embedding (thanks to the surjectivity of U) of the category of (ω-)-sheaves on Spec R into the category of (ω-)-sheaves on $\underline{\mathbb{I}\mathrm{R}}$.

2) Let R be a commutative unitary ring; the category $\mathcal{T}$ consist of all the radical folters of the form $F_{(t)} = \{n \in \mathbb{I}R \mid t^k \in n$ for some $k \geq 1\}$, $t \in R$.

Exercise. The following facts are true:

(i) $\mathcal{T}$ is a category with finite coproducts and $F_{(s)} \amalg F_{(t)} = F_{(st)}$ for any $(s,t) \in R \times R$.

(ii) The sets $V(t)$, $t \in R$, constitute a subcategory $\mathcal{B} \subset cl\,\mathrm{Spec}\,R$ closed under the (finite) coproducts, and $V(s) \cup V(t) = V(st)$. We have $V(s) = F_{(s)} \cap \mathrm{Spec}\,R$ and the map $F_{(t)} \mapsto V(t)$ defines a precositus morphism $\underline{\mathcal{B}} = (\mathcal{B}, \bar{\mathcal{B}}) \to \underline{\mathcal{T}} = (\mathcal{T}, \bar{\mathcal{T}})$ where the structure $\bar{\mathcal{B}}$ is induced by the embedding $\mathcal{B} \hookrightarrow cl\,\mathrm{Spec}\,R$ from $\underline{\mathrm{Spec}\,R}$.

(iii) For any $t \in R$ the radical filter $F_{(t)}$ consists of all $m \in \mathbb{I}R$, such that $V(m) \subset V(t)$

It follows that the map $V(t) \mapsto F_{(t)}$ is the functor, inverse to $\mathcal{F} \mapsto \mathrm{Spec}\,R \cap \mathcal{F}$.

Therefore the precositus $\underline{\mathcal{T}}$ is isomorphic to the cositus $\underline{\mathcal{B}}$, and, therefore, is a cositus itself. In particular, the categories of the presheaves and sheaves on $\underline{\mathcal{T}}$ are isomorphic to the corresponding categories on $\underline{\mathcal{B}}$. Since $V(t)=\bigcap_{i\in I} V(t_i)$ implies $V(t)=\bigcap_{i\in J} V(t_i)$ for a finite subset $J\subset I$, then $\underline{\mathcal{B}}=\underline{\mathcal{B}}_\omega$. Therefore, the categories of the sheaves and of the ω-sheaves on $\underline{\mathcal{B}}$ (and hence on $\underline{\mathcal{T}}$) coincide. The equalities $F_{(s)} \amalg F_{(t)} = F_{(st)} = F_{(s)} \circ F_{(t)}$ for any $(s, t) \in R\times R$ and $\mathcal{F}^1 \mathcal{G}^1 = (\mathcal{G} \circ \mathcal{F})^1$, true for any radical filters, imply (b) for any R-module M (see Example 3.1), and, consequently, the presheaf $\mathcal{M}_{\mathcal{T}}$ is a sheaf. With the direct image functor we get the known (in somewhat different form) fact:

For every R-module M the map $V(t) \mapsto (t)^{-1}\mathcal{M}$ naturally extends to a sheaf on the cositus $\underline{\mathcal{B}}$.

5. Quasifiniteness, bases, full quasicompactness.

We shall discuss the concepts that serve as a bridge between the sheaves and the ω-sheaves.

Definitions. Let $\underline{A} = (A, \overline{A})$ be a precositus.

1) Call $x \in Ob\, A$ a quasifinite object of $\underline{A}$, if any $\{x \to x_i \mid i \in J\}$ from $\overline{A}$ belongs to $\overline{A}_\omega$.

A precositus $\underline{A}$ will be called a quasifinite if all its objects are quasifinite; i.e., $\underline{A} = \underline{A}_\omega$.

2) A precositus $\underline{A}$ will be called quasicompact if the category A has an initial object, which is a quasifinite object of $\underline{A}$.

3) By a basis of a precositus $\underline{A}$ mean a full subcategory B of A, which is closed under the fiber coproducts and such that for every $x \in Ob\, A$ there exists $\{x \to x_i \mid i \in I\} \in \overline{A}$ with $\{x_i \mid i \in I\} \subset B$.

4) A precositus $\underline{A}$ will be called fully quasicompact, if it has a basis, all the objects of which are quasifinite.

5) A topological space will be called fully quasicompact, if the cositus $\underline{X}$ of its closed sets is fully quasicompact. □

Remark. The notions of the quasifiniteness of the basis and (full) quasicompactness are automatically dualised. In what follows they will be also applied to the presiti. □

Examples. 1) A topological space X is quasicompact if and only if the cositus $\underline{X}$ (or equivalently the situs of the open sets of X) is quasicompact.

2) Let R be a commutative ring with unit. It is well-known that Spec R is quasicompact. Besides, the subcategory $\mathfrak{B}$, formed by the sets $V(t), t \in R$, is a quasifinite base of the cositus $\underline{\text{Spec } R}$ (see Example 4.2) and therefore Spec R is a fully quasicompact space.

3) Fully quasicompact spaces (and with more reason pre(co)siti) may be not quasicompact. An example; — any non-quasicompact scheme: the collection of open affine subsets is a quasifinite basis of the situs of open sets of the topological space of a scheme.

The term "locally fully quasicompact" (pre(co)siti or spaces) would have been more to the point? But it is too cumbersome.

4) The presiti $\underline{I_\ell R}$ and $\underline{IR}$ are fully quasicompact. A quasifinite basis of I R is formed by the subcategory I R of all the finitely generated left ideals, and that of IR by the subcategory IR of all the finitely generated twosided ideals.

If R is a finitely generated left module (e.g., if R is a ring with right unit), and only in this case, the presitus $I_\varphi R$ is quasicompact. Similarly, IR is quasicompact if and only if R is a finitely generated as twosided ideal.

6. Extensions of the sheaves. Let $\underline{A} = (A, \bar{A})$ be a precositus; $G : A' \to A$ a functor, preserving the fiber coproducts and reflecting them; the last means that the existence of

$Gx \coprod_{Gy} Gz$ implies that of $x \coprod_y z$.

Then G induces a precositus structure $\bar{A}'_G \overset{def}{=} \{\{x \xrightarrow{\xi_i} x_i\}_{i \in I} \mid \{G\xi_i\} \in \in \bar{A}\}$ on A' and defines a universal, in an obvious sense, morphism $\underline{A} \to \underline{A}'_G = (A', \bar{A}'_G)$. In particular, the embedding of a basis $J_B : B \hookrightarrow A$ induces a precositus structure $\underline{B} = (B, \bar{B})$ and a morphism $\underline{J}_B : \underline{A} \longrightarrow \underline{B}$. To $\underline{J}_B$ the direct image functor $\underline{J}_{B*} : F(\underline{A}, C) \to F(\underline{B}, C)$ corresponds, the restriction onto the basis. On the other hand, it is possible to try to extend the sheaves from a basis onto $\underline{A}$.

For any functor $F : B \to C$ denote by $\delta J^R_B F$ the full subcategory of A formed by all $x \in Ob\, A$ for which there exists $J^R_B F x \overset{def}{=} \varprojlim (x\backslash B \to B \xrightarrow{F} C)$. The map $x \mapsto J^R_B F x$ uniquely extends (after the values of $J^R_B F x$, $x \in Ob\,\delta J^R_B F$, are chosen) to the functor $J^R_B F : \delta J^R_B F \longrightarrow C$, which is called the right Can extension of F.

To investigate the properties of the functors $\underline{J}_{B*}$ and of the right Can extensions of sheaves it is convenient to start with turning the set of "cocoverings" $\bar{A}$ into a category. A morphism from $\bar{x} = \{x \xrightarrow{\xi_i} x_i\}_{i \in I}$ into $\bar{y} = \{y \xrightarrow{\eta_j} y_j\}_{j \in J}$ is a triple $(f, \sigma, \{f_j\}_{j \in J})$, where $\sigma : J \to I$ is a map, $f \in A(x,y)$, and $f_j : x_{\sigma(j)} \to y_j$ are arrows from A such that for every $j \in J$ the diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\xi_{\sigma(j)}"'] & y \arrow[d, "\eta_j"] \\ x_{\sigma(j)} \arrow[r, "f_j"] & y_j \end{tikzcd}$$

commutes; the composition is $(f, \sigma, \{f_j\}) \circ (g, \tau, \{g_e\}) = (f \circ g, \tau \circ \sigma, \{f_j \circ g_{\sigma(j)}\})$

Proposition. Let B be a basis of the precositus $\underline{A} = (A, \bar{A})$.

1) If for every sheaf $G : B \to C$ on B $J^R_B G$ is a sheaf on $\underline{A}$, then J_{B*} is an equivalence of the categories, and its quasiinverse sends every sheaf G into its right Can extension $J^R_B G$.

2) Suppose that for every $x \in \mathrm{Ob}\, A$ we have

($\dagger$) for every finite subset $\{x \xrightarrow{\xi_i} b^i\}_{i \in J} \subset \mathrm{Ob}\, x\backslash B$ there exist $\bar{x} = \{x \to v_j\}_{j \in I} \in \bar{A} \cap \mathrm{Ob}\, x\backslash B$ and morphisms $\bar{\xi}_i = (\xi_i, \ldots) : \bar{x} \to \bar{b}^i = (b^i \to b^i_k)_{k \in K}$, where $\bar{b}^i \in \bar{B}$, $i \in J$.

Then the functor $J_{B*} : F(\underline{A}, C) \to F(\underline{B}, C)$ realises an equivalence of $F(\underline{A}, C)$ with the full subcategory $F(\underline{B}, \underline{A}, C)$ of $F(\underline{B}, C)$, formed by all the sheaves G for which $J^R_B G$ is a sheaf on the precositus $\underline{A}$.

3) Suppose that for every $\bar{x} = \{x \to b_i\}_{i \in I} \in \bar{A} \cap \mathrm{Ob}\, x\backslash B$ and for an arbitrary arrow $\xi : x \to b$, $b \in \mathrm{Ob}\, B$, there exists a "cocovering" $\bar{b} = \{b \to b'_j\}_{j \in J} \in \bar{B}$ and an extension of the morphism ξ to the morphism $\bar{\xi} : \bar{x} \to \bar{b}$. Then for every sheaf $G \in F(\underline{B}, C)$ the right Can extension $J^R_B G$ it exists is a sheaf on $\underline{A}$.

Sketch of the proof. (i) Denote by δJ^R_B the full subcategory of $C^B = F_o(\underline{B}, C)$, formed by all the presheaves G for which $J^R_B G$ is defined on the whole A.

For any presheaf G from δJ^R_B there exists a canonical morphism $\varepsilon_G : J_{B\#} J^R_B G \to G$, which turns out to be an isomorphism since B is full.

This fact is verified by the direct application of the formula defining $J^R_B G$. It is, however, well known (see, e.g. [10]).

(ii) For every presheaf $G : B \to C$ from $\mathcal{S} J^R_B$ and any $y \in Ob\,\mathcal{C}$ there exists (since $\mathcal{C}(y,-)$ commutes with $\varprojlim$) a natural isomorphism

$$J^R_B \mathcal{C}(y, G-) \simeq \mathcal{C}(y, J^R_B G\,\cdot)$$

Therefore we can (and will) assume that $\mathcal{C} = \mathcal{E}ns$.

(iii) Thus, let G be a sheaf of sets on $\underline{A}$; i.e. for each $\bar{x} = \{x \to x_i\}_{i \in I} \in \bar{A}$ the canonical diagram

$$G x \to \prod_{i \in I} G x_i \rightrightarrows \prod_{(i,j) \in I \times I} G(x_i \coprod_x x_j) \tag{2}$$

is exact. Since B is a basis, there exist $\{x \to b_i\}_{i \in I} \in \bar{A}$ and a collection $\{b_i \coprod_x b_j \to b_{ijt} \mid t \in J_{ij}\} \in \bar{A}$ for each pair i, j such that $\{b_i, b_{ijt} \mid \{i,j\} \subset I, t \in J_{ij}\} \in B$. Since (2) is exact for $\bar{x} = \{x \to b_i \mid i \in I\}$ and $G(b_i \coprod_x b_j) \to \prod_{t \in J_{ij}} G b_{ijt}$ is injective for each pair $(i, j) \in I \times I$, then the diagram

$$G x \to \prod_{i \in I} G b_i \rightrightarrows \prod_{(i,j) \in I \times I,\ t \in J_{ij}} G b_{ijt} \tag{3}$$

is exact. This, clearly, implies the existence of a unique morphism $\Psi_{\bar{x}} : J^R_B J_{B\#} G x \longrightarrow G x$ such that the diagram

$$\begin{tikzcd} J^R_B J_{B\#} G x \arrow[rr] \arrow[dr, "\Psi_{\bar{x}}"'] & & \prod_{i \in I} G b_i \\ & G x \arrow[ul, dashed, bend right, "\gamma_G(x)"'] \arrow[ur] & \end{tikzcd} \tag{4}$$

is commutative. Since (3) is exact, the standard arguments yield $\Psi_{\bar{x}} \circ \gamma_G(x) = 1_{Gx}$, where $\gamma_G(x)$ is a canonical arrow.

(iv) Consider the function $\hat{G}: A_{\amalg} \to \mathcal{E}ns$ (see part (ii) of the proof of Proposition 3) assigning to a collection of arrows $\{x \to x_i\}_{i \in \mathcal{Y}}$ the kernel of the pair

$$\prod_{i \in \mathcal{Y}} G x_i \rightrightarrows \prod_{(i,j) \in \mathcal{Y} \times \mathcal{Y}} G(x_i \amalg x_j)$$

The function $\hat{G}$ naturally extends to a functor ~~of~~ on $A_{\amalg}$ (and, therefore, on its full subcategory $\bar{A}$).

The verification is straightforward.

Now it is possible to start the demonstration of the ~~propositions.~~ statements.

1) If G is a sheaf of sets such that $J_B^R J_{B\#} G$ is also a sheaf, then the canonical arrow $\gamma_G : J_B^R J_{B\#} G \to G$ is an isomorphism.

Indeed, since $\varepsilon_{J_{B\#}G}$ is an isomorphism (see (i)), $J_{B\#}(J_B^R J_{B\#} G) \simeq J_{B\#} G$. On the other hand, since (3) is exact for an arbitrary G, then $J_{B\#} G' \simeq J_{B\#} G''$ for some sheaves G', G'' on A implies that $G' \simeq G''$.

Therefore, if $J_B^R F$ is a sheaf for each sheaf F, then

(a) the functor $\underline{J}_B^R \overset{def}{=} J_B^R \Big|_{F(\underline{B}, \mathcal{C})}^{F(\underline{A}, \mathcal{C})}$ is right-adjoint to the direct image functor $\underline{J}_{B*}$ with the adjointing morphisms $\varepsilon^B = \{\varepsilon_{G'}\}$, $\gamma^B = \{\gamma_G\}$;

(b) the functor $\underline{J}_{B*}$ is an equivalence of categories since both ε^B and γ^B are isomorphisms.

2) Suppose that the condition $(\dagger)$ is satisfied and show that for any $G \in Ob\, F(\underline{A}, \mathcal{E}ns)$ the presheaf $J_B^R J_{B\#} G$ is a sheaf.

Let $\{x \xrightarrow{\xi_i} \mathscr{b}^i \mid i \in J\}$ be an arbitrary finite subset of $Ob\, x \backslash B$, $\bar{x} = \{x \to v_j\}_{j \in I}$, $\bar{\mathscr{b}}^i = \{\mathscr{b}^i \to \mathscr{b}^i_k\}_{k \in K}$ and $\bar{\xi}_i = (\xi_i, \partial_i, \ldots)$ are the cocoverings and the morphisms of the cocoverings from $(\dagger)$. To them the diagram

$$\begin{tikzcd}
\hat{G}\bar{x}' \arrow[rr] \arrow[dr] \arrow[dd] & & Gb^i \arrow[dd, "\wr"] \\
& Gx \arrow[ul, dashed] \arrow[ur, dashed] \arrow[dl, "\sim"'] & \\
G\bar{x} \arrow[rr] & & \hat{G}\bar{b}^i
\end{tikzcd} \qquad (5)$$

corresponds, in which $\bar{x}' \overset{\text{def}}{=} \{x \to v_j, x \to b^i \mid i \in J, j \in I\}$ (cf. (3) from the proof of Proposition 3); ψ is the morphism from (4). The further discussion runs over the lines of the proof of Proposition 3:

- Since (5) commutes, the arrow $Gx \to \widehat{G\bar{x}'}$ is an isomorphism.

- This and diagram (3) implies that the "projections" $J^R_B J_{B\#} G x \to G b^i$ factor through the coretraction of the canonical morphism $\gamma_G(x)$. Since the set of arrows $\{x \xrightarrow{\xi^i} b^i\}_{i \in J}$ here is arbitrary, $\gamma_G(x)$ must be an isomorphism.

Thus, the functor $\underline{J}_{B*}$ takes values in the subcategory $F(\underline{B}, \underline{A}, C)$. The fact that the corestriction $\underline{J}_{B*}|^{F(\underline{B}, \underline{A}, C)}$ is an equivalence of categories with quasiinverse functor $J^R_B \Big|^{F(\underline{A}, C)}_{F(\underline{B}, \underline{A}, C)}$, is actually proved in 1).

3) Now let the condition of the third heading of the proposition hold, and $G \in Ob\, F(\underline{B}, \mathcal{E}ns)$. Since

$$\begin{tikzcd}
\widehat{J^R_B G\bar{x}} \arrow[rr, "\hat{G}\xi"] & & \hat{G}\bar{b} \\
& J^R_B G b \arrow[dr, "\sim"] & \\
J^R_B Gx \arrow[uu] \arrow[ur, dashed] \arrow[rr, "G\xi"'] & & Gb \arrow[uu, "\wr"']
\end{tikzcd}$$

commutes and $\xi : x \to b$ is arbitrary, then $J^R_B G x \to \widehat{J^R G \bar{x}}$ is an isomorphism. The care to complete the proof is left to the reader. $\square$

Remark. The following condition, which is quite sufficient for us, is a particular case of (†):

(✱) for any finite subset $\{x \xrightarrow{\xi_j} b_j\}_{j\in J} \in Ob_x \setminus B$ there exists a collection of arrows $\{x \xrightarrow{\eta_i} v_i\}_{i\in I} \in \overline{A}\cap Ob_x \setminus B$ such that if $\xi_j \notin \{\eta_i \mid i\in I\}$, then there exists a set of coprojections $\{b_j \to b_j \amalg_x v_i\}_{i\in I}$ that belongs to A. □

Corollary. Under the conditions of heading 3) of the proposition the functor J_{B*} realizes an equivalence of the categories $F(\underline{A}, C)$ and $F(\underline{B}, C)$, the functor $J^R_B \big|^{F(\underline{A}, C)}_{F(\underline{B}, C)}$ being quasiinverse.

If $\underline{A}$ is a cositus then the conditions of heading 3) hold for any B and we get a known (for siti and "nice" categories C) statement: $\underline{J}_{B*}$ is equivalence of categories.

Examples. 1) Let $\mathcal{T}$ be the category of the radical filters of the left ideals of a ring R satisfying the conditions of Example 2.2; $\underline{\mathcal{T}} = (\mathcal{T}, \overline{\mathcal{T}})$ is the associated precositus. Clearly, the condition (✱) (see Remark above) is satisfied for an arbitrary B, since adding to $\{\mathcal{F} \hookrightarrow \mathcal{F}_i \mid i\in I\} \in \overline{\mathcal{T}}$ an arbitrary family of an arrows, with $\mathcal{F}$ as the origin, does not lead out of $\overline{\mathcal{T}}$. As for the condition of heading 3) of proposition 6, a basis satisfies it if and only if the precositus $\underline{B}$, associated with it, is a cositus.

2) Let R be a commutative ring with unit; $\mathcal{B}$ the standard quasifinite basis of the cositus Spec R (see Examples 4.2 and 5.2). According to the corollary of Proposition 6, for any category C with inverse limits, the functor $J_{\mathcal{B}}$ "of the restrictions onto the basis $\mathcal{B}$ " is an equivalence of $F(\mathrm{Spec}\, R, C)$ and $F(\underline{\mathcal{B}}, C)$ with the quasiinverse

functor, which assigns to every sheaf $\mathcal{G}$ its right Can extension $J_{\mathcal{B}}^{R}\mathcal{G} : W \mapsto \varprojlim\{\mathcal{G}(V(t)) \mid W \subset V(t),\ t \in R\}$.

In particular, to each R - module M a "structural" sheaf $\mathcal{O}^{a}_{\mathcal{M}}$ corresponds, which is the unique extension of the sheaf $\widetilde{\mathcal{M}}_{\mathcal{B}}$ onto $\underline{\mathcal{S}pec R}$ (see example 5.2). It sends a closed set W into the module

$$\mathcal{O}^{a}_{\mathcal{M}}(W) = \varprojlim\{(t)^{-1}\mathcal{M} \mid W \subset V(t),\ t \in R\}. \quad \square$$

§ 4. Affine semischemes

In this section we fix an associative ring R.

1. Main theorem. Let $\{\mathcal{F}_i \mid i \in J\}$ be a family of topologizing filters; $\mathcal{F} = \bigcap\{\mathcal{F}_i \mid i \in J\}$. To the diagram $\{\mathcal{F}_i \subset \mathcal{F}_i \circ \mathcal{F}_j \supset \mathcal{F}_j \mid (i,j) \in J \times J\}$ the diagram $\{G_{\mathcal{F}_i} \to G_{\mathcal{F}_i \circ \mathcal{F}_j} \leftarrow G_{\mathcal{F}_j}\}$ and cones

$$\begin{tikzcd} & G_{\mathcal{F}_i} M \arrow[dr] & \\ G_{\mathcal{F}} M \arrow[ur] \arrow[rr] \arrow[dr] & & G_{\mathcal{F}_i \circ \mathcal{F}_j} M \\ & G_{\mathcal{F}_j} M \arrow[ur] & \end{tikzcd} , \qquad (1)$$

where $M \in \mathrm{Ob}\, R\text{-mod}$, correspond. It seems to me that the main step in the elucidation of the other side of the phenomenon we are interested in --globalization -- is the following statement:

Theorem. Let $\{\mathcal{F}_i \mid i \in J\}$ be a family of radical filters, M an arbitrary R-module.

1) The canonical R-module morphism $G_{\mathcal{F}} M \to \prod_{i \in J} G_{\mathcal{F}_i} M$ is a monomorphism.

2) If J is finite, then the cone (1) is terminal or, equivalently, the diagram

$$G_{\mathcal{F}} M \longrightarrow \prod_{i \in J} G_{\mathcal{F}_i} M \rightrightarrows \prod_{i \in J \ni j} G_{\mathcal{F}_i \circ \mathcal{F}_j} M,$$

corresponding to the cone (1), is exact.

Proof. 1) Let $x \in \bigcap\{\mathrm{Ker}\, j_{\mathcal{F}_i, M} \mid i \in J\}$. This means that for any $i \in J$ there exists $m_i \in \mathcal{F}_i$ such that $m_i x = 0$; i.e. $\mathrm{Ann}(x) \in \bigcap_{i \in J} \mathcal{F}_i = \mathcal{F}$. But $G_{\mathcal{F}} M$ is $\mathcal{F}$-torsion-free and therefore $x = 0$.

2) Let $u_i \in G_{\mathcal{F}_i} M$, $i \in J$, be elements such that $\tau_{ij}(u_i) = \tau_{ji}(u_j)$ for any $(i,j) \in J \times J$; here τ_{ij} are the natural morphisms $G_{\mathcal{F}_i} \longrightarrow G_{\mathcal{F}_j \circ \mathcal{F}_i} \simeq G_{\mathcal{F}_i} \circ G_{\mathcal{F}_j}$

Fix $i \in J$. Let $m_i \in \mathcal{F}_i$ be an ideal such that there exists a commuting diagram

$$\begin{tikzcd}
m_i \arrow[rr, "m_i \cdot u_i"] \arrow[dr, "u^i_{m_i}"'] & & G_{\mathcal{F}_i} M \\
& \mathcal{F}_i^1 M \arrow[ur, tail] &
\end{tikzcd} \qquad (2)$$

for a uniquely determined R-module morphism $u^i_{m_i}$.
For any $x \in \mathcal{P}(m_i)$ there exists an ideal $\nu_x \in \mathcal{F}_i$ such that the multiplication $u^i_{m_i}(x)$ by ν_x factors through $\phi^{\mathcal{F}_i}_M : M \to \mathcal{F}_i^1 M$; i.e. for any $\lambda \in \mathcal{P}(\nu_x)$ there exists a Z-module morphism $u^i_{\lambda,x} : \lambda \otimes x \longrightarrow M$ such that the diagram

$$\begin{tikzcd}
\lambda \otimes x \arrow[rr, "\lambda \cdot x"] \arrow[dr, two heads] \arrow[dd] & & m_i \arrow[r, "m_i \cdot u_i"] \arrow[dd, "u^i_{m_i}"] & G_{\mathcal{F}_i} M \\
& \lambda x \arrow[ur, hook] & & \\
M \arrow[rr] & & \mathcal{F}_i^1 M \arrow[uur, tail] &
\end{tikzcd} \qquad (3)$$

commutes. Here $\lambda \cdot x$ is the morphism of multiplying λ by x. The identity $\tau_{ij}(u_i) = \tau_{ji}(u_j)$ yields the existence of $m_{ij} \in \mathcal{F}_i$ such that $m_{ij} \cdot ((\lambda \cdot x) \cdot u_j) = m_{ij} \cdot (j_{\mathcal{F}_j, M} \circ u^i_{\lambda,x})$. Since by hypothesis J is finite, then $\cap\{m_{i\ell} \mid \ell \in J\} = \widetilde{m}_i \in \mathcal{F}_i$ and we can write

$$\widetilde{m}_i \cdot (\lambda \cdot x \cdot u_\ell) = j_{\mathcal{F}_\ell, M}(\widetilde{m}_i \cdot u^i_{\lambda,x}), \ \ell \in J, \quad (4).$$

Further, we stick to the scenario of the proof of Proposition 2.5. Denote by $\mathcal{H}^{\bar{u}}$, where $\bar{u} = (u_i)_{i \in J}$, the full subcategory of $I_\ell R$ formed by the ideals ν such that the morphism $\nu \cdot \bar{u}$ of multiplication by ν factors through $M \longrightarrow \prod_{\ell \in J} G_{\mathcal{F}_\ell} M$.

a) The category $\mathcal{H}^{\bar{u}}$ contains all the ideals of the form $\widetilde{m}_i \lambda x$.

NP

It suffices to look at the commuting

diagram

$$\begin{tikzcd}
\tilde{m}_i \otimes \lambda \otimes x \arrow[r, two heads, "\gamma"] \arrow[dr, "\tilde{m}_i \cdot u^i_{\lambda, x}"'] & \tilde{m}_i \lambda x \arrow[r] \arrow[d, dashed] & \prod\limits_{e \in J} G_{\mathcal{F}_e} M \\
& M \arrow[ur, tail, "j_{\{\mathcal{F}_e\}} \overset{def}{=} (j_{\mathcal{F}_e, M})_{e \in J}"'] &
\end{tikzcd} \qquad (5)$$

The dotted line here appears thanks to epimorphicy of γ and monomorphicy of $j_{\{\mathcal{F}_e\}} = (j_{\mathcal{F}_e, M})_{e \in J}$.

b) If $\{n_1, n_2\} \subset \mathcal{H}^{\bar{u}}$, then $n_1 + n_2 \in \mathcal{H}^{\bar{u}}$ as is clear from the commuting diagram

$$\begin{tikzcd}
M \arrow[rr, tail] & & \prod\limits_{e \in J} G_{\mathcal{F}_e} M \\
n_1 \amalg n_2 \arrow[u, "q_{n_1} \times q_{n_2}"] \arrow[rr, two heads] & & n_1 + n_2 \arrow[u] \arrow[ull, dashed]
\end{tikzcd}$$

where the existence of the dotted arrow is justified by the same reasons.

c) Finally, together with every ascending family of ideals $\{n_\alpha \mid \alpha \in \mathfrak{A}\}$ the category $\mathcal{H}^{\bar{u}}$ contains the union $\cup\{n_\alpha \mid \alpha \in \mathfrak{A}\}$ of its elements.

As in the proof 2.5, the last two statements allow one to deduce the existence of a final object $n^{\bar{u}}$ in the category $\mathcal{H}^{\bar{u}}$. Thanks to (a) $\tilde{m}_i \lambda x \subset n^{\bar{u}}$. Since $x \in \mathcal{P}(m_i)$ and $\lambda \in \mathcal{P}(\nu_x)$ are arbitrary and $\{m_i, \tilde{m}_i, \nu_x\}_{x \in \mathcal{P}(m_i)}$ belongs to $\mathcal{F}_i$, then $n^{\bar{u}}$ belongs to $\mathcal{F}_i \circ \mathcal{F}_i \circ \mathcal{F}_i$; i.e. $n^{\bar{u}} \in \mathcal{F}_i$ since $\mathcal{F}_i$ is a radical filter. Since $i \in J$ is arbitrary, then $n^{\bar{u}} \in \bigcap\limits_{i \in J} \mathcal{F}_i = \mathcal{F}$ as required. $\square$

2. Auxiliary facts. We need several prerequisites to pass from Theorem 1 to "geometric" corollaries.

Proposition. Let $\mathcal{F}$, $\mathcal{G}$ be radical sets. M an R-module. The following conditions are equivalent:

a) the canonical morphism $(\mathcal{F}\circ\mathcal{G})^1M \to (\mathcal{F}\vee\mathcal{G})^1M$ is isomorphism.

b) $(\mathcal{F}\circ\mathcal{G})^1M \to (\mathcal{F}\circ\mathcal{G}\circ\mathcal{F})^1M$ is isomorphism.

Proof. Clearly, a) $\Longrightarrow$ b).

b) $\Longrightarrow$ a). Let Ω be a full subcategory of the category $\mathcal{T}_e R$ of topologizing sets formed by all the filters $\mathcal{F}'$ such that $\mathcal{F}' \subset \mathcal{F}\vee\mathcal{G}$ and the natural morphism $(\mathcal{F}\circ\mathcal{G})^1M \to (\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}')^1M$ is an isomorphism. Notice that $[\{\mathcal{F}',\mathcal{F}''\}\subset\Omega] \Rightarrow [\mathcal{F}'\circ\mathcal{F}''\in\Omega]$.

In fact, if $\{\mathcal{F}',\mathcal{F}''\}\subset\Omega$, then

$$(\mathcal{F}\circ\mathcal{G})^1M \simeq (\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}')^1M \simeq \mathcal{F}'^1((\mathcal{F}\circ\mathcal{G})^1M) \simeq$$

$$\simeq \mathcal{F}'^1((\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}'')^1M) \simeq (\mathcal{F}\circ\mathcal{G}\circ(\mathcal{F}''\circ\mathcal{F}'))^1M.$$

Clearly, Ω possesses a final object $\tilde{\mathcal{F}} = \bigcup\{\mathcal{F}' \mid \mathcal{F}'\in\Omega\}$. Since $\tilde{\mathcal{F}}\circ\tilde{\mathcal{F}}\in\Omega$, then $\tilde{\mathcal{F}}$ is radical. It is not difficult to see that $\mathcal{F}\circ\mathcal{G}\in\Omega$, since, thanks to b), all the arrows in the chain

$$(\mathcal{F}\circ\mathcal{G})^1M = (\mathcal{F}\circ\mathcal{G}\circ\mathcal{G})^1M \to \mathcal{G}^1((\mathcal{F}\circ\mathcal{G})^1M) \to$$

$$\to \mathcal{G}^1((\mathcal{F}\circ\mathcal{G}\circ\mathcal{F})^1M) \to (\mathcal{F}\circ\mathcal{G}\circ(\mathcal{F}\circ\mathcal{G}))^1M$$

are isomorphisms. Therefore $\mathcal{F}\circ\mathcal{G}\subset\tilde{\mathcal{F}}$ and hence $\mathcal{G}, \mathcal{F}\subset\tilde{\mathcal{F}}$ or, equivalently, $\tilde{\mathcal{F}} = \mathcal{F}\vee\mathcal{G}$. Clearly, $\mathcal{F}\circ\mathcal{G}\circ(\mathcal{F}\vee\mathcal{G}) = \mathcal{F}\vee\mathcal{G}$.

Corollary. Let $\mathcal{F}$, $\mathcal{G}$ be radical sets, M an R-module such that $(\mathcal{F}\circ\mathcal{G})^1M$ and $(\mathcal{G}\circ\mathcal{F})^1M$ are isomorphic. Then $(\mathcal{F}\circ\mathcal{G})^1M \to (\mathcal{F}\vee\mathcal{G})^1M$ is an isomorphism.

P ¶ Proof. In fact,

$$(\mathcal{F}\circ\mathcal{G}\circ\mathcal{F})^1M \simeq \mathcal{F}^1((\mathcal{F}\circ\mathcal{G})^1M) \simeq \mathcal{F}^1((\mathcal{G}\circ\mathcal{F})^1M) \simeq$$

$$\simeq (\mathcal{G}\circ\mathcal{F}\circ\mathcal{F})^1M \simeq (\mathcal{G}\circ\mathcal{F})^1M \simeq (\mathcal{F}\circ\mathcal{G})^1M. \square$$

3. Affine semischemes. A pair $(R, \mathcal{T})$, where R is an associative ring, $\mathcal{T}$ a full subcategory of the category $\mathcal{T}\mathrm{I}_\ell R$ of radical sets with finite coproducts and with the property $[\{\mathcal{F},\mathcal{G}\}\subset\mathcal{T}] \Rightarrow [\mathcal{F}\cap\mathcal{G}\in\mathcal{T}]$, will be called a (left) affine $\amalg$-semischeme. If is a V-category, then we will skip the sign $\amalg$ of coproducts.

Let $(R, \mathcal{T})$ be a $\amalg$-semischeme. For every R-module M the map $\mathcal{F} \longmapsto G_{\mathcal{T}\mathcal{F}} M$ determines the presheaf $M_{\mathcal{T}}$ on the precositus $\underline{\mathcal{T}} = (\mathcal{T}, \tilde{\mathcal{T}})$ (see 3.2.2) with values in the category R-mod.

Proposition. Let $(R, \mathcal{T})$ be an affine $\amalg$-semischeme, M an R-module. Consider the following conditions:

a) $M_{\mathcal{T}}$ is a ω-sheaf on $\underline{\mathcal{T}}$;

b) for any $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ the natural morphism $\mathcal{F}^1 \mathcal{G}^1 M \longrightarrow (\mathcal{F}\amalg\mathcal{G})^1 M$ is monomorphism,

c) for any $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ the modules $\mathcal{F}^1 \mathcal{G}^1 M$ and $\mathcal{G}^1 \mathcal{F}^1 M$ are isomorphic.

Then the following implications hold: a) $\Longleftrightarrow$ b) $\Longrightarrow$ c). If $(R, \mathcal{T})$ is a semischeme, then c) $\Longrightarrow$ b).

Proof. b) $\Longrightarrow$ a). Let $\{\mathcal{F} \hookrightarrow \mathcal{F}_i \mid i\in J\}$ be a finite covering in $\underline{\mathcal{T}}$. Consider the diagram

$$\begin{tikzcd}
G_{\mathcal{T}\mathcal{F}} M \arrow[r] & \prod\limits_{i\in J} G_{\mathcal{T}\mathcal{F}_i} M \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, dashed] & & \prod\limits_{i\in J\ni e} G_{\mathcal{T}\mathcal{F}_i\amalg\mathcal{F}_e} M \\
& & \prod\limits_{i\in J\ni e} G_{\mathcal{T}\mathcal{F}_i\circ\mathcal{F}_e} M \arrow[ur, dashed] &
\end{tikzcd}$$

Clearly, the monomorphicy of all the arrows $G_{\mathcal{T}\mathcal{F}_i\circ\mathcal{F}_e} M \longrightarrow G_{\mathcal{T}\mathcal{F}_i\amalg\mathcal{F}_e}$ implies that of their product $\prod\limits_{i\in J\ni e} G_{\mathcal{T}\mathcal{F}_i\circ\mathcal{F}_e} M \longrightarrow \prod\limits_{i\in J\ni e} G_{\mathcal{T}\mathcal{F}_i\amalg\mathcal{F}_e} M$. Now it is clear that the exactness of the diagram $X_0 \longrightarrow X_1 \rightrightarrows X_2$ and monomorphicy of $X_2 \longrightarrow Y$ yields the exactness of $X_0 \to X_1 \rightrightarrows Y$.

It remains to make use of Theorem 1.

a) $\Longrightarrow$ b). By hypothesis $[\{\mathcal{F},\mathcal{G}\}\subset\mathcal{T}]\Rightarrow[\mathcal{F}\cap\mathcal{G}\in\mathcal{T}]$. Therefore the exactness of the diagram

$$G_{\mathcal{F}\cap\mathcal{G}}M \to G_{\mathcal{F}}M \prod G_{\mathcal{G}}M \rightrightarrows G_{\mathcal{F}\amalg\mathcal{G}}M \prod G_{\mathcal{F}\amalg\mathcal{G}}M$$

implies the monomorphicy of the arrow $\eta : (\mathcal{F}\circ\mathcal{G})^1M \to$ $\to (\mathcal{F}\amalg\mathcal{G})^1M$, since $G_{\mathcal{F}\circ\mathcal{G}}M \to G_{\mathcal{F}\amalg\mathcal{G}}M$ is monomorphism. Since $\mathcal{F}\circ\mathcal{G}\subset\mathcal{F}\amalg\mathcal{G}$, then $(\mathcal{F}\circ\mathcal{G})^1M\to(\mathcal{F}\amalg\mathcal{G})^1M$ is epimorphism, therefore the monomorphicy of η is equivalent to its isomorphicy.

We have also proved the implication b) $\Longrightarrow$ c).

If $(R,\mathcal{T})$ is a semischeme (i.e. $\amalg = \vee$), then it follows from corollary 2 that c) $\Longrightarrow$ b).

Corollary 1. *Let an R-module M be $\mathcal{F}$ -torsion-free (i.e. $\mathcal{F}M=0$) for any $\mathcal{F}\in\mathcal{T}$. Then $M_{\mathcal{T}}$ is a ω -sheaf. In particular, if R satisfies the condition $[x\in R, m\in\mathcal{F}\in\mathcal{T}$ and $m\cdot x=0]\Longrightarrow[x=0]$ then $R_{\mathcal{T}}$ is a ω -sheaf.*

Corollary 2. *Let $(R,\mathcal{T})$ be a semischeme. Then for any irreducible R-module M the presheaf $M_{\mathcal{T}}$ is a ω-sheaf.*

In fact, for any topologizing set $\mathcal{F}$ either $\mathcal{F}M=0$ or $\mathcal{F}M=M$. This immediately implies the isomorphicy of $(\mathcal{G}\circ\mathcal{F})^1M$ and $(\mathcal{F}\circ\mathcal{G})^1M$.

4. Topologizing sets and ideals. The main aim of this and the subsequent section is to make the conditions of Proposition 3 a trifle more constructive.

Let $\mathcal{G}$ be an arbitrary topologizing filter, n a left ideal of R. Denote $\{\lambda\in R\,|\,(n:\lambda)\in\mathcal{G}\}$ by $n_{\mathcal{G}}$.

Proposition. Let $\mathcal{G}$, $\mathcal{F}$ be topologizing filters, n and m left ideals of R.

1) $n_{\mathcal{G}}$ is a left ideal of R containing n.

2) $n_{\mathcal{G}}$ is a final object of the full subcategory $\Xi_{\mathcal{G}}$ of I_1R formed by the ideals m such that $n \in \mathcal{G} \circ \{m\}$.

3) $(n_{\mathcal{G}} : x) = (n : x)_{\mathcal{G}}$ for any $n \in I_1R$ and $x \in \mathcal{P}(R)$.

4) The map $(n, \mathcal{G}) \longmapsto n_{\mathcal{G}}$ depends on n and $\mathcal{G}$ functorially: if $n \subset m$ and $\mathcal{G} \subset \mathcal{F}$, then $n_{\mathcal{G}} \subset m_{\mathcal{F}}$. Moreover, if $n \longrightarrow m$, then $n_{\mathcal{G}} \longrightarrow m_{\mathcal{G}}$.

5) $(n \cap m)_{\mathcal{G}} = n_{\mathcal{G}} \cap m_{\mathcal{G}}$

and for any family $\{\mathcal{F}_i \mid i \in I\}$ of topologizing filters the equality holds:

$$n_{\cap\{\mathcal{F}_i \mid i \in I\}} = \cap\{n_{\mathcal{F}_i} \mid i \in I\}$$

6) $n_{\mathcal{F} \circ \mathcal{G}} = (n_{\mathcal{G}})_{\mathcal{F}}$. In particular, if is a radical filter.

Proof. 1) Since $\mathcal{G}$ is a cofilter, then

$$[\{\lambda_1, \lambda_2\} \subset n_{\mathcal{G}}] \Rightarrow [\mathcal{G} \ni (n:\lambda_1) \cap (n:\lambda_2) = (n:\lambda_1+\lambda_2)].$$

Thanks to the uniformity of $\mathcal{G}$

$[\lambda \in n_{\mathcal{G}}$, i.e. $(n:\lambda) \in \mathcal{G}] \Rightarrow [(n:a\lambda) = ((n:\lambda):a) \in \mathcal{G}$ for any $a \in R$, i.e. $a\lambda \in n_{\mathcal{G}}]$.

2) Statement 2) follows immediatly from the definition of $n_{\mathcal{G}}$; a strightforward verification of 5) and 6) is left to the reader.

3) Let $x \in \mathcal{P}(R)$, $\lambda \in (n_{\mathcal{G}} : x)$; i.e. $((n:z):\lambda) = (n:\lambda z) \in \mathcal{G}$ for any $z \in x$. Therefore $\lambda \in \bigcap_{z \in x} (n:z)_{\mathcal{G}}$ and $\bigcap_{z \in x} (n:z)_{\mathcal{G}} = \left(\bigcap_{z \in x} (n:z)\right)_{\mathcal{G}} = (n:x)_{\mathcal{G}}$ thanks to 5). Conversely, if $\lambda \in (n:x)_{\mathcal{G}}$, then $(n:\lambda x) = ((n:x):\lambda) \in \mathcal{G}$; i.e. $\lambda x \subset n_{\mathcal{G}}$. $\square$

4) It is easy to verify that $n_{\mathcal{G}} \subset m_{\mathcal{G}}$ if $n \subset m$ and $\mathcal{G} \subset \mathcal{F}$. Let now $n \longrightarrow m$; i.e. $n \subset m$ or $(n:x) \subset m$ for some $x \in \mathcal{P}(R)$. In the first case $n_{\mathcal{G}} \subset m_{\mathcal{G}}$ as has been just mentioned;

in the second one

$$(n_{e_j} : x) = (n : x)_{e_j} \subset m_{e_j}$$

according to the statement 3). □

5. <u>$\mathcal{T}$-symmetric and $\mathcal{T}$-admissible ideals</u>. Let $\mathcal{T}$

be a set of topologizing filters. An ideal $\mathfrak{n}$ is $\mathfrak{T}$-symmetric if $[\mathfrak{n}_{\mathcal{F}} \in \mathcal{G}] \Rightarrow [\mathfrak{n}_{\mathcal{G}} \in \mathcal{F}]$ for any $\{\mathcal{F}, \mathcal{G}\} \subset \mathfrak{T}$. Denote $\mathrm{Sym}\,\mathfrak{T}$ the set of $\mathfrak{T}$-symmetric ideals. Clearly, $\mathrm{Sym}\,\mathfrak{T}$ consists of all $\mathfrak{n} \in I_{\ell}R$ such that $[\mathfrak{n} \in \mathcal{F} \circ \mathcal{G}] \Rightarrow$ $\Rightarrow [\mathfrak{n} \in \mathcal{G} \circ \mathcal{F}]$ for any $\{\mathcal{F}, \mathcal{G}\} \subset \mathfrak{T}$ (see the end of the preceding subsection).

Now let $(R, \mathfrak{T})$ be a $\amalg$-semischeme. A left ideal $\mathfrak{n}$ will be called $\mathfrak{T}$-*admissible* if $[\mathfrak{n} \in \mathcal{F} \amalg \mathcal{G}] \Rightarrow$ $\Rightarrow [\mathfrak{n}_{\mathcal{F}} \in \mathcal{G}]$ for any $\{\mathcal{F}, \mathcal{G}\} \subset \mathfrak{T}$. It is clear from the implications $[\mathfrak{n}_{\mathcal{F}} \in \mathcal{G}] \Leftrightarrow [\mathfrak{n} \in \mathcal{F} \circ \mathcal{G}] \Rightarrow [\mathfrak{n} \in \mathcal{F} \amalg \mathcal{G}]$ that any $\mathfrak{T}$-admissible ideal is $\mathfrak{T}$-symmetric. Now we can reformulate Proposition 3 as follows:

Proposition. *Let $(R, \mathfrak{T})$ be an affine $\amalg$-semischeme and M an R-module.*

1) The following conditions are equivalent:

a) $M_{\mathfrak{T}}$ is a ω-sheaf on $\mathfrak{T}$;

b) the annihilator of any element of M is $\mathfrak{T}$-admissible.

2) If $(R, \mathfrak{T})$ is a semischeme, then $M_{\mathfrak{T}}$ is a ω-sheaf if and only if the annihilator of any element of M is $\mathfrak{T}$-symmetric.

6. Examples. 1) Let $(R, \mathfrak{T})$ be a semischeme and $\mathfrak{T} = \{F_{\mathcal{S}} \mid \mathcal{S} \in \mathfrak{m} \subset 2^{\mathfrak{P}(R)}\}$. The $\mathfrak{T}$-symmetricity of $\mathfrak{n}$ reads as follows: $[\mathfrak{n}_{\mathcal{F}_{\mathcal{S}}} \in F_{\mathcal{S}'}] \Rightarrow [\mathfrak{n}_{\mathcal{F}_{\mathcal{S}'}} \in F_{\mathcal{S}}]$ for any $\{\mathcal{S}, \mathcal{S}'\} \subset \mathfrak{m}$. By definition of $F_{\mathcal{S}}$ and $F_{\mathcal{S}'}$, the fact that $\mathfrak{n}_{F_{\mathcal{S}}} \in F_{\mathcal{S}'}$ means that there exists $s' \in \mathcal{S}'$ and for any $x \in \mathfrak{P}(R)$ there exists $s'_x \in S'$ such that $(\mathfrak{n} : s')$, $(\mathfrak{n} : s'_x x) \in F_{S}$;

i.e. there exist $s, s_x \in \mathcal{S} \ni s_x$ and for any $y \in \mathcal{P}(R)$ there exist $t_y, s_y \in \mathcal{S}'$ such that $\{ss', s_x s'_x x, s_y y s', t_y y s'_x x\} \subset \mathcal{P}(\mathfrak{n})$. Therefore the implication $[\mathfrak{n}_{F_{\mathcal{S}}} \in F_{\mathcal{S}'}] \Rightarrow [\mathfrak{n}_{F_{\mathcal{S}'}} \in F_{\mathcal{S}}]$ is equivalent to the following:

Let for any $\{x, y\} \subset \mathcal{P}(R)$ there exist $\{s_x, t_y\} \subset \mathcal{S}$ and $s'_x \in \mathcal{S}'$ such that $s_x s'_x x, \; t_y y s'_x x \subset \mathfrak{n}$ and $s'_y y s', \; ' \subset \mathfrak{n}$ for some $\{s, s_y\} \subset \mathcal{S}, s' \in \mathcal{S}'$.

Then for any $\{x, y\} \subset \mathcal{P}(R)$ there exist $\{\tilde{s}'_x, t'_y\} \subset \mathcal{S}'$ and $\tilde{s}_x \in \mathcal{S}$ such that $\tilde{s}'_x \tilde{s}_x x, \; t'_y y \tilde{s}_x x \subset \mathfrak{n}$ and $\tilde{s}'\tilde{s}, \; s'_y y \tilde{s} \subset \mathfrak{n}$ for some $\tilde{s} \in \mathcal{S}, \{\tilde{s}', s'_y\} \subset \mathcal{S}'$.

$\mathcal{T}$-symmetricity of $\mathfrak{n}$ means the fulfilment of this condition for all $\{\mathcal{S}, \mathcal{S}'\} \subset \mathfrak{m}$.

2) Now let $(R, \mathcal{T})$ be a semischeme and $\mathcal{T}$ consist of radical filters of finite type (see 1.4.6). Such semischemes will be called finite type semischemes.

By Proposition 1.4.6 $\mathcal{T}$ consists of radical filters $F_{\mathcal{S}}, \mathcal{S} \in \mathfrak{m}$, where each $\mathcal{S}$ is a multiplicative subset of $\mathcal{P}(R)$ such that

For any $x \in \mathcal{P}(R)$ and $s \in \mathcal{S}$ there exists $t \in \mathcal{S}$ such that $tx \subset (R, s) = Rs + s$. (#)

If S satisfies (#), then $[\mathfrak{m} \in F_S] \Leftrightarrow [\mathcal{P}(\mathfrak{m}) \cap S \neq \emptyset]$ which easily implies that $[\mathfrak{n}_{F_S} \in F_{S'}] \Longleftrightarrow [ss' \subset \mathfrak{n}$ for some $s \in \mathcal{S}, s' \in \mathcal{S}']$ for any $\{\mathcal{S}, \mathcal{S}'\} \subset \mathfrak{m}$. Therefore a left ideal $\mathfrak{n}$ is $\mathcal{T}$-symmetric if $[ss' \subset \mathfrak{n}$ for some $s \in \mathcal{S}, s' \in \mathcal{S}'] \Longrightarrow [t't \subset \mathfrak{n}$ for some $t \in \mathcal{S}, t' \in \mathcal{S}']$ for any $\{\mathcal{S}, \mathcal{S}'\} \subset \mathfrak{m}$. $\square$

Remark. If under the conditions of example 1) all the left ideals of R are weakly regular (e.g.

R contains a right unit), then the condition for $\mathcal{T}$ -symmetricity of n takes a more compact form:

If for any $\{x,y\}\subset \mathcal{P}(R)$ there exist $(s,s')\in S\times S'$ such that $sys'x\subset n$, then for any $\{x,y\}\subset\mathcal{P}(R)$ there exists $(t,t')\in S\times S'$ such that $t'ytx\subset n$.

7. Extensions of presheaves and fully quasicompact semischemes. Let $(R,\mathcal{T})$ be an affine $\coprod$ -semischeme, ${}^{\cap}\mathcal{T}$ a full subcategory of $\mathcal{T}I_{\ell}R$ formed by the various intersections of the filters from $\mathcal{T}$. It is not difficult to verify that ${}^{\cap}\mathcal{T}$ is a category with finite coproducts, and the embedding $i=i_{\mathcal{T}}:\mathcal{T}\hookrightarrow{}^{\cap}\mathcal{T}$ commutes with the coproducts and, therefore, determines a precositi morphism ${}^{\cap}\underline{\mathcal{T}}\longrightarrow\underline{\mathcal{T}}$. To each presheaf $F:\mathcal{T}\to\mathcal{C}$ with values in $\mathcal{C}$ one can assign the presheaf $\dot{i}^R_{\#}:{}^{\cap}\mathcal{T}\to$ $\longrightarrow\mathcal{C}$ representing the functor $\mathcal{C}^{\mathcal{T}}(i_{\#}-,F)$ ("the right inverse image" of the presheaf F). This presheaf is described by the formulas

$$\mathcal{G}\longmapsto \dot{i}^R_{\#}F(\mathcal{G})=\varprojlim\left(F(\mathcal{G}')\mid \mathcal{G}\subset\mathcal{G}'\in\mathcal{T}\right)$$

for any filter $\mathcal{G}$ of ${}^{\cap}\mathcal{T}$ that show that the canonical arrow $\quad i_{\#}\dot{i}^R_{\#}F\longrightarrow F\quad$ is an isomorphism for any presheaf F.

Proposition. The functor $i_{*}:\mathbb{F}({}^{\cap}\underline{\mathcal{T}},\mathcal{C})\longrightarrow\mathbb{F}(\underline{\mathcal{T}},\mathcal{C})$ is an equivalence of categories. Its quasiinverse is the functor

$$\dot{i}^R_{*}=\dot{i}^R_{\#}\Big|_{\mathbb{F}(\underline{\mathcal{T}},\mathcal{C})}^{\mathbb{F}({}^{\cap}\underline{\mathcal{T}},\mathcal{C})}:\ F\longmapsto \dot{i}^R_{\#}F.$$

It is left for the reader to either prove this statement or to regard it as a corollary of the general facts presented in §3 .

An affine $\amalg$-semischeme is called quasifinite (fully quasicompact) if the precositus $\underline{\mathfrak{T}} = (\mathfrak{T}, \widetilde{\mathfrak{T}})$ is quasifinite (fully quasicompact).

Clearly, quasifiniteness of the $\amalg$-semischeme $(R, \mathfrak{T})$ implies the full quasicompactness of the semischeme $(R, {}^{\cap}\mathfrak{T})$.

Proposition 5 and the above proposition imply

Corollary. If $(R, \mathfrak{T})$ is a quasifinite $\amalg$-semischeme, then for an arbitrary R-module M the presheaf $\widetilde{M}_{\mathfrak{T}} \overset{\text{def}}{=} i_{\#}^{R} M_{\mathfrak{T}} : \mathcal{G} \mapsto \varprojlim(\mathcal{F}^{-1}M \mid \mathcal{G} \subset \mathcal{F} \in \mathfrak{T})$ - the right-extension of $M_{\mathfrak{T}}$ onto ${}^{\cap}\mathfrak{T}$ - is a sheaf if and only if the annihilator of any element of M is $\mathfrak{T}$-admissible ($\mathfrak{T}$-symmetric if $(R, \mathfrak{T})$ is a semischeme).

Example. Suppose that every filter from $\mathfrak{T}$ is closed under the arbitrary intersections of ideals (or equivalently, every filter from $\mathfrak{T}$ contains a minimal ideal). Then, as is not difficult to verify, any filter of finite type of $\mathfrak{T}$ is quasifinite. In particular, if R is a left Artinian ring, then any affine $\amalg$-semischeme $(R, \mathfrak{T})$ is quasifinite: since R is an Artinian ring, then the intersection of an arbitrary set of left ideals equals the intersection of ideals of an appropriate finite subset; since R is left Noetherian, then all the cofilters of left ideals are of finite type. □

8. Spectra of affine semischemes. Let $(R, \mathfrak{T})$ be an affine $\amalg$-semischeme.

A left ideal p of R is called $\mathfrak{T}$-prime if $[p \in \mathcal{F} \amalg \mathcal{G}] \Rightarrow [p \in \mathcal{F} \cup \mathcal{G}]$ for any $\{\mathcal{F}, \mathcal{G}\} \subset \mathfrak{T}$.

The family of $\mathfrak{T}$-prime left ideals of R is denoted by $\mathrm{Spec}_{\ell}(R, \mathfrak{T})$.

Obviously, any $\mathfrak{T}$-prime ideal p is $\mathfrak{T}$-admissible. In particular, if an R-module M is such that the annihilator of any its element is $\mathfrak{T}$-prime, then $M_{\mathfrak{T}}$ is a ω-sheaf.

On $Spec_\ell(R,\mathfrak{T})$, determine a topology taking all the subsets of the form $V_{\mathcal{F}} \overset{\text{def}}{=} \mathcal{F} \cap Spec_\ell(R,\mathfrak{T})$ for a basis of closed sets. It is clear from the definition that $\mathcal{F} \longmapsto V_{\mathcal{F}}$ is a functor commuting with coproducts and therefore determining a morphism of the cositus $\underline{Spec_\ell(R,\mathfrak{T})}$ into the precositus $\underline{\mathfrak{T}}$.

Proposition. Let $(R,\mathfrak{T})$ be a $\coprod$-semischeme. The spectra of $\coprod$-semischemes $(R,\mathfrak{T})$ and $(R,{}^\cap\mathfrak{T})$ coincide as topological spaces.

Proof. Since the embedding $\mathfrak{T} \hookrightarrow {}^\cap\mathfrak{T}$ commutes with finite coproducts, then $Spec_\ell(R,{}^\cap\mathfrak{T}) \subset Spec_\ell(R,\mathfrak{T})$. Now suppose $\{\tilde{\mathcal{F}}, \tilde{\mathcal{G}}\} \subset {}^\cap\mathfrak{T}$, i.e. $\tilde{\mathcal{F}} = \bigcap_{i\in J} \mathcal{F}_i$ and $\tilde{\mathcal{G}} = \bigcap_{k\in K} \mathcal{G}_k$ for some $\{\mathcal{F}_i, \mathcal{G}_k \mid i\in J, k\in K\} \subset \mathfrak{T}$, and $p \in Spec_\ell(R,\mathfrak{T}) \cap \tilde{\mathcal{F}} \coprod \tilde{\mathcal{G}}$. Then $p \in Spec_\ell(R,\mathfrak{T}) \cap (\mathcal{F}_i \coprod \mathcal{G}_k)$ for any $(i,k) \in J \times K$. If $p \notin \mathcal{G}_k$ for some k, then $p \in \mathcal{F}_i$ for all $i \in J$ and therefore $p \in \bigcap\{\mathcal{F}_i \mid i \in J\}$. $\square$

Let $(R,\mathfrak{T})$ be a $\coprod$-semischeme. To any closed subset $W \subset Spec_\ell(R,\mathfrak{T})$ assign the filter $\hat{\mathfrak{T}}_W$, the "radical closure" of the set $\mathfrak{T}_W \overset{\text{def}}{=} \bigcup\{\mathcal{F} \in \mathfrak{T} \mid V_{\mathcal{F}} \subset W\}$. Let $\mathfrak{T}_\diamond$ be the category consisting of all the $\hat{\mathfrak{T}}_W$, $W \in cl\, Spec_\ell(R,\mathfrak{T})$. Then $\underline{{}^\cap\mathfrak{T}_\diamond}$ is a cositus isomorphic to $\underline{Spec_\ell(R,\mathfrak{T})}$. Clearly, the spectra of $(R,\mathfrak{T})$ and $(R,{}^\cap\mathfrak{T}_\diamond)$ coincide (see the just proved proposition 8).

In general case $\mathcal{T}_\diamond$ is not a **V**-category even if $\mathcal{T}$ is a **V**-category.

9. Spectra of semischemes and the left spectrum. Concerning "richness" of $\mathrm{Spec}_\ell(R,\mathcal{T})$ for an arbitrary $\amalg$-semischeme we can only say that this spectrum contains the set $\mathcal{T}^\perp = \mathrm{I}_\ell R \setminus \bigcup\{\mathcal{F} \mid \mathcal{F}\in\mathcal{T}\}$. The going is much better when $\mathcal{T}$ is a **V**-category.

Proposition. Let $(R,\mathcal{T})$ be a semischeme. Then $\mathrm{Spec}_\ell R \subset \mathrm{Spec}_\ell(R,\mathcal{T})$. In particular, $\mathrm{Spec}_\ell(R,\mathcal{T})$ contains the set $\mathrm{Max}_\ell^{w} R \overset{\mathrm{def}}{=} \mathrm{Max}_\ell R \cap \mathrm{I}_\ell^{w} R$ of all the maximal left weakly regular ideals.

Proof. Let us show that $\mathrm{Spec}_\ell R \cap \mathcal{F} = \mathrm{Spec}_\ell R \cap \hat{\mathcal{F}}$ for an arbitrary topologizing filter $\mathcal{F}$, where $\hat{\mathcal{F}}$ is the radical closure of $\mathcal{F}$.

For this consider the family $\Omega_{\mathcal{F}}$ of topologizing filters $\mathcal{G}$ such that $\mathcal{F}\cap\mathrm{Spec}_\ell R = \mathcal{G}\cap\mathrm{Spec}_\ell R$. Clearly, $[\{\mathcal{G}_1,\mathcal{G}_2\}\subset\Omega_{\mathcal{F}}] \Rightarrow [\mathcal{G}_1\circ\mathcal{G}_2\in\Omega_{\mathcal{F}}]$, since $\mathrm{Spec}_\ell R\cap\mathcal{G}_1\circ\mathcal{G}_2 = \mathrm{Spec}_\ell R\cap(\mathcal{G}_1\cup\mathcal{G}_2)$ for any topologizing filters $\mathcal{G}_1$ and $\mathcal{G}_2$. Clearly, the union $\tilde{\mathcal{F}}$ of all the filters from $\Omega_{\mathcal{F}}$ belongs to $\Omega_{\mathcal{F}}$ together with $\tilde{\mathcal{F}}\circ\tilde{\mathcal{F}}$. Thus, $\tilde{\mathcal{F}}$ is a radical filter containing $\mathcal{F}$ and therefore $\hat{\mathcal{F}}\in\Omega_{\mathcal{F}}$.

If $\mathcal{F}$ and $\mathcal{G}$ are topologizing filters, then $\hat{\mathcal{F}}\vee\hat{\mathcal{G}} = (\widehat{\mathcal{F}\circ\mathcal{G}})$ and the just above implies

$$\mathrm{Spec}_\ell R\cap(\hat{\mathcal{F}}\vee\hat{\mathcal{G}}) = \mathrm{Spec}_\ell R\cap(\widehat{\mathcal{F}\circ\mathcal{G}}) = \mathrm{Spec}_\ell R\cap\mathcal{F}\circ\mathcal{G} =$$
$$= (\mathrm{Spec}_\ell R\cap\mathcal{F})\cup(\mathrm{Spec}_\ell R\cap\mathcal{G}) = (\mathrm{Spec}_\ell R\cap\hat{\mathcal{F}})\cup(\mathrm{Spec}_\ell R\cap\hat{\mathcal{G}}).$$

Therefore $\mathrm{Spec}_\ell R \subset \mathrm{Spec}_\ell(R,\mathcal{T}\mathrm{I}_\ell R)$ **Obviously,** $\mathrm{Spec}_\ell(R,\mathcal{T}\mathrm{I}_\ell R)\subset\mathrm{Spec}_\ell(R,\mathcal{T})$ for any semischeme $(R,\mathcal{T})$. $\square$

II. The symmetric spectrum and the prime ideals.

Let $(R, \mathcal{T})$ be a $\amalg$ - semischeme. We endow the set $\mathcal{S}pec(R,\mathcal{T}) \overset{def}{=}$ $= IR \cap \mathcal{S}pec_\ell(R,\mathcal{T})$ with the topology induced from $\mathcal{S}pec_\ell(R,\mathcal{T})$, and call both the set and the space of this Spec the symmetric spectrum of the $\amalg$ - semischeme $(R,\mathcal{T})$.

Proposition. *Let $(R, \mathcal{T})$ be a $\amalg$-semischeme such that the precositus $\mathcal{T}$ has a basis $\mathcal{B}$, consisting of the filters of bifinite type. Then every ideal from $\mathcal{S}pec(R, \mathcal{T})$ is contained in some ideal from $\mathcal{S}pec\, R \cap \mathcal{S}pec(R, \mathcal{T})$.*

Proof. The condition " $\mathcal{B}$ is a basis of the precositus $\mathcal{T}$ " means literally that $(R, \mathcal{B})$ is a $\amalg$- semischeme, $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{T} \subset \mathcal{B}^\cap$ (i.e. every radical filter from $\mathcal{T}$ is the intersection of some filters from $\mathcal{B}$). From this we get $\mathcal{S}pec_\ell(R,\mathcal{B}^\cap) \subset \mathcal{S}pec_\ell(R,\mathcal{T}) \subset \mathcal{S}pec_\ell(R,\mathcal{B})$. By proposition 8 $\mathcal{S}pec_\ell(R,\mathcal{B}) = \mathcal{S}pec_\ell(R, \mathcal{B}^\cap)$ and, therefore, $\mathcal{S}pec_\ell(R,\mathcal{T}) = \mathcal{S}pec_\ell(R, \mathcal{B})$. Hence, changing $\mathcal{T}$ by $\mathcal{B}$, we may consider $\mathcal{T}$ consisting of the filters of bifinite type. For every $p \in \mathcal{S}pec(R,\mathcal{T})$ let $\mathcal{T}_p \overset{def}{=} \{\mathcal{F} \in \mathcal{T} \mid p \notin \mathcal{F}\}$. Since $\Sigma \mathcal{T}_p \overset{def}{=} \cup\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}_p\}$ is a filter of bifinite type, then by Proposition 2.8 every ideal from $IR \setminus \Sigma \mathcal{T}_p$, including p , is contained in an ideal from $Max(IR \setminus \Sigma\mathcal{T}_p)$. As $\mathcal{F} \circ \mathcal{G} \subset \mathcal{F} \amalg \mathcal{G} \subset \Sigma \mathcal{T}_p$ for any $\{\mathcal{F}, \mathcal{G}\} \subset \mathcal{T}_p$, then by Corollary of Proposition 2.8 $Max(IR \setminus \Sigma \mathcal{T}_p) \subset \mathcal{S}pec\, R$. $\square$

Corollary. *Under the conditions of the propositions*

1) *For every closed subset* W *of* Spec $(R, \mathcal{T})$ *the subset* $W \cap \mathcal{S}pec\, R$ *is dense in* W.

2) *If* $\mathcal{S}pec\, R \subset \mathcal{S}pec(R,\mathcal{T})$, *then*

$[\mathcal{F} \cap IR = \mathcal{G} \cap IR] \Leftrightarrow [V_{\mathcal{F}} \cap \mathcal{S}pec\, R = V_{\mathcal{G}} \cap \mathcal{S}pec\, R] \Rightarrow [V_{\mathcal{F}} \cap IR = V_{\mathcal{G}} \cap IR]$

Corollary. *Let M be an R-module such that the annihilator of any element of M belongs to* $\mathrm{Spec}_\ell R$. *Then for any semischeme* $(R,\mathcal{T})$ *the presheaf* $\mathcal{M}_{\mathcal{T}}$ *is a* ω-*sheaf*.

This corollary (following from $\mathcal{T}$-admissibility of $\mathrm{Spec}_\ell(R,\mathcal{T})$) can be considered as a generalization of Corollary 3.2. In fact, the annihilators of the elements of irreducible R-modules are regular maximal left ideals that, as we know, belong to $\mathrm{Spec}_\ell R$. $\square$

10. "Quasicoherent" sheaves on spectra. To a canonical morphism of precositi ${}^{\mathcal{T}}q : \mathrm{Spec}_\ell(R,\mathcal{T}) \longrightarrow \underline{\mathcal{T}}$ the "direct image" functors ${}^{\mathcal{T}}q_{\#} : \mathbb{F}_o(\mathrm{Spec}_\ell(R,\mathcal{T}),\mathcal{C}) \to \mathbb{F}_o(\underline{\mathcal{T}},\mathcal{C})$ and ${}^{\mathcal{T}}q_{*} : \mathbb{F}(\mathrm{Spec}_\ell(R,\mathcal{T}),\mathcal{C}) \to \mathbb{F}(\underline{\mathcal{T}},\mathcal{C})$ correspond. To them we can assign "left inverse image" functors, i.e. to find presheaves and sheaves copresenting functors

$\mathbb{F}_o(\underline{\mathcal{T}},\mathcal{C})(X, {}^{\mathcal{T}}q_{\#} -)$ and $\mathbb{F}(\underline{\mathcal{T}},\mathcal{C})(X, {}^{\mathcal{T}}q_{*} -)$

respectively.

For an arbitrary R-module M denote ${}^{\#}\mathcal{M}_{\mathcal{T}}$ the left inverse image of the presheaf $\mathcal{M}_{\mathcal{T}}$ and by ${}^{\#}\mathcal{M}_{\mathcal{T}}^{a}$ the sheaf associated with ${}^{\#}\mathcal{M}_{\mathcal{T}}$.

The stalk of ${}^{\#}\mathcal{M}_{\mathcal{T}}^{a}$ at a point p of the spectrum is isomorphic to the colimit of R-modules $\mathcal{F}^{-1}M = G_{\mathcal{F}}M$, i.e. $\mathcal{F}$-localizations of M, with respect to the inductive subcategory $\mathcal{T}_p$ of $\mathcal{T}$ formed by all $\mathcal{F} \in \mathcal{T}$ that do not contain p.

II. The symmetric spectrum and the prime ideals.

Let $(R,\mathcal{T})$ be a $\amalg$ - semischeme. We endow the set $\mathcal{S}pec(R,\mathcal{T}) \overset{def}{=} \mathrm{I}R \cap \mathcal{S}pec_{\ell}(R,\mathcal{T})$ with the topology induced from $\mathcal{S}pec_{\ell}(R,\mathcal{T})$, and call both the set and the space of this Spec the symmetric spectrum of the $\amalg$ - semischeme $(R,\mathcal{T})$.

Proposition. Let $(R,\mathcal{T})$ be a $\amalg$-semischeme such that the precositus $\mathcal{T}$ has a basis $\mathcal{B}$, consisting of the filters of bifinite type. Then every ideal from $\mathcal{S}pec(R,\mathcal{T})$ is contained in some ideal from $\mathcal{S}pec\, R \cap \mathcal{S}pec(R,\mathcal{T})$.

Proof. The condition " $\mathcal{B}$ is a basis of the precositus $\mathcal{T}$ " means literally that $(R,\mathcal{B})$ is a $\amalg$- semischeme, $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{T} \subset \mathcal{B}^{\cap}$ (i.e. every radical filter from $\mathcal{T}$ is the intersection of some filters from $\mathcal{B}$). From this we get $\mathcal{S}pec_{\ell}(R,\mathcal{B}^{\cap}) \subset \mathcal{S}pec_{\ell}(R,\mathcal{T}) \subset \mathcal{S}pec_{\ell}(R,\mathcal{B})$. By proposition 8 $\mathcal{S}pec_{\ell}(R,\mathcal{B}) = \mathcal{S}pec_{\ell}(R,\mathcal{B}^{\cap})$ and, therefore, $\mathcal{S}pec_{\ell}(R,\mathcal{T}) = \mathcal{S}pec_{\ell}(R,\mathcal{B})$. Hence, changing $\mathcal{T}$ by $\mathcal{B}$, we may consider $\mathcal{T}$ consisting of the filters of bifinite type. For every $p \in \mathcal{S}pec(R,\mathcal{T})$ let $\mathcal{T}_p \overset{def}{=} \{\mathcal{F} \in \mathcal{T} \mid p \notin \mathcal{F}\}$. Since $\Sigma\mathcal{T}_p \overset{def}{=} \cup\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}_p\}$ is a filter of bifinite type, then by Proposition 2.8 every ideal from $\mathrm{I}R \setminus \Sigma\mathcal{T}_p$, including p, is contained in an ideal from $Max(\mathrm{I}R \setminus \Sigma\mathcal{T}_p)$. As $\mathcal{F} \circ \mathcal{G} \subset \mathcal{F} \amalg \mathcal{G} \subset \Sigma\mathcal{T}_p$ for any $\{\mathcal{F},\mathcal{G}\} \subset \mathcal{T}_p$, then by Corollary of Proposition 2.8 $Max(\mathrm{I}R \setminus \Sigma\mathcal{T}_p) \subset \mathcal{S}pec\, R$. $\square$

Corollary. Under the conditions of the propositions

1) For every closed subset W of Spec $(R,\mathcal{T})$ the subset $W \cap \mathcal{S}pec\, R$ is dense in W.

2) If $\mathcal{S}pec\, R \subset \mathcal{S}pec(R,\mathcal{T})$, then

$$[\mathcal{F} \cap \mathrm{I}R = \mathcal{G} \cap \mathrm{I}R] \Leftrightarrow [V_{\mathcal{F}} \cap \mathcal{S}pec\, R = V_{\mathcal{G}} \cap \mathcal{S}pec\, R] \Rightarrow [V_{\mathcal{F}} \cap \mathrm{I}R = V_{\mathcal{G}} \cap \mathrm{I}R]$$

for any $\mathcal{F}, \mathcal{G} \in \mathfrak{T}$.

Proof. 1) The first statement is equivalent to the statement of Proposition 11, whose "topological" formulation is as follows: the closure of every point $p \in \mathcal{S}pec(R, \mathfrak{T})$ contains a prime ideal.

2) Now let $\{\mathcal{G}_i \mid i \in I\}$ be a family of the radical filters of bifinite type, $\mathcal{G} = \bigcap\{\mathcal{G}_i \mid i \in I\}$. Then for every filter $\mathcal{F} \subset \mathcal{I}_\ell R$

$$[\mathcal{F} \cap \operatorname{Spec} R \subset \mathcal{G} \cap \operatorname{Spec} R] \Rightarrow [\mathcal{F} \cap IR \subset \mathcal{G} \cap IR] \qquad (1)$$

Indeed, let $\alpha \in \mathcal{F} \cap IR$ and $\alpha \notin \mathcal{G}_i$. By Corollary 2.8, $\alpha \subset p$ for some $p \in Max(IR \setminus \mathcal{G}_i)$, and $Max(IR \setminus \mathcal{G}_i) \subset \operatorname{Spec} R$. So, if $\mathcal{F} \cap \operatorname{Spec} R \subset \mathcal{G} \cap \operatorname{Spec} R$, then $\mathcal{F} \cap IR \subset \mathcal{G} \cap IR$. The inverse implication is trivial.

Since, by hypothesis, we have $\operatorname{Spec} R \subset \mathcal{S}pec(R, \mathfrak{T})$, and, according to the first heading of the corollary

$$[V_{\mathcal{F}} \cap \operatorname{Spec} R = V_{\mathcal{G}} \cap \operatorname{Spec} R] \Rightarrow [V_{\mathcal{F}} \cap IR = V_{\mathcal{G}} \cap IR],$$

the desired statement follows from (1). □

In what follows we will find out the conditions under which all the prime ideals belong to the spectrum of the $\amalg$-semischeme $(R, \mathfrak{T})$.

12. The prime spectrum and the category $Sp_\ell R$.

A set $\mathcal{G}$ of left ideals will be called symmetric if $\mathcal{G} \cap IR$ is a cofinal subset of $\mathcal{G}$. A full subcategory of $\mathfrak{T}_\ell R$ formed by symmetric filters will be denoted by $Sym_\ell R$.

Let $Sp_\ell R$ be a full subcategory of $\mathfrak{T}_\ell R$ consisting of all the filters $\mathcal{F}$ such that $\operatorname{Spec} R \cap \mathcal{F} \circ X = \operatorname{Spec} R \cap (\mathcal{F} \cup \overline{X})$ for any set X of left ideals.

Proposition. $Sp_\ell R$ possesses the following properties:

1) $Sp_e R$ is closed with respect to $\circ$ and the union of directed with respect to inclusion families of filters (inductive colimits);

2) $Sym_e R \subset Sp_e R$;

3) $Sp_e R$ contains together with every filter $\mathcal{F}$ its radical closure $\hat{\mathcal{F}}$.

Proof. 1) The closedness of $Sp_e R$ with respect to the multiplication follows directly from the associativity of the multiplication in $\mathcal{T}_e R$: . For every $\{\mathcal{F}_1, \mathcal{F}_2\} \subset Sp_e R$ we have
$\operatorname{Spec} R \cap (\mathcal{F}_1 \circ \mathcal{F}_2) \circ \mathcal{G} = \operatorname{Spec} R \cap \mathcal{F}_1 \circ (\mathcal{F}_2 \circ \mathcal{G}) =$
$= \operatorname{Spec} R \cap (\mathcal{F}_1 \cup \mathcal{F}_2 \circ \mathcal{G}) = \ldots$.

Let $\{\mathcal{F}_\alpha \mid \alpha \in \mathfrak{A}\}$ be an directed with respect to inclusion family of filters of $Sp_e R$, m a left ideal, p a prime ideal of $(\bigcup_{\alpha \in \mathfrak{A}} \mathcal{F}_\alpha) \circ \{m\}$. Assume that $m \not\subset p$; i.e. there exists an element such that $x \in m$, $x \notin p$. By hypothesis $(p : x) \in \mathcal{F}_\alpha$ for some $\alpha \in \mathfrak{A}$. But then $(p : y) \in \mathcal{F}_\alpha$ for any $y \in \mathcal{P}((R, x))$, since $\mathcal{F}_\alpha$ is uniform; i.e. $p \in \mathcal{F}_\alpha \circ \{(R, x)\}$. Since $(R, x) \not\subset p$ and $\mathcal{F}_\alpha \in Sp_e R$, then $p \in \mathcal{F}_\alpha$.

2) Let $\mathcal{F}$ be a symmetric set of ideals, m an arbitrary left ideal, $p \in \operatorname{Spec} R$. If $p \in \mathcal{F} \circ \{m\}$, then for any $x \in m$ there exists an ideal $n_x \in \mathcal{F} \cap \mathcal{I} R$ such that $n_x x \subset p$. Either $m \subset p$ or $x \notin p$ for some $x \in m$. But then $n_x \subset p$ and therefore $p \in \mathcal{F}$.

3) Let $\mathcal{F} \in Sp_e R$ and $Sp_{\mathcal{F}} R$ be a full subcategory of $Sp_e R$ formed by all the filters $\mathcal{F}'$ from $Sp_e R$ such that $\operatorname{Spec} R \cap \mathcal{F}' = \operatorname{Spec} R \cap \mathcal{F}$.

The first heading yields the existence in $Sp_{\mathcal{F}}R$ of a finite object $\tilde{\mathcal{F}} = \bigcup\{\mathcal{G} \mid \mathcal{G} \in Sp_{\mathcal{F}}R\}$ which thanks to the closedness of $Sp_{\mathcal{F}}R$ with respect to $\circ$ is a radical filter. It follows, $\hat{\mathcal{F}} \in Sp_{\mathcal{F}}R$. □

Let us that $Sp_{\ell}R$ can be in general considerably wider than $Sym_{\ell}R$: e.g. the radical closure of a symmetric filter should not be necessarily symmetric.

13. Symmetric filters of finite type. The situation gets milder when dealing with filters of finite type.

Proposition. 1) Let $\mathcal{F}, \mathcal{G}$ be sets of left ideals, where $\mathcal{F}$ is symmetric and $\mathcal{G}$ is of finite type.

Then the family of ideals of the form mn, where $m \in \mathcal{F} \cap IR$, $n \in \mathcal{G}$, forms a cofinal subset in $\mathcal{F} \circ \mathcal{G}$.

2) For any symmetric set $\mathcal{F}$ of finite type its radical closure $\hat{\mathcal{F}}$ coincides with $\mathcal{F}^{(\infty)} \overset{\mathrm{def}}{=} \bigcup_{\ell \geq 1} \mathcal{F}^{(\ell)}$, where $\mathcal{F}^{(1)} = \mathcal{F}$, $\mathcal{F}^{(\ell+1)} = \mathcal{F}^{(\ell)} \circ \mathcal{F}$.

Proof. 1) Clearly, $mn \in \mathcal{F} \circ \mathcal{G}$ for any $\{\mathcal{F}, \mathcal{G}\} \subset 2^{I_\ell R}$, $n \in \mathcal{G}$ and $m \in \mathcal{F}$.

Now, let $\mathcal{F}$ be symmetric and $\nu \in \mathcal{F} \circ \mathcal{G}$. Then $\mathcal{G}$ is of finite type, hence $\nu \in \mathcal{F} \circ \{n\}$, where $n = (R, x)$ is the ideal generated by some $x \in \mathcal{P}(R)$. By hypothesis there exists an ideal $m \in \mathcal{F} \cap IR$ such that $mx \subset \nu$. But then $mn = m(R, x) \subset \nu$.

2) The just proved fact implies for $\{\mathcal{F}, \mathcal{G}\} \subset 2^{I_\ell R}$:

a) [$\mathcal{F}$ and $\mathcal{G}$ are symmetric and $\mathcal{G}$ is ~~of~~ finite type] $\Rightarrow$ [$\mathcal{F} \circ \mathcal{G}$ is symmetric];

b) [$\mathcal{F}$, $\mathcal{G}$ are sets of finite type and $\mathcal{F}$ is symmetric] $\Rightarrow$ [$\mathcal{F} \circ \mathcal{G}$ is of finite type].

In fact, in the first case the set of ideals $\{mn \mid m \in \mathcal{F} \cap IR, n \in \mathcal{G} \cap IR\}$ is cofinal in $\mathcal{F} \circ \mathcal{G}$. In the second case such is the set of ideals of the form $\nu(R,t)$, where $\nu \in \mathcal{F} \cap IR$, $t \in \mathcal{P}(R)$, $(R,t) \in \mathcal{G}$. Clearly, $\nu(R,t) = \nu t$ and by hypothesis there exists $s \in \mathcal{P}(R)$ such that $\nu \supset (R,s) \in \mathcal{F}$ and therefore $\nu t \supset (R,s)t = (R,st) \in \mathcal{F} \circ \mathcal{G}$.

Thus, if $\mathcal{F}$ is a symmetric subset of $I_e R$ of finite type, then so are all the $\mathcal{F}^{(\ell)}$ and therefore $\mathcal{F}^{(\infty)} = \bigcup_{\ell \geq 1} \mathcal{F}^{(\ell)}$. Let us show that $\mathcal{F}^{(\infty)} \circ \mathcal{F}^{(\infty)} \subset \mathcal{F}^{(\infty)}$.

Let $n \in \mathcal{F}^{\infty} \circ \mathcal{F}^{\infty}$; i.e. $n \in \mathcal{F}^{(\infty)} \circ \{m\}$, where $m \in \mathcal{F}^{(k)}$ for some k. Since $\mathcal{F}^{(\ell)}$ is finite type, we can and will assume that $m = (R,s) \overset{\text{def}}{=} Rs + s$ for some $s \in \mathcal{P}(R)$. The ideal $(n:s)$ belongs to $\mathcal{F}^{(\ell)}$ for some $\ell \geq 1$ and therefore so do all the ideals $(n:y)$, where y runs $\mathcal{P}((R,s))$. Hence $n \in \mathcal{F}^{(\ell)} \circ \mathcal{F}^{(k)} \subset \mathcal{F}^{(\infty)}$.

Clearly, all the symmetric sets of left ideals are uniform. Therefore $\mathcal{F}^{(\infty)}$ is a radical filter. □

Remark. In addition to the promised in the formulation of Proposition we have established the invariance of the family (subcategory) of symmetric subsets of finite type of $I_e R$ with respect to $\circ$ and the radical closure. □

14. Examples. 1) Consider the filters ${}^{\alpha}\mathcal{F} = \{n \in I_e R \mid \alpha \subset n\}$, where $\alpha \in IR$. It is not difficult to see that ${}^{\alpha}\mathcal{F} \circ {}^{\beta}\mathcal{F} = {}^{\alpha\beta}\mathcal{F}$ for any $\{\alpha, \beta\} \subset IR$. In particular, ${}^{\alpha}\mathcal{F}^{(\infty)} = \bigcup_{\ell \geq 1} {}^{\alpha}\mathcal{F}^{(\ell)}$ is a symmetric topo-

logizing filter. Notice that [${}^{\alpha}\mathcal{F}$ is a filter of finite type] $\Longleftrightarrow$ [the two-sided ideal α is a finitely generated left ideal].

2) For an arbitrary left ideal n of R denote n_s the ideal $n \cap (n : R)$, the maximal two-sided ideal contained in n. Let $m \in IR$; $\mathcal{L}_m = \{n \in I_e R \mid m \subset \mathcal{B}(n_s)\}$, where $\mathcal{B}(n_s) = \bigcap\{p \in \operatorname{Spec} R \mid n_s \subset p\}$ is the Baire radical of n_s. It is not difficult to see that $\mathcal{L}_m$ coincides with the union of all the symmetric sets $\mathcal{F}'$ such that $\operatorname{Spec} R \cap \mathcal{F}' = \operatorname{Spec} R \cap \mathcal{L}_m = \{p \in \operatorname{Spec} R \mid m \subset p\}$. In particular, ${}^{m}\mathcal{F}^{(\infty)} \subset \mathcal{L}_m$. Now suppose that

the filter ${}^{m}\mathcal{F}$ is of bifinite type (if R satisfies the ascending chain condition for two-sided ideals then all the $\mathcal{G} \subset I_e R$ are of bifinite type). Then $\mathcal{L}_m = (IR \cap {}^{m}\widehat{\mathcal{F}})$. In particular, $\mathcal{L}_m \subset {}^{m}\widehat{\mathcal{F}}$ and $\mathcal{L}_m = {}^{m}\widehat{\mathcal{F}}$ if and only if ${}^{m}\widehat{\mathcal{F}}$ is symmetric.

In fact, let $\alpha \in IR \setminus {}^{m}\widehat{\mathcal{F}}$. By Proposition 2.8 $\alpha \subset p$ for some $p \in \operatorname{Spec} R \setminus {}^{m}\widehat{\mathcal{F}}$. Since $m \not\subset p$, then $\alpha \notin \mathcal{L}_m$ and therefore $IR \cap \mathcal{L}_m \subset {}^{m}\widehat{\mathcal{F}}$ and $\mathcal{L}_m = (IR \cap \mathcal{L}_m) \subset {}^{m}\widehat{\mathcal{F}}$.

Conversely, since $\operatorname{Spec} R \cap {}^{m}\widehat{\mathcal{F}} = \operatorname{Spec} R \cap {}^{m}\mathcal{F}$ thanks to Proposition 12, then ${}^{m}\widehat{\mathcal{F}} \cap IR \subset \mathcal{L}_m \cap IR$ (see Corollary 11).

By Proposition 13, if a two-sided ideal m is ~~a left~~ finitely generated ideal, then ${}^{m}\mathcal{F}^{(\infty)} = {}^{m}\widehat{\mathcal{F}}$ and therefore ${}^{m}\widehat{\mathcal{F}} = \mathcal{L}_m = \{n \in I_e R \mid m \subset \mathcal{B}(n_s)\}$. $\square$

15. Symmetric semischemes. A $\amalg$-semischeme $(R, \mathcal{T})$ will be called symmetric, if $\mathcal{T}$ consists of symmetric filters.

Proposition. 1) *Let* $(R, \mathcal{T})$ *be a symmetric* $\amalg$*-semischeme. Then* $\mathrm{Spec}_\ell(R,\mathcal{T}) = \{ n \in I_\ell R \mid n_s \in \mathrm{Spec}(R,\mathcal{T})\}$, *and for any closed subset* W *of* $\mathrm{Spec}(R, \mathcal{T})$ *the set* $W \cap IR = W \cap \mathrm{Spec}(R,\mathcal{T})$ *is dense in* W.

2) *If* $(R, \mathcal{T})$ *is a semischeme and* $\mathcal{T} \subset \mathrm{Sp}_\ell R$, *then* $\mathrm{Spec}\, R \subset \mathrm{Spec}(R, \mathcal{T})$.

3) Let $(R,\mathcal{T})$ be a symmetric semischeme such that the precositus $\mathcal{T}$ has a basis consisting of filters of bifinite type. Then $\mathcal{F}\longmapsto \mathrm{Spec}\,R\cap\mathcal{F}$ is an injection from $\mathcal{T}$ into the preorder of the subsets of Spec R. In particular, $\mathcal{T}$ is a cositus.

Proof. 1) Let $n\in I_\ell R$, $n_s\in \mathrm{Spec}_\ell(R,\mathcal{T})\ni m$, $\{\mathcal{F},\mathcal{G}\}\subset\mathcal{T}$

Then $[n\in\mathcal{F}\amalg\mathcal{G}]\Leftrightarrow[n_s\in\mathcal{F}\amalg\mathcal{G}]\Leftrightarrow[n_s\in\mathcal{F}\cup\mathcal{G}]\Leftrightarrow$ $\Leftrightarrow[n\in\mathcal{F}\cup\mathcal{G}]$; $[m_s\in\mathcal{F}\amalg\mathcal{G}]\Leftrightarrow[m\in\mathcal{F}\amalg\mathcal{G}]\Leftrightarrow$ $\Leftrightarrow[m\in\mathcal{F}\cup\mathcal{G}]\Leftrightarrow[m_s\in\mathcal{F}\cup\mathcal{G}]$.

2) Proposition 12 implies that $(R,\mathcal{S}p_\ell R)$ is a semischeme and $\mathrm{Spec}\,R\subset\mathrm{Spec}(R,\mathcal{S}p_\ell R)$. Obviously, if $\mathcal{T}$ is a $\vee$-subcategory of the V-category $\mathcal{S}p_\ell R$, then $\mathrm{Spec}(R,\mathcal{S}p_\ell R)\subset\mathrm{Spec}(R,\mathcal{T})$

3) Follows directly from Corollary of Proposition 11. □

16. When is the embedding $\mathrm{Spec}\,R\hookrightarrow\mathrm{Spec}(R,\mathcal{T})$ continuous? For any pair of twosided ideals α, β the equalit~~ies~~y ${}^{\alpha}\mathcal{F}\circ{}^{\beta}\mathcal{F}={}^{\alpha\beta}\mathcal{F}$ implies $\widehat{{}^{\alpha}\mathcal{F}}\vee\widehat{{}^{\beta}\mathcal{F}}=\widehat{{}^{\alpha\beta}\mathcal{F}}$. Hence an arbitrary set Ω of twosided ideals, closed with respect to multiplication $(m,n)\longmapsto mn$, generates a $\vee$-category $\mathcal{T}_\Omega=\{{}^{m}\widehat{\mathcal{F}}\mid m\in\Omega\}$. Since $\mathcal{T}_\Omega\subset\mathcal{S}p_\ell R$, then

Proposition 1) For an arbitrary multiplicative (i.e. closed with respect to the multiplication of ideals) set Ω of twosided ideals of R the inclusion $\mathrm{Spec}\,R\hookrightarrow\mathrm{Spec}(R,\mathcal{T}_\Omega)$ is continuous (the topology on Spec R is defined, as usual, by the closed subsets $V(\alpha)=\{p\in\mathrm{Spec}\,R\mid\alpha\subset p\}$, $\alpha\in IR$).

2) If $(R,\mathcal{T})$ is a symmetric semischeme and $\mathcal{T}$ has a

basis, consisting of filters of bifinite type, then the embedding Spec R $\hookrightarrow$ Spec $(R, \mathcal{T})$ is continuous iff $\mathcal{T} = \mathcal{T}_{\Omega}$ for some multiplicative set Ω of twosided ideals.

Proof. 1) The continuity of the embedding Spec R $\hookrightarrow$ Spec $(R, \mathcal{T})$ follows from the definition of the topology on Spec R and Spec $(R, \mathcal{T})$: we have ${}^{m}\widehat{\mathcal{F}} \cap \mathrm{Spec}\, R = V(m)$ - a closed subset of Spec R - for any $m \in \mathcal{I}R$

2) Now let $(R, \mathcal{T})$ be a symmetric semischeme satisfying the conditions of heading 2); the embedding Spec R $\hookrightarrow$ Spec$(R, \mathcal{T})$ is a continuous map. The latter means literally that for every $\mathcal{F} \in \mathcal{T}$ there exists a twosided ideal α such that $\mathrm{Spec}\, R \cap \mathcal{F} = V(\alpha) \overset{\mathrm{def}}{=} \{ p \in \mathrm{Spec}\, R \mid \alpha \subset p \}$. If $\alpha \notin \mathcal{F}$, then by Proposition 2.8 $\alpha \subset q$ for some q from Spec $R \setminus \mathcal{F}$ contradicting the assumption. So, $\alpha \in \mathcal{F}$ and together with α the filter $\mathcal{F}$ contains ${}^{\alpha}\widehat{\mathcal{F}} = \widehat{\{\alpha\}}$, the radical closure of $\{\alpha\}$. Since $\mathrm{Spec}\, R \cap {}^{\alpha}\widehat{\mathcal{F}} = \mathrm{Spec}\, R \cap \mathcal{F}$, then by Corollary of Proposition 11 $\mathcal{I}R \cap \mathcal{F} = \mathcal{I}R \cap {}^{\alpha}\widehat{\mathcal{F}}$ and, consequently, $\mathcal{F} = \overline{(\mathcal{I}R \cap \mathcal{F})} \subset {}^{\alpha}\widehat{\mathcal{F}}$. Therefore, $\mathcal{F} = {}^{\alpha}\widehat{\mathcal{F}}$. Note that since $\mathcal{F}$ is symmetric, $\mathcal{F} = \{ n \in \mathcal{I}_{\ell} R \mid V(n_s) \subset \mathcal{F} \cap \mathrm{Spec}\, R = V(\alpha) \}$ (see Example 14.2). □

Corollary. Under the conditions of heading 2) of Proposition 16 $\mathcal{T}$ is a fully quasicompact cositus.

Proof. Let $\mathcal{B}$ be a basis of the cositus $\mathcal{T}$ (see Proposition 15) consisting of the filters of bifinite type. Show that all the elements of $\mathcal{B}$ are quasifinite.

Let $\{\mathcal{F}, \mathcal{F}_i \mid i \in I\}$ be a family of filters from $\mathcal{T}$ such that $\mathcal{F} = \bigcap \{\mathcal{F}_i \mid i \in I\}$. By proposition 16 there exist twosided ideals $\{\alpha, \alpha_i \mid i \in I\}$ such that $\mathcal{F} = \{ n \in \mathcal{I}_{\ell} R \mid \alpha \subset \mathfrak{p}(n_s) \}$,

$\mathcal{F}_i = \{n \mid \alpha_i \in \mathfrak{p}(n_s)\}$ and $\mathcal{F} = \bigcap\{\mathcal{F}_i \mid i \in I\}$ iff $\mathfrak{p}(\sup_{i\in I} \alpha_i) = \mathfrak{p}(\alpha)$. If $\mathcal{F}$ is a filter of bifinite type, then $[\sup_{i\in I} \alpha_i \in \mathcal{F}] \Rightarrow [\sup_{j\in I_0} \alpha_j \in \mathcal{F}$ for some finite subset $I_0 \subset I]$. Therefore $\mathfrak{p}(\{\sup \alpha_i \mid i \in I_0\}) = \mathfrak{p}(\alpha)$, hence $\bigcap\{\mathcal{F}_i \mid i \in I_0\} = \mathcal{F}$. $\square$

17. Structural presheaves on the prime spectrum. For every subset $W \subset$ Spec R let ${}_W\mathcal{F} \overset{def}{=} \{n \in I_\ell R \mid V(n_s) \subset W\}$. Clearly, ${}_W\mathcal{F}$ is a symmetric topologizing filter and for every $m \in IR$ the filter ${}_{V(m)}\mathcal{F}$ coincides with $\mathcal{L}_m$ in the notations of Example 14.2. The arguments of Example 14.2 imply that the correspondence $W \mapsto {}_W\hat{\mathcal{F}}$, where $\hat{}$ denotes as before the radical closure, is inverse to the map $\mathcal{F} \mapsto \operatorname{Spec} R \cap \mathcal{F}$ when W runs the set $cl\,\operatorname{Spec} R$ of closed subsets of the prime spectrum, and $\mathcal{F}$ runs the set of filters $\mathcal{T}_{IR} = \{{}^{\alpha}\hat{\mathcal{F}} \mid \alpha \in IR\}$. In other words, the map $W \mapsto {}_W\hat{\mathcal{F}}$ is an isomorphism of the cositi $\underline{\mathcal{T}_{IR}} \longrightarrow \underline{\operatorname{Spec} R}$, which (via the direct image functor) induces the isomorphism of the corresponding categories of presheaves and sheaves. This isomorphism sends the canonical presheaves $M_{\mathcal{T}_{IR}}$, where $M \in Ob\,R\text{-}mod$, into the presheaves $\widetilde{M}$, which send a closed subset W into $G_{{}_W\hat{\mathcal{F}}} M$.

For an arbitrary multiplicative set of twosided ideals Ω the collection of sets $\{V(\alpha) \mid \alpha \in \Omega\}$ forms a cositus, which is isomorphic to the cositus $\underline{\mathcal{T}_\Omega}$ and is a basis of the topology whose closed sets are $\tau_\Omega = \{V(\sup_i \alpha_i)\}$, where $\{\alpha_i \mid i \in I\}$ runs the subsets of Ω. The restrictions of the presheaves $\widetilde{M}$ on the topology τ_Ω will be denoted by $\widetilde{M}_{\tau_\Omega}$.

Proposition. Let R be a ring with right unit. Then for every R-module M, such that $M_{\mathcal{T}_\Omega}$ is an ω-sheaf the canonical arrow $\{R\}^1 M \to \Gamma \widetilde{M}^a_{\tau_\Omega}$ is an isomorphism ($\Gamma\widetilde{M}^a_{\tau_\Omega}$ is an R-module of the global sections of $\widetilde{M}^a_{\tau_\Omega}$, associated with the presheaf $\widetilde{M}_{\tau_\Omega}$).

Proof. 1) If there is a right unit in R, then $\{R\}$ is a radical filter of finite type. According to Proposition 2.8 (see also its Corollary) every ideal from $\mathrm{IR} \setminus \{R\}$ is contained in an ideal from $Max(\mathrm{I}R \setminus \{R\}) \overset{def}{=} Max\, R$, and all the maximal twosided ideals of R are prime . In particular,

$$[\alpha \in \mathrm{I}R,\ V(\alpha) = \varnothing\ ] \Longleftrightarrow [\alpha = R\,]. \tag{1}$$

If $\{\alpha_i \mid i \in I\} \subset \mathrm{I}R$ and $\bigcap_{i\in I} V(\alpha_i) = V(\sup \alpha_i) = \varnothing$, then, as follows from (1), $\sup\{\alpha_i \mid i \in I\} = R$. Then (due to the existence of right unit in R) $\sup\{\alpha_j \mid j \in J\} = R$ for a finite subset $J \subset I$; consequently, $\bigcap\{V(\alpha_j) \mid j \in J\} = \varnothing$. Therefore, the existence of right unit in R implies the quasicompactness of Spec R.

2) If R is a ring with right unit e, then every morphism $f: R \to M$ of left R-modules is the right multiplication by f(e); i.e. the canonical morphism $M \mapsto Hom_R(R, M)$ is an epimorphism of the R-modules with the kernel $\{R\}M$. From this (and from the radicality of the trivial filter $\{R\}$) it follows that the canonical morphism $\{R\}^1 M \to \{R\}^{-1} M = G_{\{R\}} M = Hom_R(R, \{R\}^1 M)$ is an isomorphism.

3) Let F be an ω-sheaf over a topological space X and F^a the associated sheaf. Then for every quasifinite closed set W (i.e. $W \in \mathcal{C}\ell X$ such that the complement to it

is quasifinite) the natural arrow $F(W) \to F^a(W)$ is an isomorphism. In particular, if X is a quasicompact space then $F(\emptyset) \longrightarrow F^a(\emptyset) \overset{\text{def}}{=} \Gamma F^a$ is an isomorphism.

4) Let M be an R-module such that $\mathcal{M}_{\mathcal{T}_\Omega}$ is an ω-sheaf. Then $\mathcal{M}_{\mathcal{T}_\Omega^\cap}$ is also an ω-sheaf. Since the map $W \mapsto$ $\mapsto {}_W\hat{\mathcal{F}}$ induces an isomorphism of the category of the closed sets of the space $(\mathrm{Spec}\, R,\ \tau_\Omega)$ with the category $\mathcal{T}_\Omega^\cap$, the sheaf $\widetilde{\mathcal{M}}_{\tau_\Omega}$ is an ω-sheaf. According to heading 3) of the proof, for every quasifinite closed set W from τ_Ω the natural morphism of R-modules $G_{{}_W\hat{\mathcal{F}}}\mathcal{M} \to$ $\to \widetilde{\mathcal{M}}^a_{\tau_\Omega}(W)$ is an isomorphism. In particular, since (Spec R, τ_Ω) is quasicompact (Spec R is quasicompact according to step 1) of the proof) $\widetilde{\mathcal{M}}_{\tau_\Omega}(\emptyset) = G_{\{R\}}\mathcal{M} \to \Gamma\widetilde{\mathcal{M}}^a_{\tau_\Omega}$ is an isomorphism. As we found out in step 2), $G_{\{R\}}\mathcal{M} \simeq \{R\}^{-1}\mathcal{M}$. □

<u>Example</u>. Let $\Omega = \mathcal{I}^\ell R$ be the set of all twosided ideals of R finitely generated as left ideals. It is easy to verify that the set $\mathcal{I}^\ell R$ is multiplicative.

Indeed, let $\{s,t\} \subset \mathcal{P}(R)$ and the left ideals generated by s and t, say $\alpha = (R,s) \overset{\text{def}}{=} Rs+s$, $\beta = (R,t)$, are twosided. Clearly, $(R,st) \subset \alpha\beta$. On the other hand,

$$\alpha\beta = (Rs+s)(Rt+t) = (RsR)t + (sR)t + Rst + st \subset$$
$$\subset (Rs+s)t + (Rs+s)t + Rst + st = Rst + st \overset{\text{def}}{=} (R,st).$$

Every filter ${}_\alpha\hat{\mathcal{F}}$, $\alpha \in \mathcal{I}^\ell R$, is symmetric and is of finite type (see Proposition 13 and Examples 14). The proof of Corollary of Proposition 16 implies the quasifiniteness of the cositus $\mathcal{T}_{\mathcal{I}^\ell R}$, and, therefore, the full quasicompactness of the space $(\mathrm{Spec}\, R,\ \tau_{\mathcal{I}^\ell R})$. Therefore for every ω-sheaf F on (Spec R, $\tau_{I^\ell R}$) and for any $V(\alpha)$, $\alpha \in I^\ell R$, the canonical arrow $F(V(\alpha)) \to F^a(V(\alpha))$ is an isomorphism, and for an arbitrary closed subset W of $(Spec R, \tau_{I^\ell R})$ we have $F^a(W) \simeq \varprojlim \{F(V(\alpha)) \mid W \subset V(\alpha), \alpha \in I^\ell R\}$ (see Propositions 7, 3.6 and Example 3.6.1). In particular, if M is an R-module such that $M_{I^\ell R}$ is an ω-sheaf, then $\widetilde{M}^a_{\tau_{I^\ell R}}(V(\alpha)) \simeq G_{\alpha\hat{\mathcal{F}}} M$ for every $\alpha \in I^\ell R$ and $\widetilde{M}^a_{\tau_{I^\ell R}}(W) \simeq \varprojlim (G_{\alpha\hat{\mathcal{F}}} M \mid W \subset V(\alpha), \alpha \in I^\ell R)$ for any $W \in \tau_{I^\ell R}$.

Note that $(\text{Spec R}, \tau_{I^\ell R})$ = Spec R, if R is commutative or left noetherian. In general, the set I^ℓ_R (and, consequently, $\tau_{I^\ell R}$) may be rather meagre.

18. Semiprime rings and modules. Denote by $I_\ell^{\natural} R$ the set of all the proper ideals $n \in I_\ell R$ such that n_s is a semiprime ideal; i.e. n_s coincides with its lower Baire radical $\natural(n_s) = \cap\{p \mid p \in V(n_s)\}$. Denote by $R\text{-}\natural mod$ the full subcategory of the category $R\text{-}mod$, formed by all the modules M such that $Ann\,\xi \in I_\ell^{\natural} R$ for every $\xi \in M \setminus \{0\}$. The modules from $R\text{-}\natural mod$ will be called semiprime .

Proposition. 1) *The set $I_\ell^{\natural} R$ contains together with any family of ideals their intersection.*

2) *The category $R\text{-}\natural mod$ contains together with any family of modules its product and all the submodules of every module.*

3) *The following properties of a twosided ideal n are equivalent:*

(a) n *is semiprime;*

(b) $(n : R) = n$ and $(n : x) \in I_\ell^{\natural} R$

for every $x \in R$;

(c) $(n : R) = n$ and the left module R/n belongs to $R\text{-}\mathfrak{P}\mathrm{mod}$.

Proof. 1) (i) $(\bigcap\{m^j \mid j \in J\})_s = \bigcap\{m^j_s \mid j \in J\}$ for every family of left ideals $\{m^j \mid j \in J\}$.

Indeed, $((\bigcap\{m^j \mid j\in J\}) : R) = \bigcap\{(m^j : R) \mid j \in J\}$;

consequently,

$$(\bigcap_{j\in J} m^j)_s = (\bigcap_{j\in J} m^j) \cap ((\bigcap_{j\in J} m^j) : R) = \bigcap_{j\in J}(m^j \cap (m^j : R)) = \bigcap_{j\in J} m^j_s .$$

(ii) Clearly, $\bigcap_{j\in J} m^j_s \subset \mathfrak{P}(\bigcap_{j\in J} m^j_s) \subset \bigcap_{j\in J} \mathfrak{P}(m^j_s)$.
Therefore, since $m^j_s = \mathfrak{P}(m^j_s)$ for all $j \in J$, then $\bigcap\{m^j_s \mid j \in J\}$ is semiprime

2) Let $\{M_i \mid i \in I\}$ be a family of R-modules and $\xi = (\xi_i) \in \prod_{i\in I} M_i$. If $\mathrm{Ann}\,\xi_i \in \mathcal{I}_\ell^{\mathfrak{P}} R$ for every $i \in I$, then $\mathrm{Ann}\,\xi = \bigcap_{i\in I} \mathrm{Ann}\,\xi_i \in \mathcal{I}_\ell^{\mathfrak{P}} R$ due to 1).

The semiprimeness of the submodules of a semiprime module is obvious.

3) (a)$\Rightarrow$(b). It is clear that $V(n') = V((n' : R))$ for every $n' \in \mathcal{I}R$. Therefore $n = \mathfrak{P}((n : R))$ and, consequently, $(n : R) = n$, if $n = \mathfrak{P}(n)$.

Let $n \in \mathcal{I}R$ and $x \in R$; set $U(x) \overset{def}{=} \{p \in \mathrm{Spec}R \mid x \notin p\}$.
It is easy to see that $V(n) \cap U(x) = V((n : (R,x))) \cap U(x$; consequently,

$$V(n) = (V(n) \cap U(x)) \cup (V(n) \cap V(x)) = (V((n : (R,x))) \cap U(x)) \cup (V(n) \cup V(x)) \quad (1)$$

For any $W \subset \mathrm{Spec}R$ let $\tau(W) = \bigcap\{p \mid p \in W\}$. It follows from (1) that

$$\mathfrak{P}(n) = \tau(V((n : (R,x))) \cap U(x)) \cap \tau(V(n) \cap V(x)) \quad (2)$$

Now note that $(n : (R,x)) = (n : x)_s$ and

$$\mathfrak{P}((n : x)_s) \subset \tau(V((n : x)_s) \cap U(x)),\ (R,x) \subset \tau(V(n) \cap V(x)).$$

This and (2) imply that $\mathfrak{F}((n:x)_s)\cdot(R,x)\subset\subset\mathfrak{F}(n)$. This means, that if $n=\mathfrak{F}(n)$, then $\mathfrak{F}((n:x)_s)\subset(n:(R,x))=(n:x)_s$; i.e. $(n:x)_s$ is a semiprime ideal.

(b) $\Rightarrow$ (a). If $(n:x)\in I_\ell^{\mathfrak{F}}R$ for every $x\in R$, then $(n:R)=\bigcap\{(n:x)\mid x\in R\}\subset I_\ell^{\mathfrak{F}}R$, as claimed in 1).

The implications (b)$\Longleftrightarrow$(c) follow from the definition of $R-{}^{\mathfrak{F}}\mathrm{mod}$. □

Corollary. The left R-module R is semiprime if and only if the ring R is semiprime (i.e. when 0 is a semiprime ideal).

Proof Follows from heading 3) of Proposition 18. □

19. Semiprime ideals and ω-sheaves.

Lemma. Let $(R,\mathcal{T})$ be a $\amalg$-semischeme; $\mathfrak{Ad}\,\mathcal{T}$ the set of all the $\mathcal{T}$-admissible left ideals;
$\wedge\mathcal{T}\overset{def}{=}\bigcap\{\mathcal{F}\mid\mathcal{F}\in\mathcal{T}\}$.

Then $\mathfrak{Ad}\,\mathcal{T}\circ\wedge\mathcal{T}\subset\mathfrak{Ad}\,\mathcal{T}$.

Proof. Let $\nu\in\mathfrak{Ad}\,\mathcal{T}\circ\{m\}$, where $m\in\wedge\mathcal{T}$.

Then $[\nu\in\mathcal{F}\amalg\mathcal{G}]\Rightarrow[(\nu:x)\in\mathcal{F}\amalg\mathcal{G}$ for any $x\in\mathcal{P}(m)]\Rightarrow$
$\Rightarrow(\nu:x)\in\mathcal{F}\circ\mathcal{G}$ for any $x\in\mathcal{P}(m)$; i.e., $\nu\in\mathcal{F}\circ\mathcal{G}\circ\{m\}\subset\mathcal{F}\circ\mathcal{G}$, since $\wedge\mathcal{T}\subset\mathcal{G}$, and, therefore $\mathcal{G}\circ\wedge\mathcal{T}=\mathcal{G}]$. □

Proposition. Let Ω be a multiplicative set of twosided ideals of R such that $(R,\mathcal{T}_\Omega)$ is a symmetric semischeme. Then all the ideals from $I_\ell^{\mathfrak{F}}R$ are admissible.

Proof. Let α and β be twosided ideals such that the radical filter ${}^{\alpha\beta}\hat{\mathcal{F}}$, generated by $\alpha\beta$, is symmetric.

Then for every $\nu \in I_\ell R$ we have $[\nu \in {}^{\alpha}\hat{\mathcal{F}} \vee {}^{\beta}\hat{\mathcal{F}} = {}^{\alpha\beta}\hat{\mathcal{F}}] \Leftrightarrow$
$\Leftrightarrow [\nu_\xi \in {}^{\alpha\beta}\hat{\mathcal{F}}] \Leftrightarrow [\alpha\beta \subset \natural(\nu_\xi)]$. If $\nu \in I_\ell^{\natural} R$, i.e. $\nu_\xi = \natural(\nu_\xi)$,
then these implications may be extended:
$\Leftrightarrow [\nu \in {}^{\alpha\beta}\mathcal{F} = {}^{\alpha}\mathcal{F} \circ {}^{\beta}\mathcal{F}] \Rightarrow [\nu \in {}^{\alpha}\hat{\mathcal{F}} \circ {}^{\beta}\hat{\mathcal{F}}]$. □

Corollary 1. Under the conditions of the proposition 19 all the ideals from $I_\ell^{\natural} R \cup I_\ell^{\natural} R \circ \Lambda \mathcal{T}_\Omega$ are $\mathcal{T}_\Omega$-admissible.

This fact follows directly from the proposition and from the lemma. □

Corollary 2. Suppose that the conditions of Proposition 19 are satisfied.

If M is an R-module, such that $Ann\, \xi \in I_\ell^{\natural} R \cup I_\ell^{\natural} R \circ \Lambda \mathcal{T}_\Omega$ for every $\xi \in M \setminus \{0\}$, then $\tilde{M}_{\tau_\Omega}$ is an ω-sheaf. In particular, M_{τ_Ω} is an ω-sheaf, if M is a semiprime module, and $\tilde{R}_{\tau_\Omega}$ is an ω-sheaf when R is a semiprime ring.

The statement follows from Propositions 5 and 19 . □

Corollary 3. Let R be a ring with right unit, Ω a multiplicative subset of IR such that ${}^{\alpha}\hat{\mathcal{F}}$ is a symmetric filter for every $\alpha \in \Omega$. Then for every semiprime R-module M the canonical arrow $M \longrightarrow \Gamma \tilde{M}^{a}_{\tau_\Omega}$ is an isomorphism.

Proof. If M is a semiprime module, then $\tilde{M}_{\tau_\Omega}$ is an ω-sheaf according to Corollary 2. Proposition 17 implies that the camonical map $M \longrightarrow \Gamma \tilde{M}^{a}_{\tau_\Omega}$ is an isomorphism. Now note that $\{R\}^{1} M \simeq M$ if M is a semiprime module. □

Corollary 4. Let R be a ring such that $\tau_{I'R}$ coincides with the topology of Spec R (see Example 17); i.e., for every $\alpha \in \mathrm{I}R$ there exists a family $\{\alpha_i \mid i \in J\} \subset \mathrm{I}^\ell R$ such that $\alpha = \sup\{\alpha_i \mid i \in J\}$.

Then for any semiprime R-module M there exists a unique up to isomorphism sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ such that $\widetilde{M}(V(\alpha)) \simeq G_{\alpha\widehat{\mathfrak{F}}} M$ for every $\alpha \in \mathrm{I}^\ell R$.

Corollary 5. Let R be a left noetherian ring. Then for every semiprime R-module M the canonical presheaf $\widetilde{M}$ is a sheaf. In particular, $\widetilde{R}$ is a sheaf, if R is semiprime.

Corollary 4 follows from Corollary 2 and Example 17; Corollary 5 is a particular case of Corollary 4. □

Remark. The strongest, as far as I know, of the published by now statements on the structural presheaves on a prime spectrum is due to Van Oystaeyen and Vershoren; used it in their monograph [4] as a starting point for developing of a non-commutative analogue of algebraic geometry. In our notation their statement sounds as follows:

Theorem. Let R be a left noetherian ring. If M is an R-module such that $(\mathrm{Ann}\,\xi)_s = 0$ for every $\xi \in M \setminus \{0\}$, then the structural presheaf $\widetilde{M}$ is a sheaf. In particular, R is a sheaf if R is a prime ring (recall that the ring R is called prim if its zero ideal is prim).

This theorem follows in fact from the symmetricity of the filters $\widehat{\mathfrak{F}}_\alpha$, $\alpha \in \mathrm{I}R$, and Corollary 1 of Proposition 3. Corollaries of Proposition 18, and, in particular, Corollary 5 are considerably stronger. □

Example. As is well known [6] the universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of zero characteristic is left- (and right-) noetherian and semiprime. Therefore, to $U(\mathfrak{g})$ Corollary 5 is applicable and we see that the canonical presheaf $\widetilde{\mathcal{U}(\mathfrak{g})}$ over $\operatorname{Spec} \mathcal{U}(\mathfrak{g})$ is a sheaf. Since $\mathcal{U}(\mathfrak{g})$ is unitary, the natural map $\mathcal{U}(\mathfrak{g}) \to \Gamma \widetilde{\mathcal{U}} (\mathfrak{g})$ is a ring isomorphism. $\square$

§ 5. Geometrizations of the left spectrum.

1. Localizations at points of $\mathrm{Spec}_l R$. The left spectrum $\mathrm{Spec}_l R$ that had come into existence in examples of section 1 and had appeared until now only occasionally, will be playing hereafter (together with its subset $\widehat{\mathrm{Spec}}_l R$) the most considerable role. For the convenience of the reader list the relations, the principal part of which is established in section 1:

$$\begin{array}{ccccccc} I_\ell^w R & \supset & \mathrm{Spec}_\ell R & \supset & \widehat{\mathrm{Spec}}_\ell R & \supset & \widehat{\widehat{\mathrm{Spec}}}_\ell R \\ \cup & & & & \cup & & \\ I_\ell^{reg} R & \supset & \mathrm{Max}_\ell^{reg} R & \subset & \mathrm{Max}_\ell R \cap I_\ell^w R & & \end{array}$$

$$\widehat{\mathrm{Spec}}_\ell R \cap IR = \mathrm{Spec}_\ell R \cap IR \subset \mathrm{Spec} R$$

Here $I_\ell^{reg} R$ is the set of all the regular left ideals of R, $\mathrm{Max}_\ell^{reg} R = \mathrm{Max}_\ell R \cap I_\ell^{reg} R$ the set of the maximal regular left ideals--the annihilators of non-zero elements of the simple R-modules.

For any $p \in \mathrm{Spec}_\ell R$ denote by j_p the canonical ring morphism $R \longrightarrow G_{\mathcal{F}_p} R$.

Proposition. 1) Let $\mathcal{F}$ be a radical filter of left ideals of R. For any $n \in I_\ell R$ the preimage $j_{\mathcal{F},R}^{-1}(G_{\mathcal{F}} n)$ of the ideal $G_{\mathcal{F}} n$ coincides with the $\mathcal{F}$-saturation $n_{\mathcal{F}} \overset{def}{=} \{\lambda \in R \mid (n:\lambda) \in \mathcal{F}\}$ of n (see 4.4); in particular, $G_{\mathcal{F}} n = G_{\mathcal{F}} n_{\mathcal{F}}$.

2) For any $p \in \mathrm{Spec}_\ell R$ the ideal $j_p^{-1}(G_{\mathcal{F}_p} p)$ coincides with $\hat{p} \overset{def}{=} \{\lambda \in R \mid (p:\lambda) \not\succ p\}$ (by Proposition 1.6 $\hat{p}$ belongs to $\widehat{\mathrm{Spec}}_\ell R$ and is isomorphic to p).

Proof. 1) For any submodule N of an R-module M the submodule $j_{\mathcal{F},M}^{-1}(G_{\mathcal{F}} N)$ is equal to the $\mathcal{F}$-saturation $N_{\mathcal{F}} = \{\xi \in M \mid m\cdot\xi \subset N$ for some $m \in \mathcal{F}\}$ of N.

Clearly, the image of $N_{\mathcal{F}}$ by the canonical morphism $M \to H_{\mathcal{F}} M$ belongs to the submodule $H_{\mathcal{F}} N$; therefore $j_{\mathcal{F},M}(N_{\mathcal{F}}) \subset G_{\mathcal{F}} N$.

For any $\xi\in j^{-1}_{\mathcal{F},M}(G_{\mathcal{F}}\mathcal{N})$ one can find an ideal $m\in\mathcal{F}$ such that $m\cdot j_{\mathcal{F},M}(\xi)\subset j_{\mathcal{F},M}(\mathcal{N})$. Therefore for every $x\in\mathcal{P}(m)$ there exists $n_x\in\mathcal{F}$ such that $n_x x\xi\in\mathcal{N}$. This means that the left ideal $(\mathcal{N}:\xi)\overset{\mathrm{def}}{=}\{z\in R\mid z\cdot\xi\in\mathcal{N}\}$ belongs to $\mathcal{F}\circ\{m\}\subset\mathcal{F}\circ\mathcal{F}$. Since $\mathcal{F}$ is radical, $(\mathcal{N}:\xi)\in\mathcal{F}$; i.e. $\xi\in\mathcal{N}_{\mathcal{F}}$.

2) $j_p^{-1}(G_{\mathcal{F}_p}p)=\widehat{p}$, since $\widehat{p}=p_{\mathcal{F}_p}$. According to Proposition 1.6 the inclusion $p\subset\widehat{p}$ is an isomorphism in $I^{\succ}_{\ell}R$ for every $p\in\mathrm{Spec}_1R$. □

2. Left quasilocal rings. One may consider the contents of this subsection as a sequel of 2.10.

Definitions. 1) A left ideal m will be called *final* if $\mathcal{F}_m=\{R\}$ and *quasifinal* if $\mathcal{F}_m$ consist of q-non-proper ideals (see 2.10).

2) A ring possessing a (quasi)final ideal will be called a *left (quasi)local* ring.

3) Rings with the unique maximal two-sided ideal will be called *symmetrically local*. (In literature such rings are called local as well as the rings with the unique maximal one-sided ideal. In the topics that we study here the difference is too essential to be ignored.)

4) A ring possessing a proper ideal wich contains any q-proper two-sided ideal will be called *symmetrically quasilocal*. □

Therefore, the final ideals of R are the finite objects of the full subcategory of the category $\mathcal{I}_\ell^{\flat} R$ formed by the proper ideals. This means that all of them are isomorphic to each other and belong to $\mathrm{Spec}_\ell R$.

The quasifinal ideals are exactly the left ideals $\mu \neq R$ such that $R \simeq G_{\mathcal{F}_\mu} R$; or, equivalently, such that μ majorates any q-proper ideal n, i.e. $n \to \mu$. Note that a quasifinite ideal itself may be not q-proper.

If μ is a (quasi)finite ideal of R and m is an arbitrary q-proper ideal, then $m_s \overset{\mathrm{def}}{=} m \cap (m : R) \subset \mu$.
In other words, every ring (quasi)local from the left is (quasi)-locally symmetric.

Denote by $\mathrm{Spec}_\ell^{s\ell} R$ the set of all $p \in \mathrm{Spec}_\ell R$ such that for every proper left ideal m of R the ideal $G_{\mathcal{F}_p} m$ is also proper; and let $\mathrm{Spec}_\ell^{bi} R$ be the subset of all the ideals of the left spectrum of R such that for every proper twosided ideal m of $G_{\mathcal{F}_p} R$ the ideal $G_{\mathcal{F}_p} m$ is also proper. Obviously, $\mathrm{Spec}_\ell^{s\ell} R \subset \mathrm{Spec}_\ell^{bi} R$.

Proposition. 1) *For any* $p \in \mathrm{Spec}_\ell R$ *the ring* $G_{\mathcal{F}_p} R$ *is quasilocal with quasifinal ideal.*

2) *The following properties of an ideal* $p \in \mathrm{Spec}_\ell R$ *are equivalent:*

(a) $p \in \mathrm{Spec}_\ell^{s\ell} R$;

(b) $G_{\mathcal{F}_p} m = m$ *for every left ideal* m *of* $G_{\mathcal{F}} R$;

(c) *if* $\mu \in \mathrm{Max}_\ell G_{\mathcal{F}_p} R$, *then* $G_{\mathcal{F}_p} \mu = \mu$;

(d) *the functor* $G_{\mathcal{F}_p}$ *is exact;*

(e) $G_{\mathcal{F}_p} R$ *is a left local ring with a final ideal* μ *such that* $G_{\mathcal{F}_p} \mu$ *is a proper ideal;*

(f) $G_{\mathcal{F}_p}R$ is a left local ring with final ideal $G_{\mathcal{F}_p}p$.

3) The following properties of an ideal $p \in \mathrm{Spec}_\ell R$ are equivalent:

(g) $p \in \mathcal{S}pec_\ell^{Bi} R$;

(h) $G_{\mathcal{F}_p}R$ is a symmetrically local ring with maximal twosided ideal $(G_{\mathcal{F}_p}p)_s$.

Proof. A) Let m be a left ideal of $G_{\mathcal{F}_p}R$ and $G_{\mathcal{F}_p}m$ be a proper ideal. By Proposition 2.7 the latter statement is equivalent to the relation $j_p^{-1}m \in \mathcal{F}_p$; i.e., $j_p^{-1}m \to p$. By Corollary 1 this (and the equality $G_{\mathcal{F}_p}m = G_{\mathcal{F}_p}(j_p^{-1}m)$) imply $m \subset G_{\mathcal{F}_p}(j_p^{-1}m) \to G_{\mathcal{F}_p}p$.

B) If under the conditions of step A) m is a twosided ideal, then $j_p^{-1}m \subset p$ (since $j_p^{-1}m$ is a twosided ideal) and, consequently, $m \subset G_{\mathcal{F}_p}(j_p^{-1}m) \subset G_{\mathcal{F}_p}p$.

1) If m is a q-proper left ideal of $G_{\mathcal{F}_p}R$, then $G_{\mathcal{F}_p}m$ is a proper ideal by Proposition 2.10. Therefore (see A)) $m \to G_{\mathcal{F}_p}p$.

2) The equivalence of the properties (a) - (d) is a specialization of Proposition 11; the implication (f)$\Rightarrow$(e) is trivial; (b)$\Rightarrow$(f) follows from A).

(e)$\Rightarrow$(a). Let $\mathcal{M}$ be a final ideal for which $G_{\mathcal{F}_p}\mathcal{M}$ is a proper ideal. According to Corollary 1 of proposition 2.9, $[n \to \mathcal{M}] \Rightarrow [G_{\mathcal{F}_p}n \to G_{\mathcal{F}_p}\mathcal{M}]$. This means that $G_{\mathcal{F}_p}n$ is a proper ideal, if so is n .

3) The third statement follows from step B) of the proof. □

3. Left radical. For any subset $W \subset \mathrm{Spec}_\ell R$ denote by $r(W)$ the intersection of all the ideals from W if $W \neq \emptyset$; let $r(\emptyset) = R$. For every left ideal n denote by $V_\ell(n)$ the set $\{p \in \mathrm{Spec}_\ell R \mid n \to p\}$. The left radical of R is the function assigning to an ideal $n \in I_\ell R$ the ideal $\mathrm{rad}_\ell(n) \overset{\mathrm{def}}{=} r(V_\ell(n))$.

Similarly we may consider the composition of r with the map $\hat{V}_\ell : n \mapsto \hat{V}_\ell(n) \stackrel{\text{def}}{=} V_\ell(n) \cap \hat{\mathrm{Spec}}_\ell R$. But in this way we get nothing new as to the following Proposition states

Proposition. $\mathrm{rad}_\ell(n) = \bigcap\{p_s \mid p \in V_\ell(n)\} = r(\hat{V}_\ell(n))$ for any $n \in I_\ell R$, and the map $n \mapsto \mathrm{rad}_\ell(n)$ is a functor from $I_\ell^{\succ} R$ into IR.

Proof. Since together with every ideal p the set $\hat{V}_\ell(n)$ contains all the ideals $(p:x)$, $x \in R \setminus p$, then

$$r(\hat{V}_\ell(n)) = \bigcap\{p \mid p \in \hat{V}_\ell(n)\} = \bigcap_{p \in \hat{V}_\ell(n)} \bigcap\{(p:x) \mid x \in R\} = \bigcap\{p_s \mid p \in V_\ell(n)\},$$

since $p'_s = (p' : R)$ for every $p' \in I_\ell^w R$, and $\mathrm{Spec}_\ell R \subset I_\ell^w R$.

Clearly, $r(\hat{V}_\ell(n)) \supset r(V_\ell(n)) \stackrel{\text{def}}{=} \mathrm{rad}_\ell(n)$. On the other hand, $r(\hat{V}_\ell(n)) \subset \bigcap\{\hat{p}_s \mid p \in V_\ell(n)\}$, where $\hat{p} = \{\lambda \in R \mid (p:\lambda) \not\to p\}$ and $\hat{p}_s \subset p$, since $\hat{p} \to p$ according to Proposition 1.6. Therefore $\hat{p}_s = p_s$, and, consequently,

$$r(\hat{V}_\ell(n)) = \bigcap\{p_s \mid p \in V_\ell(n)\} = \mathrm{rad}_\ell(n)$$

The second statement of the proposition is now obvious. □

Corollary 1. If n is a twosided ideal, then

$$\mathfrak{B}(n) \subset \mathrm{rad}_\ell(n) \subset J(n)$$

(Here J(n) is Jackobson radical of an ideal n , i.e. the intersection of all the maximal left regular ideals of R, containing n, and $\mathfrak{B}(n) = \bigcap\{p \mid p \in \mathrm{Spec} R,\ n \subset p\}$ is the lower Baire radical of n).

2) If R is a Jackobson ring then $\mathfrak{B}(n)$, $\mathrm{rad}_\ell(n)$ and $J(n)$ coincide for every $n \in$ IR.

Proof. 1) For any $p \in \mathrm{Spec}_\ell R$ the ideal p_s belongs to Spec R.

Let $\{\alpha, \beta\} \subset IR$, $\alpha\beta \subset p_s$ and $\alpha \not\subset p_s$. Then

$(p:x)\not\subset p$ for every $x \in \mathcal{P}(\beta)$. It follows $\beta \subset p$, since $p \in \mathrm{Spec}_\ell R$ and β is a twosided ideal.

Thus, $\beta(n) \subset \mathrm{rad}_\ell(n)$. On the other hand, since all the regular maximal left ideals of R belong to Spec_ℓ R, then $J(n) \subset \mathrm{rad}_\ell(n)$.

2) Recall that R is called a Jackobson ring, if every prime ideal in it equals the intersection of the prime ideals (i.e. the annihilation of irreducible R-modules). The primitive ideals, in their turn, are the intersections of maximal left regular ideals (annihilators of elements of irreducible R-modules). Therefore, for the Jackobson rings $\beta(n) = J(n)$ for all $n \in$ IR. □

4. Topologies on the left spectrum.

Lemma. *The following properties of the subset* W *of the left spectrum are equivalent:*

(i) $W = \mathrm{Spec}_\ell R \cap \mathcal{F}$ *for a uniform filter* $\mathcal{F}$ *of left ideals;*

(ii) $W = \mathrm{Spec}_\ell R \cap \mathcal{F}'$ *for a radical filter* $\mathcal{F}'$;

(iii) $W = \bigcup\{V_\ell(p) \mid p \in W\}$.

Proof. (iii)$\Rightarrow$(i). Clearly, the union of an arbitrary family of the uniform filters is a uniform filter; and a set W, satisfying (iii) represents in the form $\mathrm{Spec}_\ell R \cap (\bigcup\{{}^p\mathcal{F} \mid p \in W\})$.

(i)$\Rightarrow$(ii). $\mathrm{Spec}_\ell R \cap \mathcal{F} = \mathrm{Spec}_\ell R \cap \hat{\mathcal{F}}$

for any uniform filter $\mathcal{F}$, where $\hat{\mathcal{F}}$ is the radical closure of $\mathcal{F}$. To see it, one can just follow the arguments from the demonstration of Proposition 4.9 replacing of "topologizing" by "uniform" (or note that for every uniform filter $\mathcal{F}$ the set $\mathcal{F}^{(\infty)} \stackrel{\mathrm{def}}{=} \bigcup_{n \geq 1} \mathcal{F} \circ \ldots \circ \mathcal{F}$ is a topologizing filter, as can be seen from Proposition 1.3, note also that $\hat{\mathcal{F}} = \hat{\mathcal{F}}^{\infty}$ and refer to Proposition 4.9).

(ii) $\Rightarrow$ (iii). Obvious. $\square$

Corollary. If a set $W \subset \mathrm{Spec}_\ell R$ satisfies any of the conditions of Proposition 4, then

$$\tau(W) = \bigcap\{p_s \mid p \in W\}.$$

Indeed, since $W = \bigcup\{V_\ell(p) \mid p \in W\}$, then $\tau(W) = \bigcap\{\tau(V_\ell(p)) \mid p \in W\}$ and $\tau(V_\ell(p)) = p_s$. $\square$

Denote by $\mathfrak{T}_0$ the collection of the subsets of Spec R enjoying the equivalent properties of Proposition 4. It is directly verified that $\mathfrak{T}_0$ is closed with respect to arbitrary intersections and unions. In particular, $\mathfrak{T}_0$ may be considered as the collection of the closed sets of a topology which will be also denoted by $\mathfrak{T}_0$.

Clearly, the closure of a point $p \in \mathrm{Spec}_\ell R$ in the topology $\mathfrak{T}_0$ coincides with $V_\ell(p) = \{p' \in \mathrm{Spec}_\ell R \mid p \to p'\}$, and the closure $\overline{W} = {}^{\mathfrak{T}_0}\overline{W}$ of an arbitrary subset $W \subset \mathrm{Spec}_\ell R$ is the union of the closures of its points. This implies that $\mathfrak{T}_0$ does not distinguish points isomorphic in $\mathrm{I}_\ell^{\,t} R$; in particular, the embedding $\mathrm{Sp\hat{e}c}_\ell R \hookrightarrow \mathrm{Spec}_\ell R$ and the map $\mathrm{Spec}_\ell R \longrightarrow \mathrm{Sp\hat{e}c}_\ell R$, $p \mapsto \hat{p}$ (see 1.6) are quasihomeomorphisms. Recall, that a continuous map $f : X \to Y$ is called a quasihomeomorphism, if the "inverse" map of the sets $W \mapsto f^{-1}W$ induces a bijection $\mathrm{cl}\,Y \xrightarrow{\sim} \mathrm{cl}\,X$.

Topology $\mathfrak{T}_1$. Consider the family $\mathfrak{T}_\ell$ of all the sets of the form $V_\ell(n) = \mathrm{Spec}_\ell R \cap {}^n\mathcal{F}$, $n \in \mathrm{I}_\ell R$. Note that

$$V_\ell(n) \cup V_\ell(m) = V_\ell(n \cap m) \qquad (1)$$

for any pair n, m of left ideals of R. Indeed, ${}^{n}\mathcal{F} \cup {}^{m}\mathcal{F} =$
$= \{\nu \in I_\ell R \mid n \to \nu$, or $m \to \nu\} \subset \{\nu \in I_\ell R \mid n \cap m \to \nu\} \stackrel{\mathrm{def}}{=} {}^{n\cap m}\mathcal{F}$
(here, as usual, $\to$ is a morphism of $I_\ell^{\succ} R$). This implies
$V_\ell(n) \cup V_\ell(m) \subset V_\ell(n \cap m)$. On the other hand, since $n \cap m \in$
$\in {}^{n}\mathcal{F} \circ {}^{m}\mathcal{F}$, we have
$V_\ell(n \cap m) = Spec_\ell R \cap {}^{n\cap m}\mathcal{F} \subset Spec_\ell R \cap {}^{n}\mathcal{F} \circ {}^{m}\mathcal{F} =$
$= (Spec_\ell R \cap {}^{n}\mathcal{F}) \cup (Spec_\ell R \cap {}^{m}\mathcal{F}) = V_\ell(n) \cup V_\ell(m)$.

Let $\mathfrak{T}_1$ be a collection of all the possible intersections of the sets from $\mathfrak{T}_\ell$. Since $\mathfrak{T}_\ell$ is closed with respect to finite unions, then so is $\mathfrak{T}_1$. Consequently, $\mathfrak{T}_1$ is a basis of closed sets of a topology, which will be also denoted by $\mathfrak{T}_1$.

Though $\mathfrak{T}_1$ is usually weaker than $\mathfrak{T}_0$, the closures of a point in both topologies coincide: $\overline{\{p\}} = V_\ell(p) = \{p' \mid p \to p'\}$.

Topology $\mathfrak{T}$. Denote by $\mathfrak{T}$ the subset of $\mathfrak{T}_\ell$, whose elements are the sets $V_\ell(\alpha)$, $\alpha \in IR$. Since $V_\ell(\alpha) = \{p \in Spec_\ell R \mid \alpha \subset p\}$ for every $\alpha \in IR$, then $\bigcap_{i \in J} V_\ell(\alpha_i) = V_\ell(\sup_{i\in J} \alpha_i)$ for any family $\{\alpha_i \mid i \in J\} \subset IR$. This and (1) imply that $\mathfrak{T}$ is the collection of the closed set of a topology, which will be also denoted by $\mathfrak{T}$. The closure of a point p in $\mathfrak{T}$ coincides with V (p_1). For any $\{\alpha, \alpha'\} \subset IR$ we have

$$[V_\ell(\alpha) \subset V_\ell(\alpha')] \Longleftrightarrow [rad_\ell(\alpha) \supset \alpha'].$$

It follows that for any $\alpha \in IR$ the set $V_\ell(\alpha)$ coincides with $V_\ell(rad_\ell(\alpha))$, and the assigning $\alpha \mapsto Spec_\ell R \setminus V_\ell(\alpha) \stackrel{\mathrm{def}}{=} U_\ell(\alpha)$ and $W \mapsto r(W) \stackrel{\mathrm{def}}{=} \bigcap\{p \mid p \in W\}$ we define mutually inverse isomorphisms between the preorder of the radical ideals of R (i.e. of the ideals, that are equal to their left radical ~~ideal~~) and the preorder of the open sets of $(\mathrm{Spec}_\ell R, \mathfrak{T})$.

So, on $\mathrm{Spec}_\ell\, R$ the three topologies are defined, namely, $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$. In general case the majorations $\mathfrak{T} \prec \mathfrak{T}_1 \prec \mathfrak{T}_0$ are strict. Let for every $\nu \in I_\ell R$ there exists $x \in \mathcal{P}(R)$ such that $n_s = (n : x)$ (the rings with this property are one of the characters of § 8); e.g. the quotient of R modulo a (commutative) twosided ideal, i.e. a left-artinian ring. Then $\mathfrak{T}_1$ and $\mathfrak{T}$ coincide. If R is left artinian, then all the three topologies coincide, since left artinian property is equivalent to the following one :

any topologizing filter of left ideals is of the form ${}^m\mathfrak{F} = \{ n \in I_\ell R \mid m \subset n \}$ for a twosided ideal m.

5. Structural presheaves and sheaves. For an arbitrary subset X of $\mathrm{Spec}_\ell\, R$ denote by $\mathfrak{F}_X$ the intersection $\bigcap \{ \mathfrak{F}_p \mid p \in \mathrm{Spec}_\ell R \setminus X = X^\perp \}$. The sets $\mathfrak{F}_X$ are the radical filters (since the intersection of any family of radical filters is a radical filter), which are maximal among the uniform sets $\mathcal{G}$ such that $\mathcal{G} \cap \mathrm{Spec}_\ell R \subset X$. Clearly, the map $W \mapsto \mathfrak{F}_W$, $W \in \mathfrak{T}_0$, is a section of the projection $\mathfrak{F} \mapsto V(\mathfrak{F}) = \mathrm{Spec}_\ell R \cap \mathfrak{F}$

To each left R-module M we assign the presheaf ${}^0\mathcal{O}_M : W \mapsto G_{\mathfrak{F}_W}$ over $(\mathrm{Spec}_\ell\, R, \mathfrak{T}_0)$. By the symbols ${}^1\mathcal{O}_M$ and $\mathcal{O}_M$ denote the restrictions of the presheaf ${}^0\mathcal{O}_M$ on $\mathfrak{T}_1$ and $\mathfrak{T}$, respectively. Let us find out what do the stalks of ${}^0\mathcal{O}_M^a$, ${}^1\mathcal{O}_M^a$ and $\mathcal{O}_M^a$ at the points of $\mathrm{Spec}_\ell\, R$ look like. (For any presheaf F is denoted by F^a the sheaf associated with F; recall, that the stalk of F^a at a point x is calculated by the formula $F_x^a = \varinjlim\{FV \mid x \in V, V \in \mathcal{c}\ell X\}$.)

A) The stalk of ${}^0\mathcal{O}_{\mathcal{M}}^a$ at any point $p \in \mathrm{Spec}_\ell\, R$ is isomorphic to the R-module $G_{\mathcal{F}_p}\mathcal{M}$.

Indeed, ${}^0\mathcal{O}_{\mathcal{M},p}^a = \varinjlim\{G_{\mathcal{F}_{V(\mathcal{G})}}\mathcal{M} \mid p \in \mathcal{G}\} = \varinjlim\{G_{\mathcal{F}_{V(\mathcal{G})}}\mathcal{M} \mid \mathcal{G} \subset \mathcal{F}_p\} = G_{\mathcal{F}_p}\mathcal{M}$, since $\mathcal{F}_{V(\mathcal{F}_p)} = \mathcal{F}_p$.

B) The stalks of ${}^1\mathcal{O}_{\mathcal{M}}^a$. Here the situation is somewhat more complicated. First of all, note that $\mathcal{F}_p = \bigcup\{\mathcal{F}_{V_\ell(n)} \mid n \in \mathcal{F}_p\}$, since $n \in \mathcal{F}_{V_\ell(n)}$ for any $n \in I_\ell R$ and $\mathcal{F}_{V_\ell(n)} \subset \mathcal{F}_p$ if $n \in \mathcal{F}_p$. Unfortunately, this does not imply that the canonical arrow ${}^1\mathcal{O}_{\mathcal{M},p}^a = \varinjlim\{G_{\mathcal{F}_{V_\ell(n)}}\mathcal{M} \mid n \in \mathcal{F}_p\} \to G_{\mathcal{F}_p}\mathcal{M}$ is an isomorphism. We have at our disposal only Proposition 2.12, which enables us to claim the following:

The canonical arrow ${}^1\mathcal{O}_{\mathcal{M},p}^a \to G_{\mathcal{F}_p}\mathcal{M}$ is injective. It is an isomorphism if $G_{\mathcal{F}_p}\mathcal{M} \simeq H_{\mathcal{F}_p}\mathcal{F}^1_{V_\ell(n)}\mathcal{M}$ for some $m \in \mathcal{F}_p$. The latter surely happens if the directed with respect to the inclusion set of the submodules $\{\mathcal{F}_{V_\ell(n)}\mathcal{M} \mid n \in \mathcal{F}_p\}$ of M stabilizes; i.e. $\bigcup\{\mathcal{F}_{V_\ell(n)}\mathcal{M} \mid n \in \mathcal{F}_p\} = \mathcal{F}_{V_\ell(m)}\mathcal{M}$ for some $m \in \mathcal{F}_p$. In particular, ${}^1\mathcal{O}_{\mathcal{M},p}^a \simeq G_{\mathcal{F}_p}\mathcal{M}$ for all $p \in \mathrm{Spec}_\ell R$ if M is noetherian.

C) The stalks of $\mathcal{O}_{\mathcal{M}}^a$. For every $\underline{\nu} \in IR$ let $\mathcal{F}_{(\underline{\nu})} \stackrel{\mathrm{def}}{=} \bigcap\{\mathcal{F}_p \mid p \in \mathrm{Spec}_\ell R,\ p_s \subset \underline{\nu}\}$.

It is not difficult to verify that

$$\mathcal{F}_{(\underline{\nu})} = \bigcup\{\mathcal{F}_{V_\ell(\alpha)} \mid \alpha \in IR,\ \alpha \not\subset \nu\}.$$

Indeed, $\mathcal{F}_{V_\ell(\alpha)} = \{n \in I_\ell R \mid p \in \mathrm{Spec}_\ell R$ and $n \to p$ imply $\alpha \subset p\}$ for any $\alpha \in IR$. Hence $\bigcup\{\mathcal{F}_{V_\ell(\alpha)} \mid \alpha \in IR, \alpha \not\subset \nu\} = \{n \in I_\ell R \mid$ if $\mu \in \mathrm{Spec}_\ell R$ and $n \to \mu$, then $\alpha \subset \mu$ for some $\alpha \in \mathcal{F}_\nu \cap IR\} = \{n \in I_\ell R \mid$ if $\mu \in \mathrm{Spec}_\ell R$ and $n \to \mu$, then $\mu_s \not\subset \nu\} = \mathcal{F}_{(\nu)}$.

Thus, as in the case of the sheaf ${}^1\mathcal{O}^a_M$, Proposition 2.12 yields:

If an R-module M satisfies $G_{\mathcal{F}(p_s)}M \simeq H_{\mathcal{F}(p_s)}\mathcal{F}^1_{V_\ell(\alpha)}M$ for some $\alpha \in \mathrm{I}R \cap \mathcal{F}_p$, then the canonical arrow

$$\mathcal{O}^a_{M,p} = \varinjlim \{G_{\mathcal{F}_{V_\ell(\beta)}}M \mid \beta \in \mathrm{I}R \cap \mathcal{F}_p\} \to G_{\mathcal{F}(p_s)}M$$

is an isomorphism. It follows $\mathcal{O}^a_{M,p} \simeq G_{\mathcal{F}(p_s)}M$, when the directed set of submodules $\{\mathcal{F}_{V_\ell(\alpha)}M \mid \alpha \in \mathrm{I}R \cap \mathcal{F}_p\}$ stabilizes. In particular, $\mathcal{O}^a_{M,p} \simeq G_{\mathcal{F}(p_s)}M$ for all the points $p \in \mathrm{Spec}_\ell R$, if M is a noetherian module ,

Proposition. Let R be left-noetherian. Then for every point $p \in \mathrm{Spec}_\ell R$ and every R-module M the canonical monomorphisms ${}^1\mathcal{O}^a_{M,p} \to G_{\mathcal{F}_p}M$ and $\mathcal{O}^a_{M,p} \to G_{\mathcal{F}(p_s)}M$ are isomorphisms.

After the mentioned above it is clear that this statement is a special case of Corollary 2. □

6. Structural sheaves and quasifinite sets. Denote by ${}_{\mathrm{I}R}\mathrm{Mod}$ the full subcategory of R-mod, formed by R-modules M such that the presheaf $\mathcal{O}_M$ is an ω-sheaf. The symbol $\mathcal{Y}^a_R$: R-mod $\to \mathcal{O}^a_R$-mod will stay for the functor $M \mapsto \mathcal{O}^a_M$.

Proposition. The following properties of a left R-module M are equivalent:

(a) for any $\{\alpha, \beta\} \subset \mathrm{I}R$ the $\mathcal{F}_{V_\ell(\alpha)} \bullet \mathcal{F}_{V_\ell(\beta)}$-torsion of M coincides with its $\mathcal{F}_{V_\ell(\alpha\cap\beta)}$-torsion;

(b) $[\{\alpha, \beta\} \subset \mathrm{I}R$ and the ideal $\{x \in R \mid \mathrm{Ann}(x\xi) \in \mathcal{F}_{V_\ell(\alpha)}\}$ belongs to the filter $\mathcal{F}_{V_\ell(\beta)}] \Rightarrow [\mathrm{Ann}\,\xi \in \mathcal{F}_{V_\ell}(\alpha\cap\beta)]$ for any $\xi \in M$;

(c) $M \in \mathrm{Ob}\,{}_R\mathrm{Mod}$

2) If M satisfies the equivalent conditions of the previous heading, then for every quasifinite closed subset W of $(\mathrm{Spec}_{\ell} R, \mathfrak{T})$ the canonical morphism $G_{\mathcal{F}_W} M \to \mathcal{O}^a_M(W)$ is an isomorphism.

If, besides, the topology $\mathfrak{T}$ possesses a base $\mathcal{B}$ of quasifinite sets, then $\mathcal{O}^a_M(V) = \varprojlim (G_{\mathcal{F}_W} M \mid V \subset W, W \in \mathcal{B})$ for every $V \in \mathfrak{T}$.

3) Let R be a ring with right unit. Then the restriction of the functors $\mathcal{G}_R : M \longmapsto \mathcal{O}_M$ and $\mathcal{G}^a_R : M \mapsto \mathcal{O}^a_M$ onto $R\text{-mod}^{\{R\}}$ are full and faithful functors.

Proof. 1), 2). The first statement is a specialization of Proposition 4.5; the second one follows from the arguments of step 3) of the proof of Proposition 4.17 and from Proposition 3.6 (see also Corollary of Proposition 7).

3) For any morphism of R-modules $f : M \to M'$ and any radical filter $\mathcal{F} \subset I_{\ell} R$ there exists a unique morphism of R-modules $\tilde{f} : G_{\mathcal{F}} M \to G_{\mathcal{F}} M'$, such that the diagram

$$\begin{array}{ccc} G_{\mathcal{F}} M & \xrightarrow{\tilde{f}} & G_{\mathcal{F}} M' \\ {\scriptstyle j_{\mathcal{F},M}} \uparrow & & \uparrow {\scriptstyle j_{\mathcal{F},M'}} \\ M & \xrightarrow{f} & M' \end{array}$$

commutes.

(The uniqueness follows from the "absence" of $\mathcal{F}$-torsion by $G_{\mathcal{F}} M'$). This implies the faithfulness of the restriction of the functor $\mathcal{G}_R : M \longmapsto \mathcal{O}_M$ onto the subcategory $R\text{-mod}^{\mathcal{F}_\varnothing}$ of $\mathcal{F}_\varnothing$-torsion-free modules and the fullness and faithfulness of the restriction of $\mathcal{G}_R$ onto $R\text{-mod}_{\mathcal{F}_\varnothing}$.

Now let R be a ring with right unit. Then the set $Max_\ell R$ of the maximal left ideals of R belongs to Spec_ℓ R and every proper left ideal of R is contained in a maximal left ideal. Therefore the filter $\mathfrak{F}_\emptyset = \{n \in I_\ell R \mid n \not\to p$ for all $p \in \mathrm{Spec}_\ell R\}$ only consists of one ring, R. Besides, as was shown at the second step of the proof of Proposition 4.17, the existence of a right unit in R implies that the canonical arrow $\{R\}^\perp M \longrightarrow G_{\{R\}} M$ is an isomorphism for any M. Thus, $R\text{-}mod_{\mathfrak{F}_\emptyset}$ and $R\text{-}mod^{\mathfrak{F}_\emptyset}$ coincide with the subcategory $R\text{-}mod^{\{R\}}$ of $\{R\}$-torsion-free modules, or, equivalently, with the subcategory of the unitary R-modules (see 2.). The established above fullness and faithfulness of the restriction of the functor $\mathcal{G}_R$ on $R\text{-}mod_{\mathfrak{F}_\emptyset}$ turns out to be the ~~strict~~ fullness and faithfulness of the restriction of the functor $\mathcal{G}_R$ onto the subcategory of the unitary R-modules.

In the following subsection we will show, among other things, that $(\mathrm{Spec}_\ell R, \mathfrak{T})$ is quasicompact. This and heading 2) imply the fullness and faithfulness of the restriction of $\mathcal{G}^a_R$ onto the subcategory $R\text{-}mod^{\{R\}}$. $\square$

A similar statement holds for the geometric representation $M \longmapsto {}^1\mathcal{O}^a_M$ (with IR replaced by $I_\ell R$ in the formulation). Note however, that the category ${}_R{}^1Mod$, whose objects are all the modules M, for which ${}^1\mathcal{O}_M$ is an $\underline{\omega}$-sheaf, is contained in ${}_R Mod$ and usually is much more pour than ${}_R Mod$. Besides, as will be clearly demonstrated in what follows, in general, the amount of quasicompact open subsets of the space $(\mathrm{Spec}_\ell R, \mathfrak{T}_1)$ is considerably smaller than that of $(\mathrm{Spec}_\ell R, \mathfrak{T})$, and it itself is more seldom quasicompact.

7. Quasicompact open sets of $(\mathrm{Spec}_\ell R, \mathfrak{T})$.

Proposition. 1) Let R be a ring with right unit. Then the space $(\mathrm{Spec}_\ell R, \mathfrak{T})$ is quasicompact.

2) The following properties of a twosided ideal α of R are equivalent:

(a) $V_\ell(\alpha)$ is quasifinite;

(b) If $\{\alpha_i \mid i \in J\} \subset IR$ and $\alpha \subset \mathrm{rad}_\ell(\sup\{\alpha_j \mid j \in J\})$, then $\alpha \subset \mathrm{rad}_\ell(\sup\{\alpha_j \mid j \in J_0\})$ for a finite subset $J_0 \subset J$;

(c) if $\{\alpha_i \mid i \in J\}$ is an increasing chain of twosided ideals such that $\alpha_i \subset \alpha$ for all i and $\alpha \subset \mathrm{rad}_\ell(\sup\{\alpha_j \mid j \in J\})$, then $\alpha \subset \mathrm{rad}_\ell(\alpha_{i_0})$ for some i_0;

(d) if $\{\alpha_i \mid i \in J\}$ is an increasing chain of ideals from $IR \setminus \mathfrak{F}_{V_\ell(\alpha)}$, then $\sup\{\alpha_i \mid i \in J\} \notin \mathfrak{F}_{V_\ell(\alpha)}$.

Proof. 1) (i) For a regular ideal n of an arbitrary ring R the following implications hold:

$$[V_\ell(n) = \varnothing] \Leftrightarrow [n = R].$$

Indeed, every regular left ideal n is contained, if it is proper, in a maximal left regular ideal, and maximal regular left ideals belong to $\mathrm{Spec}_\ell R$ (see 1.4.3). So, if $V_\ell(n) = \varnothing$, then $n = R$. Clearly, $V_\ell(R) = \varnothing$.

(ii) If R is a ring with right unit, then all its left ideals are regular. Therefore the equality

$$\cap\{V_\ell(\alpha_i) \mid i \in J\} = V_\ell(\sup\{\alpha_i \mid i \in J\})$$

implies that

$$[\cap\{V_\ell(\alpha_i) \mid i \in J\} = \varnothing] \Leftrightarrow [\sup\{\alpha_i \mid i \in J\} = R].$$

Due to the presence of a right unit $\sup\{\alpha_i \mid i \in J\} = R$ if and only if $\sup\{\alpha_j \mid j\in J_0\}=R$ for a finite subset $J_0 \subset J$. Therefore, $[\cap\{V_\ell(\alpha_i) \mid i\in J\}=\emptyset] \Leftrightarrow [\cap\{V_\ell(\alpha_j) \mid j\in J_0\}=\emptyset]$.

2) (a)$\Rightarrow$(b). According to subsection 4 (see the discussion of the topology $\mathfrak{T}$) for any family $\{\alpha_i \mid i\in J\}\subset IR$ we have $[\alpha\subset \mathrm{rad}_\ell(\sup\{\alpha_i \mid i\in J\})] \Leftrightarrow [\bigcap_{i\in J} V_\ell(\alpha_i)\subset V_\ell(\alpha)]$. Therefore, if the set $V_\ell(\alpha)$ is quasifinite, then

$$[\alpha\subset \mathrm{rad}_\ell(\sup\{\alpha_i \mid i\in J\})] \Leftrightarrow [V_\ell(\alpha)=(\bigcap_{i\in J} V_\ell(\alpha_i))\cup V_\ell(\alpha)=$$
$$=\bigcap_{i\in J}(V_\ell(\alpha_i)\cup V_\ell(\alpha))=\bigcap_{i\in J_0}(V_\ell(\alpha_i)\cup V_\ell(\alpha))=(\bigcap_{i\in J_0} V_\ell(\alpha_i))\cup V_\ell(\alpha)$$

for a finite set $J_0\subset J] \Leftrightarrow [\bigcap_{i\in J_0} V_\ell(\alpha_i)\subset V_\ell(\alpha)] \Leftrightarrow [\alpha\subset \mathrm{rad}_\ell(\sup_{i\in J_0}\alpha_i)]$.

(c)$\Rightarrow$(b). It is clear that the condition (b) holds if and only if it holds for an increasing chains of ideals. Besides, as was already mentioned in the course of the proof of (a)$\Rightarrow$(b),

$$[\alpha\subset \mathrm{rad}_\ell(\sup\{\alpha_i \mid i\in J\})] \Leftrightarrow [V_\ell(\alpha)=\bigcap_{i\in J}(V_\ell(\alpha_i)\cup V_\ell(\alpha))=$$
$$=\bigcap_{i\in J} V_\ell(\alpha_i\cap\alpha)] \Leftrightarrow [\alpha\subset \mathrm{rad}_\ell(\sup\{\alpha_i\cap\alpha \mid i\in J\})].$$

(d)$\Rightarrow$ (c) and (a)$\Rightarrow$(d), since

$$\mathfrak{F}_{V_\ell(\alpha)}=\{n\in I_\ell R \mid \alpha\subset \mathrm{rad}_\ell(n)\}. \ \square$$

Corollary 1. Consider the following properties of a closed set W of $(\mathrm{Spec}_\ell R, \mathfrak{T})$:

(i) W is quasifinite;

(ii) $W=V_\ell(\alpha)$ for a finitely generated twosided ideal α;

(iii) $G_{\mathfrak{F}_W}\sup_{i\in J}\alpha_i \simeq \sup_{i\in J} G_{\mathfrak{F}_W}\alpha_i$ for any increasing family $\{\alpha_i \mid i\in J\}$ of twosided ideals of R;

(iv) the functor $G_{\mathfrak{F}_W}$ commutes with the colimits (and therefore is isomorphic to $G_{\mathfrak{F}_W}R^{(1)}\otimes_R \sim$, where, as usual, $R^{(1)}$ is a ring, obtained from R by adjusting the unit).

Then (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

If R is commutative, all these properties are equivalent.

Proof. (i) $\Rightarrow$ (ii). Every ideal $m \in IR$ presents in the form $\sup\{\alpha_i \mid i \in J\}$, where $\{\alpha_i \mid i \in J\}$ is a directed family of finitely generated ideals. If $V_\ell(m)$ is quasifinite, then according to heading 2) of Proposition 7 $V_\ell(m) = V_\ell(\alpha_j)$ for some $j \in J$.

(iii) $\Rightarrow$ (i). Let $\{\alpha_i \mid i \in J\}$ be an increasing chain of ideals from $IR \setminus \mathfrak{F}_W$. Then $\{G_{\mathfrak{F}_W}\alpha_i \mid i \in J\}$ is an increasing chain of proper ideals of $G_{\mathfrak{F}_W}R$. Since $G_{\mathfrak{F}_W}R$ is unitary, $\sup\limits_{i\in J} G_{\mathfrak{F}_W}\alpha_i$ is also a proper ideal. By hypothesis $\sup\limits_i G_{\mathfrak{F}_W}\alpha_i = G_{\mathfrak{F}_W}(\sup\limits_i \alpha_i)$, therefore $G_{\mathfrak{F}_W}(\sup\limits_i \alpha_i)$ is a proper ideal. By Proposition 2.7 this means, that $\sup\limits_i \alpha_i \notin \mathfrak{F}_W$. It only remains to make use of the implication (d) $\Rightarrow$ (a) of heading 2) of Proposition 7.

(iv) $\Rightarrow$ (iii) is obvious.

If R is a commutative ring, then the set $V_\ell(\alpha) = V(\alpha)$ is quasifinite, if α is finitely generated; and $V(\alpha) = V(s)$ for a suitable $s \in \alpha$. The ring $G_{\mathfrak{F}_{V(s)}}R^{(1)}$ coincides with the quotient ring $(s)^{-1}R^{(1)}$ and $G_{\mathfrak{F}_{V(s)}} \simeq (s)^{-1}R^{(1)} \otimes_R \sim$. $\square$

Corollary 2. Let R be a noetherian ring (i.e. the ascending chain condition for twosided ideals is satisfied).

Then (Spec_ℓ R, $\mathfrak{T}$) is quasifinite. In particular, for any module M from ${}_R Mod$ (see Proposition 6) the canonical morphism $\mathcal{O}_M \to \mathcal{O}_M^a$ is an isomorphism.

8. Quasicompactness of the space $(\mathcal{S}pec_\ell R, \mathfrak{T}_1)$.

Proposition. Let R be a ring with right unit. Then the following properties are equvalent:

(a) $(\mathcal{S}pec_\ell R, \mathfrak{T}_1)$ is quasicompact

(b) if $\{n_i\}_{i\in J}$ is a family of left ideals of R such that $\sup\{(n_i : x_i) | i \in J\} = R$ for any family $\{x_i | i \in J\} \subset \mathcal{P}(R)$, then there exists a finite subset $J_0 \subset J$ such, that $\sup\{(n_j : x_j') | j \in J_0\} = R$ for all $\{x_j' | j \in J_0\} \subset \subset \mathcal{P}(R)$.

Proof. 1) Show, that

$$\bigcap_{i\in J} V_\ell(n_i) = \bigcup\{V_\ell(\sup_{i\in J}(n_i : x_i)) | \{x_i | i \in J\} \subset \mathcal{P}(R)\} \qquad (1)$$

for any family $\{n_i | i \in J\}$ of left ideals of R.

Indeed, the inclusions $\bigcap_{i\in J} V_\ell(n_i) \supset V_\ell(\sup\{n_i | i \in J\})$ and $V_\ell((m : y)) \subset V_\ell(m)$ are valid for any $m \in I_\ell R$ and $y \in \mathcal{P}(R)$. Therefore $V_\ell(\sup_{i\in J}(n_i : x_i)) \subset \bigcap\{V_\ell(n_i) | i \in J\}$ for every family $\{x_i | i \in J\} \subset \mathcal{P}(R)$. On the other hand, let p be an ideal from $\bigcap_{i\in J} V_\ell(n_i)$. By definition this means that $n_i \to p$ for every $i \in J$. Since p is a weakly regular ideal, then $n_i \to p$ if $(n_i : t_i) \subset p$ for some t_i from $\mathcal{P}(R)$; therefore $\sup\{(n_i : t_i) | i \in J\} \subset p$. Thus, we have the inclusion $\bigcap\{V_\ell(n_i) | i \in J\} \subset \bigcup\{V_\ell(\sup_{i\in J}(n_i : x_i)) | \{x_i | i \in J\} \subset \mathcal{P}(R)\}$ which is inverse to the above established.

2) From formula (1) we get implications combined with the obvious

$$[\bigcap\{V_\ell(n_i) | i \in I\} = \emptyset] \Leftrightarrow [\sup\{(n_i : x_i) | i \in J\} = R \text{ for any family } \{x_i | i \in J\} \subset \mathcal{P}(R)] \qquad (2)$$

which

implications

$$\left[(\mathrm{Spec}_\ell R, \mathfrak{T}_1) \text{ is quasicompact} \right] \Leftrightarrow$$

$\Leftrightarrow$ [if $\{n_i \mid i \in I\}$ is a family of left ideal, such that $\cap\{V_\ell(n_i) \mid i \in I\} = \emptyset$, then $\cap\{V_\ell(n_j) \mid j \in J_0\} = \emptyset$ for a finite subset $J_0 \subset J$], prove the equivalence of (a) and (b). □

Thus, everything is much more complicated when the topology $\mathfrak{T}$ is replaced by the topology $\mathfrak{T}_1$, and the geometric representation of the modules $M \mapsto {}^1\mathcal{O}^a_M$ turns out to be an effective tool for the investigation of the modules themselves only for the special classes of rings, the first step to the description of which is given by Proposition 8. The topology $\mathfrak{T}$, as we have seen, is more universal.

9. Spectra of quotient-rings and ideals. The aim of this and of several next sections is to prove the statements on the properties of Spec_ℓ and $\widehat{\mathrm{Spec}}_\ell$, similar to the Jackobson theorems about the structure of the spaces Prim of the primitive ideals (see [5], chapter IX).

The left spectrum is somewhat abundant relative to all the topologies, even $\mathfrak{T}_0$, since $\mathfrak{T}_0$ does not separate the points isomorphic in the category $\mathrm{I}^>_\ell R$. Therefore we introduce a new character: the set ${}^\sim\mathrm{Spec}_\ell R$, the quotient of $\mathrm{Spec}_\ell R$ modulo the equivalence relation "$p \simeq p'$ in $\mathrm{I}^>_\ell R$". For any subset $X \subset \mathrm{Spec}_\ell R$ the notation ${}^\sim X$ will stay for its image under the factorisation $\mathrm{Spec}_\ell R \twoheadrightarrow {}^\sim\mathrm{Spec}_\ell R$.

For an arbitrary subset $Y \subset R$ set $U_\ell(Y)=\{p \in \mathcal{S}pec_\ell R \mid Y \not\subset p\}$ and $\hat{U}_\ell(Y) = U_\ell(Y) \cap \mathcal{S}\hat{p}ec_\ell R$. Obviously, $U_\ell(Y)$ and $\hat{U}_\ell(Y)$ are an open subsets of $(\mathcal{S}pec_\ell R, \mathfrak{T})$ and its subspace $\mathcal{S}\hat{p}ec_\ell R$ respectively, if Y is a two-sided ideal of R.

Proposition. Let α be a proper twosided ideal of R.

1) The map $\mu \mapsto \mu/\alpha$ induces bijections $V_\ell(\alpha) \rightarrow \mathcal{S}pec_\ell R/\alpha$ and $\hat{V}_\ell(\alpha) \stackrel{def}{=} V_\ell(\alpha) \cap \mathcal{S}\hat{p}ec_\ell R \rightarrow \mathcal{S}\hat{p}ec_\ell R/\alpha$, which are homeomorphisms in the topology $\mathfrak{T}$.

2) The map $\mu \mapsto \mu \cap \alpha$ induces the maps $u_\alpha : U_\ell(\alpha) \rightarrow \mathcal{S}pec_\ell \alpha$ and $\hat{u}_\alpha : \hat{U}_\ell(\alpha) \rightarrow \mathcal{S}\hat{p}ec_\ell \alpha$. The map $\hat{u}_\alpha$ is a homeomorphism in the topologies $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$; the map u_α is a quasihomeomorphism in the same topologies. The "reduced" map $\tilde{u}_\alpha$, corresponding to the map u_α, is bijection, and, therefore, homeomorphism in $\tilde{\mathfrak{T}}_0$, $\tilde{\mathfrak{T}}_1$ and $\tilde{\mathfrak{T}}$.

Proof. 1) Let $\mu \in V_\ell(\alpha)$, $\bar{\mu} \stackrel{def}{=} \mu/\alpha$, $\bar{n}$ a left ideal of R/α, n its preimage in R.

Suppose that $(\bar{\mu} : \bar{x}) \not\subset \bar{\mu}$ for every $\bar{x} \in \mathcal{P}(\bar{n})$. It is equivalent to the fact, that $(\mu : x) \not\subset \mu$ for any $x \in \mathcal{P}(n)$. Since $\mu \in \mathcal{S}pec_\ell R$, then $n \rightarrow \mu$; i.e. either $n \subset \mu$, or $(n : y) \subset \mu$ for some $y \in \mathcal{P}(R)$. Clearly, $(n : y) \supset \alpha$ (since $\alpha \subset n$ and α is a twosided ideal) and $\overline{(n : y)} = (\bar{n} : \bar{y})$; the bar denotes the image of the corresponding sets in R/α. Thus, either $(\bar{n} : \bar{y}) \subset \bar{\mu}$ or $\bar{n} \subset \bar{\mu}$; i.e., $\bar{n} \rightarrow \bar{\mu}$.

If $\mu \in \mathcal{S}\hat{p}ec_\ell R$, then a part of the above arguments show that $\bar{\mu} \in \mathcal{S}\hat{p}ec_\ell R/\alpha$.

Now let $\bar{\mu} \in Spec_\ell R$; μ the preimage in R of $\bar{\mu}$, and $n \in I_\ell$
Suppose that $(\mu : x) \not\subset \mu$ for every $x \in \mathcal{P}(n)$. Then $(\mu : y) \not\subset \mu$ for every $y \in \mathcal{P}(n+\alpha)$ since $\alpha \subset \mu$. Let $\bar{n}_\alpha$ be the image of an ideal $n_\alpha = n+\alpha$ in R/α. Clearly, $(\mu : y) \not\subset \mu$ if and only it $(\bar{\mu} : \bar{y}) \not\subset \bar{\mu}$. Therefore, since $\bar{\mu} \in Spec_\ell R/\alpha$, then either $\bar{n} \subset \bar{\mu}$, or $(\bar{n} : \bar{x}) \subset \bar{\mu}$ for some $x \in \mathcal{P}(R)$; if $\bar{\mu} \in \widehat{Spec}_\ell R/\alpha$, then $\bar{n} \subset \bar{\mu}$. Obviously, it follows, that $n \subset n_\alpha \longrightarrow \mu$ (that $n \subset \mu$ if $\bar{\mu} \in \widehat{Spec}_\ell R/\alpha$).

Thus the map $\mu \mapsto \mu/\alpha$ induces the bijections $V_\ell(\alpha) \xrightarrow{\sim} Spec_\ell R/\alpha$, $\hat{V}_\ell(\alpha) \xrightarrow{\sim} \widehat{Spec}_\ell R/\alpha$. The fact that they are homeomorphisms in $\mathfrak{Z}$ follows directly from the definition of the topologies of the corresponding spaces.

2) (a) Let us show, that for every $\mu \in V_\ell(\alpha)$ the ideal $\mu \cap \alpha$ belongs to $Spec_\ell \alpha$.

Let $n \in I_\ell \alpha$ and suppose, that $(\mu \cap \alpha : x)_\alpha = (\mu \cap \alpha : x) \cap \alpha \not\subset \mu \cap \alpha$ for any $x \in \mathcal{P}(n)$. It follows (since $(\mu \cap \alpha : y) = (\mu : y) \cap \alpha$ for any $y \subset R$), that $(\mu : x) \not\subset \mu$ for any $x \in \mathcal{P}(\alpha n)$. Since $\mu \in Spec_\ell R$ and αn is a left ideal in R, then the "inclusion" $\alpha n \to \mu$ holds; i.e. either $\alpha n \subset \mu$ or $(\alpha n : y) \subset \mu$ for some $y \in \mathcal{P}(R)$. Consider each of these cases.

(i) Let $\alpha n \subset \mu$. Then $(R, n) \to \mu$.

Indeed, if $(R, n) \not\to \mu$, then $\{\alpha, (R, n)\} \subset \mathfrak{F}_\mu$, since by hypothesis $\alpha \not\subset \mu$; hence, $\alpha n = \alpha(R, n) \subset \mathfrak{F}_\mu \circ \mathfrak{F}_\mu \subset \mathfrak{F}_\mu$ — contradiction.

(ii) Now suppose, that $(\alpha n : y) \subset \mu$ for some $y \in \mathcal{P}(R)$. Then $\alpha((R, n) : y) \subset \mu$, since, obviously, $\alpha((R, n) : y) \subset$

$\subset (\alpha(R,n):y)$. As was established in (i), $\alpha((R,n):y)\subset\mu$ implies that $((R,n):y)\to\mu$; hence, $(R,n)\to\mu$.

Thus, the both cases lead to one relation: $(R,n)\to\mu$ This means that either $(R,n)\subset\mu$, and then $n\subset\mu$, or $((R,n):x)\subset\mu$ for some $x\in\mathcal{P}(R)$. Let us show that in the latter case x can be selected from $\mathcal{P}(\alpha)$.

Indeed, there exists $z_\mu\in\mathcal{P}(\alpha)$ such that $(\mu:z_\mu)\subset\mu$ (if this fails, then $\alpha\to\mu$; but, since the ideal α is twosided, $[\alpha\to\mu]\Leftrightarrow[\alpha\subset\mu]$). The inclusion $((R,n):x)\subset\mu$ implies that $((R,n):z_\mu x)=$ $=(((R,n):x):z_\mu)\subset(\mu:z_\mu)\subset\mu$.

It is not difficult to see, that $(n:z_\mu x)\cap\alpha\subset$ $\subset((R,n):z_\mu x)\cap\alpha\subset\mu\cap\alpha$. These inclusions mean, that $n\longrightarrow\mu\cap\alpha$ in $I_\ell^{\succ}\alpha$.

($\hat{a}$) If $\mu\in\hat{\mathcal{U}}_\ell(\alpha)$, then $\mu\cap\alpha\in \widehat{Spec}_\ell\alpha$.

The implication $[\,n\in I_\ell\alpha$ and $(\mu\cap\alpha:x)_\alpha\not\subset\mu$ for any $x\in\mathcal{P}(n)]\Rightarrow[(\mu:x)\not\subset\mu$ for any $x\in\mathcal{P}(\alpha n)]$, established in the beginning of (a), for $\mu\in\widehat{Spec}_\ell R$ can be continued as follows: $\Rightarrow[\alpha n=\alpha(R,n)\subset\mu]$. If $(R,n)\not\subset\mu$, then $\alpha(R,n)\in\mathfrak{F}_\mu\circ\mathfrak{F}_\mu\subset\mathfrak{F}_\mu$; i.e. $\alpha(R,n)\not\subset\mu$. Therefore $\alpha\subset(R,n)\subset\mu$. QED.

(b) If $\mu\in\mathcal{U}_\ell(\alpha)$, $\nu\in I_\ell R$ and $\alpha\cap\nu\to\mu$, then $\nu\to\mu$.

Indeed, if $\nu\not\to\mu$, since $\alpha\not\to\mu$, then $\nu\cap\alpha\in\mathfrak{F}_\mu\circ\mathfrak{F}_\mu\subset\mathfrak{F}_\mu$; i.e. $\nu\cap\alpha\not\to\mu$.

In particular, the map $u_\alpha:\mathcal{U}_\ell(\alpha)\to Spec_\ell\alpha$, $\mu\mapsto\mu\cap\alpha$, is compatible with the preorders on $\mathcal{U}_\ell(\alpha)$ and $Spec_\ell\alpha$ induced from $I_\ell^{\succ}R$ and $I_\ell^{\succ}\alpha$ respectively.

It follows, in turn, that u_α uniquely defines a map $\tilde{u}_\alpha : \tilde{U}_\ell(\alpha) \to \tilde{S}pec_\ell \alpha$ such that the diagram

$$\begin{array}{ccc} U_\ell(\alpha) & \xrightarrow{u_\alpha} & Spec_\ell \alpha \\ \downarrow & & \downarrow \\ \tilde{U}_\ell(\alpha) & \xrightarrow{\tilde{u}_\alpha} & \tilde{S}pec_\ell \alpha \end{array} \qquad (1)$$

is commutative, and this map is injective.

(b̂) If $\mu \in U_\ell(\alpha)$, $\nu \in I_\ell R$ and $\nu \cap \alpha \subset \mu$ then $\nu \subset \mu$, since if $\nu \not\subset \mu$, then $\nu \cap \alpha \in \hat{\mathfrak{F}}_\mu \circ \hat{\mathfrak{F}}_\mu \subset \hat{\mathfrak{F}}_\mu \overset{\text{def}}{=} \{n \in I_\ell R \mid n \not\subset \mu\}$. This implies the injectivity of $\hat{u}_\alpha$.

(c1) The map u_α is continuous in the topology $\mathfrak{T}$.

We have $[\beta \subset u_\alpha(\mu) = \mu \cap \alpha] \Leftrightarrow [\alpha\beta\alpha \subset \mu \cap \alpha] \Leftrightarrow$ for any $\beta \in I\alpha$, whence it is clear that $u_\alpha^{-1}(V_\ell^\alpha(\beta)) = V_\ell^R(\alpha\beta\alpha) \cap U_\ell(\alpha)$.

(c2) The map u_α is continuous in the topology $\mathfrak{T}_1$.

Let $n \in I_\ell \alpha$ and $n \to \mu \cap \alpha$, where $\mu \in U_\ell(\alpha)$ this means that either $n \subset \mu \cap \alpha$, and then $(R,n) \subset \mu$, or $(n:y) \subset \mu \cap \alpha$ for some $y \in \mathcal{P}(\alpha)$. In the latter case $((R,n):y) \to \mu$, which, as we will show, implies $(R,n) \to \mu$.

If $((R,n):y) \not\to \mu$, then $\alpha((R,n):y) \not\to \mu$. But $\alpha((R,n):y) \subset \subset (\alpha(R,n):y) = (\alpha n:y) \subset (n:y)$ and $(n:y) \subset \mu$ by hypothesis.

Thus, if $\mu \in U_\ell(\alpha)$ and $n \to \mu \cap \alpha$, then $(R,n) \to \mu$; i.e. $u_\alpha^{-1}(V_\ell^\alpha(n)) \in V_\ell^R((R,n))$. In step (I) of the proof it was shown that $n \to \mu \cap \alpha$ if $(R,n) \to \mu$; i.e. the inverse inclusion $V_\ell^R((R,n)) \cap U_\ell(\alpha) \subset u_\alpha^{-1}(V_\ell^\alpha(n))$ holds. Any closed subset of the space $(Spec_\ell \alpha, \mathfrak{T}_1)$ is of the form $\bigcap\{V_\ell^\alpha(n_i) \mid i \in I\}$ for some family $\{n_i \mid i \in I\} \subset I_\ell \alpha$, and the equalities $U_\ell(\alpha) \cap V_\ell^R((R,n_i)) = u_\alpha^{-1}(V_\ell^\alpha(n_i))$ imply $u_\alpha^{-1}(\bigcap_{i \in I} V_\ell^\alpha(n_i)) = U_\ell(\alpha) \cap \cap (\bigcap\{V_\ell^R((R,n_i)) \mid i \in I\})$.

(c3) $[\mu \in V_\ell^R(m) \cap U_\ell(\alpha)] \Leftrightarrow [\mu \cap \alpha \in V_\ell^\alpha(m \cap \alpha)]$ for any $m \in I_\ell R$.

Let $\mu \in V_\ell(m) \cap U_\ell(\alpha)$. If $m \subset \mu$, then certainly $\mu \cap \alpha \in V_\ell^\alpha(m \cap \alpha)$. If $(m:x) \subset \mu$ for some $x \in \mathcal{P}(R)$, then

$(m:z_\mu x)\subset\mu$ for some $z_\mu\in\mathcal{P}(\alpha)$ (see the arguments in (i));
Therefore, $(m\cap\alpha : z_\mu x)_\alpha = (m:z_\mu x)\cap\alpha \subset \mu\cap\alpha$
and $z_\mu x\in\mathcal{P}(\alpha)$. So, in this case also $\mu\cap\alpha\in V_\ell^\alpha(m\cap\alpha)$.

Thus, the implication $[\mu\in V_\ell^R(m)\cap U_\ell(\alpha)]\Rightarrow$
$\Rightarrow[\mu\cap\alpha\in V_\ell^\alpha(m\cap\alpha)]$ is established. The inverse implication is the topic of step (b) above.

(c4) From (c3) we may deduce that the map $\mathcal{U}_\alpha$ is closed (sends the closed subsets into closed subsets of the image of $U_\ell(\alpha)$) in the topologies $\mathfrak{T}$ and $\mathfrak{T}_1$;
the map $\hat{\mathcal{U}}_\alpha$ is also closed in these topologies.

(d) For an arbitrary ideal $\mu\in\mathrm{Spec}_\ell\,\alpha$ set $\mu_\alpha \overset{\mathrm{def}}{=} \{z\in R \mid \alpha z\subset\mu\}$. It is not difficult to see, that μ_α is a left ideal (due to the fact, that α is a twosided ideal). Let us show that

$$\mu_\alpha\in\mathrm{Spec}_\ell R,\ \text{and}\ \mu_\alpha\in\widehat{\mathrm{Spec}}_\ell R\quad,\ \text{if}$$
$\mu\in\mathrm{Spec}_\ell\alpha$.

Let $n\in I_\ell R$ and $(\mu_\alpha : x)\not\subset\mu_\alpha$ for any $x\in\mathcal{P}(n)$. From the definition of the ideal μ_α it is clear that $(\mu_\alpha : x)\not\subset\mu_\alpha$ if and only if there exists $\lambda_x\in R$ such that $\alpha\lambda_x x\subset\mu$ and $\alpha\lambda_x\not\subset\mu$.
Hence, $\alpha\lambda_x\not\subset\mu$, $\alpha\lambda_x x\subset\mu$ for all $x\in\mathcal{P}(n)$, and, consequently, $(\mu:x)\cap\alpha\not\subset\mu$ for any $x\in\mathcal{P}(n)$; in particular, $(\mu:x)\cap\alpha\subset\mu$ for any $x\in\mathcal{P}(\alpha n)$. If $\mu\in\widehat{\mathrm{Spec}}_\ell\alpha$, then it implies $\alpha n\subset\mu$. By definition of μ_α this means exactly, that $n\subset\mu$.

Now, if $\mu\in\mathrm{Spec}_\ell\alpha\setminus\widehat{\mathrm{Spec}}_\ell\alpha$, then either $\alpha n\subset\mu$, and hence $n\subset\mu$; or $(\alpha n:y)\cap\alpha\subset\mu$ for some $y\in\mathcal{P}(\alpha)$. Since $\alpha(n:y)\subset(\alpha n:y)\cap\alpha\subset\mu$, then $(n:y)\subset\mu_\alpha$.

So in either case $n \to \mathcal{M}_\alpha$.

Obviously, $\mathcal{M} \subseteq \mathcal{M}_\alpha \cap \alpha$. Since $\mathcal{M}_\alpha \cap \alpha = \{z \in \alpha \mid (\mathcal{M} : z)_\alpha = \alpha\}$, then $\mathcal{M} \in \mathrm{Spec}_\ell\, \alpha$ implies $\mathcal{M}_\alpha \cap \alpha \to \mathcal{M}$; and if $\mathcal{M} \in \widehat{\mathcal{S}pec}_\ell\, \alpha$, then $\mathcal{M}_\alpha \cap \alpha \subset \mathcal{M}$. Therefore in the first case the ideals $\mathcal{M}_\alpha \cap \alpha$ and $\mathcal{M}$ are isomorphic in $\mathcal{I}_\ell^{\mathcal{L}}\alpha$; in the second one they coincide.

This means, that the maps $\widehat{\mathcal{U}}_\alpha$ and $^{\sim}\mathcal{U}_\alpha$ are surjective and therefore bijective. As was earlier clarified, these maps are continuous and closed in the topologies $\mathfrak{T}$ and $\mathfrak{T}_1$ (restricted onto $\widehat{\mathcal{V}}_\ell(\alpha)$ and $\widehat{\mathcal{S}pec}_\ell\, \alpha$) and in the topologies $^{\sim}\mathfrak{T}$ and $^{\sim}\mathfrak{T}_1$ induced by the topologies $\mathfrak{T}$ and $\mathfrak{T}_1$ respectively on $^{\sim}\mathcal{V}_\ell(\alpha)$ and $^{\sim}\mathcal{S}pec_\ell\, \alpha$. The homeomorphisms are continuous closed bijections.

It remains to verify that $\mathcal{U}_\alpha$ is a quasihomeomorphism (and, consequently, $^{\sim}\mathcal{U}_\alpha$ is a homeomorphism) in the topologies $\mathfrak{T}_0$ (respectively in the induced topologies $^{\sim}\mathfrak{T}_0$).

According to (c3) the map $\mathcal{U}_\alpha$ sends the set $V_\ell^R(m) \cap \mathcal{V}_\ell(\alpha)$ into $V_\ell^\alpha(m \cap \alpha)$ for every $m \in \mathcal{I}_\ell R$. It follows that $\mathcal{U}_\alpha$ sends the set $\mathfrak{T}_0\overline{W} \cap \mathcal{V}_\ell(\alpha) = \bigcup\{V_\ell^R(p) \cap \mathcal{V}_\ell(\alpha) \mid p \in W\}$ into $\bigcup\{V_\ell^\alpha(p \cap \alpha) \mid p \in \mathcal{V}_\ell(\alpha) \cap W\} = \mathfrak{T}_0\overline{\mathcal{U}_\alpha(W)}$ for any $W \subset \mathcal{S}pec_\ell R$, and realises a bijection of $^{\sim}(\mathfrak{T}_0\overline{W} \cap \mathcal{V}_\ell(\alpha))$ onto $^{\sim}(\mathfrak{T}_0\overline{\mathcal{U}_\alpha(W)})$ (see part (d) of the proof). Therefore the bijection $^{\sim}\mathcal{U}_\alpha$ commutes with passing to the closure in the topologies induced by $\mathfrak{T}_0$ on $^{\sim}\mathcal{V}_\ell(\alpha)$ and $^{\sim}\mathcal{S}pec_\ell\, \alpha$ respectively. This means, that $^{\sim}\mathcal{U}_\alpha$ is a homeomorphism in the topologies $^{\sim}\mathfrak{T}_0$. Therefore $\mathcal{U}_\alpha$ is a quasihomeomorphism $(\mathcal{V}_\ell(\alpha), \mathfrak{T}_0) \to (\mathcal{S}pec_\ell\, \alpha, \mathfrak{T}_0)$. $\square$

Let R be an arbitrary associative ring. By heading 1) of Proposition 9, the map $\mu \mapsto \mu/R$ induces a bijection $V_\ell^{R^{(1)}}(R) \xrightarrow{\sim} \operatorname{Spec} \mathbf{Z}$, since $R^{(1)}/R \simeq \mathbf{Z}$ (as aside, this bijection turn out to be a homeomorphism in all the topologies used here). So, we can (and will) identify $V_\ell^{R^{(1)}}(R)$ with Spec Z, and, therefore, $U_\ell^{R^{(1)}}(R)$ with $\operatorname{Spec}_\ell R^{(1)} \setminus \operatorname{Spec} \mathbf{Z}$.

Evidently, the canonical projection $\operatorname{Spec}_\ell R^{(1)} \twoheadrightarrow {}^\sim\operatorname{Spec}_\ell R^{(1)}$ induces a bijection $V_\ell^{R^{(1)}}(R) \xrightarrow{\sim} {}^\sim V_\ell^{R^{(1)}}(R)$, that allows to identify, also, ${}^\sim V_\ell^{R^{(1)}}(R)$ with Spec Z.

After these prelimineries we can use the second part of Proposition 9.

Corollary 1. For any associative ring R the map $\mu \mapsto \mu \cap R$ realizes a bijection $\hat{u}_R : \operatorname{S\hat{p}ec}_\ell R^{(1)} \setminus \operatorname{Spec} \mathbf{Z} \xrightarrow{\sim} \operatorname{S\hat{p}ec}_\ell R$ and induces a bijection ${}^\sim u_R : {}^\sim\operatorname{Spec}_\ell R^{(1)} \setminus \operatorname{Spec} \mathbf{Z} \xrightarrow{\sim} {}^\sim\operatorname{Spec}_\ell R$.

The maps $\hat{u}_R$ and ${}^\sim u_R$ are homeomorphisms in the topologies induced by the topologies $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$ on $\operatorname{Spec}_\ell R$ and $\operatorname{Spec}_\ell R^{(1)} \setminus \operatorname{Spec} \mathbf{Z}$ respectively.

Corollary 2. Let α be a two-sided ideal of R. Then

$$rad_\ell^\alpha(n \cap \alpha) = rad_\ell^R(n) \cap \alpha \qquad (1)$$

for every $n \in I_\ell R$.

Proof. By proposition 9 the map $\mu \mapsto \mu \cap \alpha$ realizes a bijection $\hat{U}_\ell(\alpha) \xrightarrow{\sim} \operatorname{S\hat{p}ec}_\ell \alpha$; and for any $n \in I_\ell R$ the set $\hat{V}_\ell^\alpha(n \cap \alpha) = \{\mu \in \operatorname{S\hat{p}ec}_\ell \alpha \mid n \cap \alpha \to \mu\}$ is the image of the set $\hat{V}_\ell^R(n) \cap U_\ell(\alpha)$ (see the implications (c3) in the proof of Proposition 9): $\hat{V}_\ell^\alpha(n \cap \alpha) = \{\mu \cap \alpha \mid \mu \in \hat{V}_\ell^R(n) \cap U_\ell(\alpha)\}$. Therefore

$$rad_\ell^\alpha(n \cap \alpha) = \bigcap\{\mu \cap \alpha \mid \mu \in \hat{V}_\ell(n) \cap U_\ell(\alpha)\} = \bigcap\{\mu \cap \alpha \mid \mu \in V_\ell(n)\} =$$
$$= \bigcap\{\mu \mid \mu \in \hat{V}_\ell(n)\} \cap \alpha = rad_\ell^R(n) \cap \alpha \ . \ \square$$

Corollary 3. Let α be a two-sided ideal of R. Then for any left ideal m of the ring α

$$rad_\ell^{\alpha}(m) = rad_\ell^{\alpha}((R,m)\cap\underline{\alpha}) = rad_\ell^{R}((R,m))\cap\alpha.$$

Proof. According to (c2) of the proof of Proposition 9 for every $m \in I_\ell \underline{\alpha}$

$$\hat{V}_\ell^{\alpha}(m) = \{\underline{\mu}\cap\underline{\alpha} \mid \underline{\mu} \in V_\ell^{R}((R,m))\cap\hat{U}_\ell(\underline{\alpha})\}.$$

This and Corollary 2 yield the desired equalities. □

10. Digression: the hereditary properties of the prime spectrum. For the prime spectrum a statement similar to Proposition 9 holds:

Proposition. Let α be a two-sided ideal of R.

1) The map $p \mapsto p/\alpha$ realises a homeomorphism of the closed subset $V(\alpha)$ onto $Spec\, R/\alpha$.

2) The map $p \mapsto p \cap \alpha$ induces a homeomorhpism of an open subset $U(\alpha) \overset{def}{=} Spec\, R \setminus V(\alpha) = \{p \mid \alpha \not\subset p\}$ of $Spec\, R$ onto $Spec\, \alpha$.

Sketch of the proof 1) It is clear that the map $\beta \mapsto \bar{\beta} = \beta + \alpha/\alpha$ is a surjection of the set of twosided ideals of R onto the set of the twosided ideals of $\bar{R} = R/\alpha$. Let $\{\beta_1, \beta_2\} \subset IR$, $p \in V(\alpha)$, $\underbrace{\text{and } \mathfrak{p}}_{\mathfrak{p} \in Spec\, \bar{R}}$, $\tilde{\mathfrak{p}}$ the premage of $\mathfrak{p}$ in IR. Then $[\bar{\beta}_1 \bar{\beta}_2 \subset \bar{p}] \Leftrightarrow [\beta_1 \beta_2 \subset p] \Leftrightarrow$ [either $\beta_1 \subset p$ or $\beta_2 \subset p$] $\Leftrightarrow$ [respectively, either $\bar{\beta}_1 \subset \bar{p}$ or $\bar{\beta}_2 \subset \bar{p}$];
$[\beta_1 \beta_2 \subset \tilde{\mathfrak{p}}] \Leftrightarrow [\bar{\beta}_1 \bar{\beta}_2 \subset \mathfrak{p}] \Leftrightarrow$ [either $\bar{\beta}_1 \subset \mathfrak{p}$ or $\bar{\beta}_2 \subset \mathfrak{p}$] $\Longleftrightarrow$ [respectively, either $\beta_1 \subset \tilde{\mathfrak{p}}$ or $\beta_2 \subset \tilde{\mathfrak{p}}$]. This makes it clear that the map $p \mapsto \bar{p} = p/\alpha$ realizes a bijection $\varphi_\alpha : V(\alpha) \xrightarrow{\sim} Spec\, R/\alpha$, and $\varphi_\alpha(V(m) \cap V(\alpha)) = \varphi_\alpha(V(m+\alpha)) = V(\bar{m})$ for every $m \in IR$; i.e. φ_α is a homeomorphism.

2) (i) Let us show that $p \cap \alpha \in Spec\, \alpha$ for every $p \in U(\alpha)$.

Let $\{\beta, \beta'\} \subset I\alpha$, and $\beta\beta' \subset p$. Then $\beta R \alpha \beta' \subset p$. This means, that either $\beta \subset p$ or $\alpha\beta' \subset p$. Then, $[\alpha\beta' \subset p] \Rightarrow [\alpha(R, \beta') \subset p] \Rightarrow$

$\Rightarrow [(R, \beta') \subset p$ (since $\alpha \not\subset p$ by the hypothesis)$] \Rightarrow [\beta' \subset p \cap \alpha]$.

(ii) Every ideal $\mathfrak{p} \in Spec\, \alpha$ is a twosided ideal of R.

Indeed, $\alpha R \mathfrak{p} \subset \mathfrak{p}$ (since $\alpha R \subset \alpha$) and, therefore $R\mathfrak{p} \subset \mathfrak{p}$. Similarly, the inclusion $\mathfrak{p} R \alpha \subset \mathfrak{p}$ implies

(iii) For any $\mathfrak{p} \in \operatorname{Spec} R$ the left ideal $\mathfrak{p}_\alpha = \{z \in R \mid \alpha z \subset \mathfrak{p}\}$ is a prime ideal of R. The maps

$p \mapsto p \cap \alpha$ and $\mathfrak{p} \mapsto \mathfrak{p}_\alpha$ induce the mutually inverse bijections

$$\mathcal{U}(\alpha) \underset{\sim}{\overset{\sim}{\rightleftarrows}} \operatorname{Spec} \alpha$$

As it was shown in step (ii), $\mathfrak{p}R \subset \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} \alpha$; it follows, that $\mathfrak{p}_\alpha$ is a twosided ideal in R. Let $\{m, n\} \subset IR$ and $nm \subset \mathfrak{p}_\alpha$; i.e. $\alpha nm \subset \mathfrak{p}$. Then $(\alpha n)(\alpha m) \subset \mathfrak{p}$ and, therefore either $\alpha n \subset \mathfrak{p}$ or $\alpha m \subset \mathfrak{p}$. By definition of $\mathfrak{p}_\alpha$ these inclusions are equvalent to the inclusions $n \subset \mathfrak{p}_\alpha$ and $m \subset \mathfrak{p}_\alpha$, respectively.

It is easy to see, that $\mathfrak{p}_\alpha \cap \alpha = \{z \in \alpha \mid \alpha z \subset \mathfrak{p}\} = \mathfrak{p}$, if $\mathfrak{p}$ is prime. Conversely, if $p \in \mathcal{U}(\alpha)$, then $(p \cap \alpha)_\alpha = p$.

We leave to the reader the task to verify that the bijection $\mathcal{U}(\alpha) \xrightarrow{\sim} \operatorname{Spec} \alpha$ is homeomorphism. □

Corollary 1. *For an arbitary associative ring R the map* $\mu \mapsto \mu \cap R$ *realizes a homeomorphism*

$$\operatorname{Spec} R^{(1)} \setminus \operatorname{Spec} Z \xrightarrow{\sim} \operatorname{Spec} R$$

An analogue of corollary 2 of Proposition 9 is a known (for the unitary rings) fact concerning the "hereditarity" of the lower Baire Radical:

Corollary 2. *Let* α *and* m *be twosided ideals of* R. *Then*

$$\mathfrak{B}^\alpha(m \cap \alpha) = \mathfrak{B}^R(m) \cap \alpha .$$

Proof. By definition of the lower Baire radical

$$\mathfrak{B}^\alpha(m \cap \alpha) = \cap \{\mathfrak{p} \mid \mathfrak{p} \in V^\alpha(m \cap \alpha)\}$$

The map $\mathcal{U}(\alpha) \to \operatorname{Spec} \alpha$ being a homeomorphism means that $V^\alpha(m \cap \alpha) = \{p \cap \alpha \mid p \in V(m) \cap \mathcal{U}(\alpha)\}$. Hence $\mathfrak{B}^\alpha(m \cap \alpha) = \cap \{p \cap \alpha \mid p \in V(m) \cap \mathcal{U}(\alpha)\} = \cap \{p \cap \alpha \mid p \in V(m)\} = \mathfrak{B}^R(m) \cap \alpha$. □

Corollary 3. Let $\alpha \in IR$, $n \in I\alpha$. Then

$$\mathfrak{F}^{\alpha}(n) = \mathfrak{F}^{\alpha}((R,n,R)\cap\alpha) = \mathfrak{F}^{R}((R,n,R))\cap\alpha.$$

The statement follows from the equality

and corollary 2. □

11. Homeomorphisms connected with the idempotents.

Let e be a non-zero idempotent of R and $\widehat{U}_\ell(eRe) = \{\mu \in \widehat{Spec}_\ell R \mid eRe \not\subset \mu\}$.

Proposition. The map $\mu \mapsto \mu \cap eRe$ induces a homeomorphism $u_{eRe} : \widehat{U}_\ell(eRe) \to \widehat{Spec}_\ell eRe$ in the topologies, induced by $\mathfrak{T}$.

Proof. (a) If $\mu \in \widehat{U}_\ell(eRe)$, then interesection $\mu \cap eRe$ belongs to $\widehat{Spec}_\ell eRe$.

(a1) Note, first of all, that $(\nu : x)\cap Re = (\nu : ex)\cap Re$ for any subset $x \subset R$ and for any Z-submodule $\nu \subset R$; and therefore $(\nu\cap eRe : x)\cap eRe = (\nu : x)\cap eRe$ if $x \subset Re$.

(a2) Let $n \in I_\ell eRe$ and $(\mu\cap eRe : x)_{eRe} = (\mu\cap eRe : x)\cap eRe \not\subset \mu$ for every $x \in \mathcal{P}(n)$. The formulas of (a1) imply that $(\mu : y)\cap eRe = (\mu : ey)\cap eRe = (\mu\cap eRe : ey)\cap \cap eRe \not\subset \mu$ for any $y \in \mathcal{P}((R,n))$. In particular, $(\mu : y) \not\subset \not\subset \mu$ for any $y \in \mathcal{P}((R,n))$. Since $\mu \in \in \widehat{Spec}_\ell R$, then $(R,n) \subset \mu$; hence, $n \subset \mu\cap eRe$.

(b) If $\mu \in \widehat{U}_\ell(eRe)$, $\nu \in I_\ell R$ and $\nu\cap eRe \subset \mu$, then $\nu \subset \mu$.

Indeed, $[\nu\cap eRe \subset \mu] \Rightarrow [Re(\nu\cap Re)\subset\mu] \Rightarrow [\nu\cap Re \subset \mu$ (since by hypothesis $Re \not\subset \mu$ and $\mu \in \widehat{Spec}_\ell R$; see part (a) of the proof of Proposition 9)$] \Rightarrow [\nu \subset \mu$ (by the same reason, see part (b) of the proof of Proposition 9)$]$.

This implies the injectivity of the map u_{eRe} and the implications $[\mu \in \widehat{U}_\ell(eRe) \cap V_\ell^R(\beta),\ \beta \in IR] \Leftrightarrow [\mu \cap eRe \in \in \widehat{Spec}_\ell eRe \cap V_\ell^{eRe}(e\beta e)]$, which make it clear that u_{eRe} sends the closed sets of the space $\widehat{U}_\ell(eRe)$ into the closed sets of $\widehat{Spec}_\ell eRe$.

Then the map $\mu \mapsto \mu \cap e\alpha e$ induces the homeomorphism $\widehat{U}_\ell(eRe) \cap U_\ell(\alpha) \xrightarrow{\sim} \widehat{Spec}_\ell\, e\alpha e$.

Proof. The homeomorphism $u_{eRe} : \widehat{U}_\ell(eRe) \xrightarrow{\sim} \widehat{Spec}_\ell\, eRe$ realises a bijection, hence, a homeomorphism of the open subset $\widehat{U}_\ell(eRe) \cap U_\ell(\alpha)$ of $\widehat{U}_\ell(eRe)$ onto the open subset $\widehat{U}_\ell(e\alpha e)$ of $\widehat{Spec}_\ell\, eRe$. By Proposition 9 the map $p \mapsto e\alpha e \cap p$ realises a homeomorphism of $\widehat{U}_\ell(e\alpha e)$ onto $\widehat{Spec}_\ell\, e\alpha e$. □

Corollary 3. Let e be an idempotent in R, which is not the unit. Then for any $\alpha \in IR$ the map $\mu \mapsto \mu \cap (1-e)\alpha(1-e)$ induces a homeomorphism of $\widehat{U}_\ell((1-e)R(1-e)) \cap \widehat{U}_\ell(\alpha)$ onto $\widehat{Spec}_\ell(1-e)\alpha(1-e)$. (Here $(1-e)X(1-e)$ is the image of $X \subset R$ under the map $x \mapsto (1-e)x(1-e) \overset{def}{=} x - ex - xe + exe$.)

Proof. If R is a ring with unit, then we only need a specialisation of corollary 2 for the idempotent 1-e.

in general case replace R by $R^{(1)}$. □

Corollary 4. Let e be a nonzero idempotent in R.

1) $rad_\ell^{eRe}(\beta \cap eRe) = rad_\ell^R(\beta) \cap eRe$ for any $\beta \subset IR$.

2) For every twosided ideal β_1 of eRe:

$$rad_\ell^{eRe}(\beta_1) = rad_\ell^{eRe}((R,\beta_1,R) \cap eRe) = rad_\ell^R((R,\beta_1,R)) \cap eRe.$$

Proof. 1) By proposition II (see step (6) of its proof) the homeomorphism $\hat{u}_{eRe} : \widehat{U}_\ell(eRe) \xrightarrow{\sim} \widehat{Spec}_\ell\, eRe$ sends the set $V_\ell(\beta) \cap \widehat{U}_\ell(eRe)$ into the set $\widehat{V}_\ell^{eRe}(\beta \cap eRe)$. Therefore

$$rad_\ell^{eRe}(\beta \cap eRe) = \cap\{p' \mid p' \in \widehat{V}_\ell^{eRe}(\beta \cap eRe)\} = \cap\{p \cap eRe \mid p \in V_\ell(\beta) \cap \widehat{U}_\ell(eRe)\} = \cap\{p \cap eRe \mid p \in \widehat{V}_\ell(\beta)\} = rad_\ell^R(\beta) \cap eRe$$

2) $\beta_1 = e(R,\beta_1,R)e = (R,\beta_1,R) \cap eRe$ for any $\beta_1 \in IeRe$

This and statement 1) imply 2). □

Remark. Proposition II is clearly a particular case of Proposition 9 if the idempotent, mentioned there, is central.

As is known, if there is no non-zero nilpotents in R, all its idempotents are central. Indeed, for an arbitrary idempotent e and for any x the squares of $(1-e)xe=xe-exe$ and $ex(1-e)=ex-exe$ vanish. Therefore the lack of non-trivial nilpotents in R leads to the identities $xe=exe$, $ex=exe$, i.e. $ex=xe$. □

12. Idempotents and the prime spectrum.

Proposition. *Let e be a nonzero idempotent of R. The map $p \mapsto p\cap eRe$ induces a homeomorphism of the subspace $\mathcal{U}(eRe)=\{p \mid p \not\supset eRe\}=\mathcal{U}(e)$ of the prime spectrum of R onto Spec eRe.*

Sketch of the proof. (a) Let $p\in\mathcal{U}(e)$ and twosided ideals of eRe, such that $\beta_1\beta_2\subset p\cap eRe$. Since $\beta_1\beta_2=\beta_1 eRe\beta_2=\beta_1 R\beta_2$, then either $\beta_1\subset p$, or $\beta_2\subset p$.

(b) Let $\mathfrak{p}\in \mathrm{Spec}\, eRe$; $\mathfrak{p}_{\langle e\rangle}\overset{def}{=}\{z\in R \mid e(R,z,R)e\subset\mathfrak{p}\}$; $\{\alpha_2,\alpha_2\}\subset IR$, and $\alpha_1\alpha_2\subset\mathfrak{p}_{\langle e\rangle}$. The latter means exactly, that $e\alpha_1\alpha_2 e\subset\mathfrak{p}$. This implies $(e\alpha_1 e)(e\alpha_2 e)\subset\mathfrak{p}$, which in turn implies (since $e\alpha' e\in IeRe$ for any $\alpha'\in IR$ and $\mathfrak{p}$ is prime) that $e\alpha_i e\subset\mathfrak{p}$ for $i=1$ or 2. So $\mathfrak{p}_{\langle e\rangle}\in \mathrm{Spec}\,R$.

(c) Obviously, $\mathfrak{p}_{\langle e\rangle}\cap eRe=\{z\in eRe \mid e(R,z,R)e\subset\mathfrak{p}\}=\mathfrak{p}$, and $(p\cap eRe)_{\langle e\rangle}=\{z\in R \mid e(R,z,R)e\subset p\}=p$ for any $p\in\mathcal{U}(e)$, since $[eR(zRe)\subset p]\Rightarrow[$ (because $e\notin p$) $zRe\subset p]\Rightarrow[z\in p]$. Therefore the maps $p\mapsto p\cap eRe$ and $\mathfrak{p}\mapsto\mathfrak{p}_{\langle e\rangle}$ induce mutually inverse bijections

$u_{\langle e\rangle}: \mathcal{U}(e)\xrightarrow{\sim}\mathrm{Spec}\,(eRe)$ and $v_{\langle e\rangle}: \mathrm{Spec}\,(eRe)\xrightarrow{\sim}\mathcal{U}(e)$. The verification of the continuity and of the closedness of $u_{\langle e\rangle}$ is left to the reader. □

Corollary 1. *For every idempotent e of R the subspace $\mathcal{U}(e)$ of the prime spectrum is quasicompact.*

Corollary 2. Let e be a nonzero idempotent in R and $\alpha \in IR$. The map $p \mapsto p \cap e\alpha e$ defines a homeomorphism

$$U(e) \cap U(\alpha) \xrightarrow{\sim} \operatorname{Spec} e\alpha e$$

Corollary 3. Let e be a non-unit idempotent in R. Then for any $\alpha \in IR$ the map $p \mapsto p \cap (1-e)\alpha(1-e)$ induces a homeomorphism of the subspace $U((1-e)R(1-e)) \cap U(\alpha)$ onto Spec $(1-e)\alpha(1-e)$.

Corollary 4. Let e be a non-zero idempotent in R.

1) $\mathfrak{P}^{eRe}(\alpha \cap eRe) = \mathfrak{P}^R(\alpha) \cap eRe$ for every $\alpha \in IR$.

2) For every twosided ideal α_1 of eRe;

$$\mathfrak{P}^{eRe}(\alpha_1) = \mathfrak{P}^{eRe}((R,\alpha_1,R) \cap eRe) = \mathfrak{P}^R((R,\alpha_1,R)) \cap eRe.$$

The proof of these statements is as similar to the proof of the corresponding corollaries of Proposition 11, as their formulations are. $\square$

13. Digression: the left extension of the Jackobson radical. For every $n \in I_\ell R$ set $J_\ell(n) = \tau(\dot{V}_\ell(n)) \overset{def}{=} \bigcap \{p \mid p \in \dot{V}_\ell(n)\}$ where $\dot{V}_\ell(n) \overset{def}{=} V_\ell(n) \cap Max_\ell^{reg} R$ is the set of all the maximal regular ideals μ such that $n \to \mu$.

Since for any $x \in R \setminus \mu$ the ideal $(\mu : x)$ is maximal in $I_\ell R$ and regular, if so is μ, then $J_\ell(n) = \bigcap \{\mu_s \mid \mu \in \dot{V}_\ell(n)\}$ which makes it clear, that J_ℓ takes values in IR, same as rad_ℓ. Together with rad_ℓ, J_ℓ is a functor from $I_\ell^{\succ} R$ into IR. There is an obvious embedding $rad_\ell \hookrightarrow J_\ell$

For any subset $x \subset R$ put $\dot{U}_\ell(x) \overset{def}{=} U_\ell(x) \cap Max_\ell^{reg} R$.

Proposition. 1) The map $\mu \mapsto \mu \cap \alpha$ determines a bijection $\dot{U}_\ell(\alpha) \xrightarrow{\sim} Max_\ell^{reg} \alpha$ which is a homeomorphism in the topologies induced by $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$.

2) For any idempotent e in R the map $\mu \mapsto \mu \cap eRe$ induces a homeomorphism $\dot{\mathcal{U}}_\ell(eRe) \xrightarrow{\sim} Max_\ell\, eRe$ in the topology $\hat{\mathfrak{T}}$

Proof 1) Let $\alpha \in \mathcal{I}R$, $\mu \in Max_\ell^{reg} R$ and $\alpha \not\subset \mu$. It is easy to see that $[\alpha \not\subset \mu] \Leftrightarrow [\alpha \not\subset (\mu : x)$ for every $x \in R \setminus \mu]$. This and the maximality of all $(\mu : x)$, $x \in R \setminus \mu$, yields that R/μ is an irreducible α-module. The canonical epimorphism $\alpha \longrightarrow R/\mu$ induces an isomorphism of α-modules $\alpha/\alpha \cap \mu \xrightarrow{\sim} R/\mu$. This means that $\alpha/\alpha \cap \mu$ is an irreducible α-module and therefore $\alpha \cap \mu$ is a maximal regular left ideal of the ring α.

Now show that the map $p \mapsto p_\alpha \overset{def}{=} \{z \in R \mid \alpha z \subset p\}$, inverse to the bijection $\hat{\mathcal{U}}_\ell(\alpha) \xrightarrow{\sim} \hat{Spec}_\ell\, \alpha$, $\mu \mapsto \mu \cap \alpha$, (see step (d) of the proof of Proposition 9), sends $Max_\ell^{reg} \alpha$ into $Max_\ell^{reg} R$

Let $m \in Max_\ell^{reg} \alpha$ and d an element of the ring α such that $xd - x \in m$ for every $x \in \alpha$. Since α is a twosided ideal, then $yd - y \in m_\alpha$ for any $y \in R$; i.e. m_α is a regular ideal with a "right unit" d. Consequently, m_α is contained in a maximal left ideal μ which is necessarily regular.

Since $d \notin \mu$, then $\mu \in \hat{\mathcal{U}}_\ell(\alpha)$; hence, $\mu \cap \alpha \in \hat{Spec}_\ell \alpha$. Since $m \subset \mu \cap \alpha$ and m is maximal, then $\mu \cap \alpha = m$. The injectivity of $\hat{u}_\alpha : \hat{\mathcal{U}}_\ell(\alpha) \longrightarrow \hat{Spec}_\ell \alpha$ implies that $\mu = m_\alpha$.

Thus, there is a commutative diagram

$$\begin{tikzcd} \dot{\mathcal{U}}_\ell(\alpha) \arrow[r, hook] \arrow[d, "\dot{u}_\alpha"', "\wr"] & \hat{\mathcal{U}}_\ell(\alpha) \arrow[d, "\wr"', "\hat{u}_\alpha"] \\ Max_\ell^{reg} \alpha \arrow[r, hook] & \hat{Spec}_\ell \alpha \end{tikzcd}$$

in which the vertical arrows are bijections. It is clear, that since $\hat{u}_\alpha$ is a homeomorphism in some of the topologies, then $\dot{u}_\alpha$ is a homeomorphism in the induces topologies.

2) All the ideals from $\hat{U}_\ell(eRe)$ are regular; moreover, e is right unit for all the ideals from $\hat{U}_\ell(eRe)$.

In fact e is right unit for all the ideals of the form $\mathcal{M}(e) \overset{def}{=} \{z \in R \mid eze \in \mathcal{M}\}$, where $\mathcal{M} \in I_\ell eRe$, since $e(xe-x)e=0$ for every $x \in R$. By Proposition 10 (see step (d) of the proof) $\hat{U}_\ell(eRe) = \{\mathcal{M}(e) \mid \mathcal{M} \in \hat{Spec}_\ell eRe\}$.

All the ideals of the ring eRe are regular, since eRe is a ring with unit.

The standard considerations needed to terminate the proof are left to the reader. □

Corollary 1) Let $\alpha \in IR$, $n \in I_\ell R$, and $m \in I_\ell$. Then $J_\ell^\alpha(n \cap \alpha) = J_\ell^R(n) \cap \alpha$ and $J_\ell^\alpha(m) = J_\ell^R((R,m)) \cap \alpha$.

2) Let e be an idempotent in R; $\beta \in IR$ and $\beta' \in IeRe$. Then $J^{eRe}(\beta \cap eRe) = J^R(\beta) \cap eRe$ and $J^{eRe}(\beta') = J^R((R,\beta',R)) \cap eRe$.

Here $J_\ell^R(\cdot)$ is the radical J_ℓ in R; and $J=J^R$ the usual Jackobson radical.

This statement is deduced from Proposition 13 in the same way, as analogous corollaries were deduced form Propositions 9 and 11. The second statement (and the first, if we confine ourselves to twosided ideals and replace (R,m) by (R,m,R) in the second formula) is well-known (see [5], Ch. IX).

14. The torsion $\hat{rad}_\ell$. Recall several basic concepts of the radical theory (see [11]). Fix a category $\mathfrak{A}$ of rings closed with respect to twosided ideals and homomorphic images. Let r be the map $Ob\,\mathfrak{A} \to Ob\,\mathfrak{A}$ which to any

ring $R \in \mathrm{Ob}\,\mathfrak{A}$ assigns a two-sided ideal $r(R)$ of R. A ring R is called r-radical if $r(R)=R$ and r-semisimple if $r(R)=0$.

The map r is called the torsion or the ideally hereditary radical if the following conditions are satisfied:

(T1) $f(r(A)) \subset r(f(A))$ for any ring morphism $f\colon A \longrightarrow B$ from $\mathfrak{A}$;

(T2) $r(R/r(R))=0$ for any ring $R \in \mathrm{Ob}\,\mathfrak{A}$;

(T3) if $\mathcal{B}$ is a two-sided ideal of R, then $r(\mathcal{B})= = r(R) \cap \mathcal{B}$ (the heredity property).

Take as $\mathfrak{A}$ the category Rings and for any ring R set $\widehat{\mathrm{rad}}_1(R) = \mathrm{rad}_1^R(0) = \bigcap\{p \mid p \in \mathrm{Spec}_1 R\}$.

Proposition. The map $\widehat{\mathrm{rad}}_1\colon R \longmapsto \widehat{\mathrm{rad}}_1(R)$ is the torsion in Rings.

Proof. 1) Let $f\colon A \longrightarrow B$ be a ring morphism. By Proposition 9 the map $m \longmapsto f(m)$ realizes a bijection $V_1^A(\mathrm{Ker} f) \xrightarrow{\sim} \xrightarrow{\sim} \mathrm{Spec}_1 f(A)$ yielding

$$\mathrm{rad}_1(f(A)) = \bigcap\{p' \mid p' \in \mathrm{Spec}_1 f(A)\} = \bigcap\{f(p) \mid p \in V_1^A(\mathrm{Ker} f)\} = = f(\mathrm{rad}_1^A(\mathrm{Ker} f)).$$

On the other hand,

$$\widehat{\mathrm{rad}}_1(A) = r(\mathrm{Spec}_1 A) = \mathrm{rad}_1^A(\mathrm{Ker} f) \cap r(U_1^A(\mathrm{Ker} f)).$$

2) Obvious.

3) An equality $\widehat{\mathrm{rad}}_1(\mathcal{B}) = \widehat{\mathrm{rad}}_1(R) \cap \mathcal{B}$, $\mathcal{B} \in \mathrm{I}R$, is a direct corollary of the definition of $\widehat{\mathrm{rad}}_1$ and Corollary 2 of Proposirion 9. $\square$

Remark. It follows from Proposition 9 that $\widehat{\mathrm{rad}}_1(R/\mathcal{B}) \simeq \simeq \mathrm{rad}_1^R(\mathcal{B})/\mathcal{B}$ for every associative ring R and any $\mathcal{B} \in \mathrm{I}R$. This implies that $\widehat{\mathrm{rad}}_1$ and restrictions of rad_1 onto two-sided ideals are equivalent. However, replacing rad_1 by $\widehat{\mathrm{rad}}_1$, we loose information on the values of rad_1 at the ideals of $I_1R \smallsetminus I^1R$, where I^1R is the set of all the left ideals of R isomorphic (in $I_1^{\succ}R$) to the two-sided ideals; it is easy to see that $m \in I^1R$ iff either $m = m_s$ or $(m:x) = m_s$ for some $x \in \mathfrak{P}(R)$. □

15. Torsion rad_1 and locally nilpotent radical.

An ideal is called *locally nilpotent* iff any finite subset of its elements generates a nilpotent subring. Every associative ring R possesses a two-sided locally nilpotent ideal $\mathcal{L}(R)$ wich contains any left or right locally nilpotent ideal of R; this ideal $\mathcal{L}(R)$ is called *locally nilpotent radical* or *Levitzky radical* of R. As is proved in Section I of Appendix, the radicals $\widehat{rad}_1$ and $\mathcal{L} : R \longmapsto \mathcal{L}(R)$ coincide.

16. Discontinuity of the left spectrum and decomposition of ring into the direct sum of two-sided ideals.

Proposition. *The following properties of R are equivalent:*

1) *There exists a family* $\{\alpha_i \mid i \in I\}$ *of two-sided ideals of* R *such that*

(a) $\alpha_i \cap \alpha_j \subset \widehat{rad}_\ell(R)$ *if* $i \neq j$;

(b) *the ring* $R/\sup\{\alpha_i \mid i \in I\}$ is $\widehat{rad}_1$-radical.

2) $Spec_1R$ *is homeomorphic to the disjoint union of a family of topological spaces:*

$$(Spec_\ell R, \mathfrak{T}) \simeq \bigvee_{i \in I} X_i .$$

Proof. 1)$\Longrightarrow$2). Let $\{\alpha_i \mid i \in I\}$ be a family of ideals satisfying (a), (b). Then

$$U_\ell(\alpha_i) \cap U_\ell(\alpha_j) = U_\ell(\alpha_i \cap \alpha_j) = \emptyset$$

for any $(i,j) \in I \times I$ such that $i \neq j$;

$$\cup\{\mathcal{V}_\ell(\alpha_i)\mid i\in I\}=\mathcal{V}_\ell(\sup\{\alpha_i\mid i\in I\})\simeq \mathrm{Spec}_\ell(\sup\{\alpha_i\mid i\in I\})$$

(by Proposition)) and $\mathrm{Spec}_\ell(\sup\{\alpha_i\mid i\in I)\simeq$ $\simeq \mathrm{Spec}_\ell R$ by condition (b), since $\mathcal{V}_\ell(\sup_i \alpha_i)\simeq \mathrm{Spec}_\ell(R/\sup_i \alpha_i)$.

2) $\Longrightarrow$ 1) Conversely, let $\mathrm{Spec}_\ell R$ be homeomorphic to the disjoint union of a family $\{X_i\mid i\in I\}$ of topological spaces. This means, that $\mathrm{Spec}_\ell R=\bigcup_{i\in I}\mathcal{U}_i$, where all the subsets $\mathcal{U}_i$ are both open and closed and do not intersect with each other. Choose for every $i\in I$ a twosided ideal α_i such that $\mathcal{U}_i=\mathcal{V}_\ell(\alpha_i)$. Since $\mathcal{V}_\ell(\alpha_i\cap\alpha_j)=\mathcal{V}_\ell(\alpha_i)\cap\mathcal{V}_\ell(\alpha_j)=\emptyset$ (for $i\neq j$) and

$$\mathcal{V}_\ell(\sup\{\alpha_i\mid i\in I\})=\cup\{\mathcal{V}_\ell(\alpha_i)\mid i\in I\}=\mathrm{Spec}_\ell R,$$

then (a) and (b) respectively follow. $\square$

Corollary. Let R be a semisimple ring such that for any proper twosided ideal α the set $\mathcal{V}_\ell(\alpha)$ is nonempty. Then the following properties are equivalent:

1) The ring R represents as a direct sum of a family $\{\alpha_i\mid i\in I\}$ of nonzero (hence necessarily $\widehat{rad}_\ell$-semisimple) rings;

2) Spec_ℓ R represents as the union of a family $\{\mathcal{U}_i\mid i\in I\}$ of disjoint non-empty sets that are both open and closed.

The decomposition of R into the coproduct of a family of rings $\{\alpha_i\mid i\in I\}$ (ideals of R) is related with the corresponding decomposition of the left spectrum by the following relations:

$$\alpha_i=rad_\ell(\alpha_i)=r\left(\cup\{\mathcal{U}_j\mid j\in I\setminus\{i\}\}\right),\quad \mathcal{U}_i=\mathcal{V}_\ell(\alpha_i) \tag{1}$$

Proof Let $\{\alpha_i\mid i\in I\}$ be a family of twosided ideals of R , satisfying the conditions (a), (b) of the proposition. Since R is $\widehat{rad}_\ell$ -semisimple, then (a) means, that $\alpha_i\cap\alpha_j=0$

if $i \neq j$. Since the rad_ℓ - radicality of a ring R is equivalent to the fact: $\mathrm{Spec}_\ell R = \emptyset$, then (b) (due to the bijection $V_\ell(\alpha) \simeq \mathrm{Spec}_\ell R/\alpha$) is equivalent to emptiness of $V_\ell(\sup_i \alpha_i)$. By hypothesis condition

By Proposition 14 the $\widehat{\mathrm{rad}}_\ell$ -semisimplicity of R implies that of all the twosided ideals: $\widehat{\mathrm{rad}}_\ell(\alpha) = \widehat{\mathrm{rad}}_\ell(R) \cap \alpha = 0$. In particular, for every $i \in I$ the ideal $\alpha^{(i)} = \sup\{\alpha_j \mid j \in I \setminus \{i\}\}$ is $\widehat{\mathrm{rad}}_\ell$-semisimple; consequently, the ring R/α_i is $\widehat{\mathrm{rad}}_\ell$-semisimple. This means that $\alpha_i = \mathrm{rad}_\ell(\alpha_i) = \mathrm{r}(V_\ell(\alpha_i))$. But it is easy to see that $V_\ell(\alpha_i) = U_\ell(\alpha^{(i)})$. Therefore

$$\alpha_i = \mathrm{r}(U_\ell(\alpha^{(i)})) = \mathrm{r}(\bigcup\{U_\ell(\alpha_j) \mid j \in I \setminus \{i\}\}) = \mathrm{r}(\bigcup\{\mathcal{U}_j \mid j \in I \setminus \{i\}\}).$$

The implications $[U_\ell(\alpha) = \emptyset] \Leftrightarrow [\alpha \subset \widehat{\mathrm{rad}}_\ell(R)]$ imply that if R is $\widehat{\mathrm{rad}}_\ell$ -semisimple, then $[U_\ell(\alpha) = \emptyset] \Leftrightarrow [\alpha = 0]$. This yields that $\mathcal{U}_i = U_\ell(\alpha_i) \neq \emptyset, i \in I$. To complete the proof is suffices to refer to Proposition 16 and the relations, appearing therein. □

Remark. The condition "$V_\ell(\alpha) \neq \emptyset$ for a proper twosided ideal α" is surely satisfied, if R is a ring with right unit (see Proposition7). Besides, the ring R is surely rad_1 -semisimple, if it is semisimple in the sence of Ja :obson, i.e. is J-semisimple.

17. The central idempotents. The set $\mathfrak{Z}I(R)$ of the central idempotents of a ring R is traditionally ordered as follows: $e \leqslant f$, if $ef = e$. It is known (and easily verified), that $(\mathfrak{Z}I(R), \leqslant)$ is a structure and $e \wedge f = ef$, $e \vee f = e \circ f \overset{\mathrm{def}}{=} e + f - ef$ (the cyclic composition of e and f).

Proposition. Let R be a rad_ℓ -semisimple ring such that $V(\alpha) \neq \emptyset$ for any proper twosided ideal α,

1) For any $e \in \mathfrak{ZI}(R)$ the subset $\mathcal{U}_\ell(e)$ is quasicompact, open and closed.

2) The map $e \longmapsto \mathcal{U}_\ell(e) := \mathcal{U}_\ell(eR)$ is an injective morphism of the structure of central idempotents into the structure of the open-closed subsets of $Spec_\ell\, R$

3) If $\mathfrak{U}$ is an arbitrary open and closed set in $\mathcal{U}_\ell(e)$, where $e \in \mathfrak{ZI}(R)$, then there exists a (unique) central idempotent f such that $\mathfrak{U} = U(f)$.

Proof 1) Let $e \in \mathfrak{ZI}(R)$. Then there is a decomposition of R into the direct sum of its ideals $R = eR + (1-e)R$ (recall that $(1-e)R = \{ x - ex \mid x \in R \}$). By Proposition 16 to this decomposition a representation of $Spec_\ell\, R$ as a disjoint union of an open and a closed set corresponds: $Spec_\ell R = \mathcal{U}_\ell(eR) \cup \cup \mathcal{U}_\ell((1-e)R)$. The quasicompactness of $\mathcal{U}_\ell(e) = \mathcal{U}_\ell(eR)$ follows from the unitarity of eR (see Proposition 7) and from the existence of the canonical homeomorphism $\hat{\mathcal{U}}_\ell(eR) \simeq \simeq \widehat{Spec}_\ell eR$ (Proposition 9 or Proposition 11)

2) Since the conditions of Corollary 2 of Proposition 16 are satisfied, then $eR = rad_\ell^R(eR) = \mathfrak{r}(\mathcal{U}_\ell((1-e)R))$ for any central idempotent e. Therefore, if $\mathcal{U}_\ell(eR) = \mathcal{U}_\ell(fR)$ for some central idempotent f then $eR = fR$. Since e and f are the units of the rings eR and fR respectively, then $[eR = fR] \Longleftrightarrow [e = f]$.

For any pair of central idempotents e and f we have $efR = eR \cap fR$, $(1 - e \circ f)R = (1-e)(1-f)R = (1-e)R \cap (1-f)R$ (the last equalities may be reduced to the first one by passing to $R^{(1)}$; the first is verified straighforwardly). This implies the desired relations:

$$\mathcal{U}_\ell(efR)=\mathcal{U}_\ell(eR)\cap\mathcal{U}_\ell(fR);\ \mathcal{U}_\ell(e\circ fR)=V_\ell((1-e\circ f)R)=$$
$$=V_\ell((1-e)R)\cup V_\ell((1-f)R)=\mathcal{U}_\ell(eR)\cup\mathcal{U}_\ell(fR).$$

3) Since $\hat{\mathcal{U}}_\ell(eR)$ is homeomorphic to $\hat{\mathcal{S}pec}_\ell\, eR$ by Proposition 9 we may consider $\mathcal{U}$ as an open and closed subset of Spec_ℓ R. Since eR is $\widehat{\mathrm{rad}}_\ell$ -semisimple (as a two-sided ideal of a $\widehat{\mathrm{rad}}_\ell$ -semisimple ring) and unitary (see Remark in 16), then the conditions of Corollary of Proposition 18 are satisfied, according to which the ring eR is the direct sum of its ideals α, β such that $\mathcal{U}=\mathcal{U}_\ell^{eR}(\alpha)$, $\mathcal{U}^\perp=\mathcal{S}pec_\ell eR\setminus\mathcal{U}=\mathcal{U}_\ell^{eR}(\beta)$.

From the unitarity of eR the unitarity of α and β follows; hence $\alpha=feR=fR$ and $\beta=$ $=geR=gR$ for some (uniquely defined, as was already verified in subsect 2)) central idempotents f and g of eR.Now note that $\mathfrak{Z}I(eR)\subseteq\mathfrak{Z}I(R)$, since the centre of eR belongs to the centre of the ring R . $\square$

Corollary I Let R be a $\widehat{\mathrm{rad}}_\ell$ -semisimple ring with unit. Then the correspondence $e\mapsto\mathcal{U}_\ell(eR)$ is an isomorphism of the structure $(\mathfrak{Z}I(R),\leq)$ of ~~the~~ central idempotents onto the structure of open-closed subsets of Spec_ℓ R.

The statement follows directly from headings 2) and 3) of Proposition 17. $\square$

Corollary 2 Let R be a unitary ring. Then the following conditions are equivalent:

1) Spec_ℓ R represents as the union of a family of disjoint open (and therefore closed) sets.

2) In R there exists a family $\{e_i \mid i\in I\}$ of orthogonal idempotents such that

(a) $\sum_{i\in I} e_i = 1$ — is the unit of R ;

(b) $[e_i, x] \overset{\mathrm{def}}{=} e_i x - x e_i$ belong to $\widehat{\mathrm{rad}}_\ell R$ for all $i\in I$ and $x\in R$.

The family of sets from 1) and the family of idempotents from 2) are related via

$$\mathcal{U}_i = \mathcal{V}_\ell(e_i R e_i) \simeq \mathrm{Spec}_\ell\, e_i R e_i\,,\quad R e_i + \widehat{\mathrm{rad}}_\ell(R) = \Big(\bigcup_{j\in I\setminus\{i\}} \mathcal{U}_j\Big) \qquad (1)$$

Proof. By Proposition 16 and Corollary I the condition 1) is equivalent to the representability of $\widehat{\mathrm{rad}}_\ell$ -semisimple ring $\overline{R} = R/\widehat{\mathrm{rad}}_\ell(R)$ as $\sum_{i\in I} \bar{e}_i \overline{R}$ where $\{\bar{e}_i \mid i\in I\}$ are the central orthogonal idempotents in $\overline{R}$. Since $\widehat{\mathrm{rad}}_\ell(R)$ is a nil-ideal, then according to the Jackobson theorem the family of ortogonal idempotents $\{\bar{e}_i \mid i\in I\}$ can be lifted to a family $\{e_i \mid i\in I\}$ of orthogonal idempotents in R (see [5] , Chapter III, §8). Set $e = \sum_{i\in I} e_i$. The orthogonality of $\{e_i \mid i\in I\}$ implies that e is an idempotent, and $\bar{e} = \sum_{i\in I} \bar{e}_i = \bar{1}$ implies $xe - x \in \widehat{\mathrm{rad}}_\ell(R)$ for any $x\in R$. The latter means that $R = Re + \widehat{\mathrm{rad}}_\ell(R)$, which implies by Nakayama lemma the equation $R = Re$. A similar "right" consideration shows that $R = eR$. Obviously, $[R = Re\ (R = eR)] \Rightarrow [e$ is right (respectively left) unit of R$]$. Consequently, e is the unit.

(b) is satisfied, since all the $\bar{e}_i$ belong to the centre of R. Thus we have shown, that 1) $\Longrightarrow$ 2)

A canonical epimorphism $R \twoheadrightarrow \overline{R}$ induces a homeomorphism $\mathrm{Spec}_\ell R \xrightarrow{\sim} \mathrm{Spec}_\ell \overline{R}$ of the left spectrums that sends $U_\ell(e_i R e_i)$ into $U_\ell(\bar e_i \bar R \bar e_i) = U_\ell(\bar e_i)$.

The equality $\bar R \bar e_i = \tau(\bigcup_{j\neq i} \bar{\mathcal U}_j)$, where $\bar{\mathcal U}_j$ is the image of $\mathcal U_j$ in $\mathrm{Spec}_\ell \bar R$, implies, obviously,

$$Re_i + \widehat{rad}_\ell(R) = \tau(\bigcup\{\mathcal U_j \mid j\in I\setminus\{i\}\})$$

It is clear now that the relations (1) follow from the same relations for a $\widehat{rad}_1$-semisimple ring R, which, in turn, follow from Corollary of Proposition 16.

The implication 2) $\Longrightarrow$ 1) follows from Proposition 16. □

18. Closed points. Irreducible spaces. Dimensions.

A. Let $|X|$ denote the set of closed points of topological space X.

Proposition. 1) $|(\mathrm{Spec}_\ell R, \mathfrak{T}_0)|$ consists of all the two-sided ideals that are simultaneously the maximal left regular ideals.

2) $|(\mathrm{Spec}_\ell R, \mathfrak{T})| = |(\mathrm{Spec}_\ell R, \mathfrak{T}_0)| = \mathrm{Max}_\ell^{reg} R \cap IR$.

Proof. 1) (i) First, suppose that R is a ring with right unit. Let $\mu \in |(\mathrm{Spec}_\ell R, \mathfrak{T}_0)|$, i.e. $V_\ell(\mu) = \{\mu\}$. Since R possesses a right unit, $\mu \subset m$ for some $m \in \mathrm{Max}_\ell R$ and $\mathrm{Max}_\ell R \subset \widehat{\mathrm{Spec}}_\ell R$. The closedness of μ implies that $\mu = m$. Besides, since $\mu \to (\mu : x)$ for any $x \in R$, then $\mu = (\mu : x)$ for all $x \in R \setminus \mu$. It follows that $\mu = \mu_s$, i.e. μ is a two-sided ideal. It is clear that for any μ from $\mathrm{Max}_\ell R \cap IR$ the set $V_\ell(\mu)$ consists of one point.

(ii) Lemma. The map $\nu \longmapsto \nu \cap R$ realizes a bijective correspondence between the set of all left ideals of $R^{(1)}$, that contain the elements of the form $1-\alpha$, $\alpha \in R$, and all the regular left ideals of R.

Proof. Let $n \in I_\ell R^{(1)}$ and $1-a \in n$ for some $a \in R$. Then $x - xa = x(1-a) \in n$ for every $x \in R$; i.e. $n \cap R$ is regular.

Conversely, let $m \in I_\ell R$ and $x - xa \in m$ for some $a \in R$ and every $x \in R$. Obviously, $(m:a) = m$. Since $(m:a) = (m:a)_{R^{(1)}} \cap R$ (we consider m as a left ideal of $R^{(1)}$), it means that $m = (m:a)_{R^{(1)}} \cap R$. Clearly, $(1-a)a = a - a \cdot a \in m$, i.e. $1-a \in (m:a)_{R^{(1)}}$.

(iii) We are interested in the following corollary of just proved Lemma:

The map $\mathcal{M} \mapsto \mathcal{M} \cap R$ realizes a bijective correspondence between the set of all the left maximal ideals of $R^{(1)}$, that do not contain R, and the set of all the regular maximal left ideals of R.

Indeed, if $\mathcal{M} \in Max_\ell R^{(1)}$ and $R \not\subset \mathcal{M}$, then $\mathcal{M} + R = R^{(1)}$. This means that $1 = y + a$ for some $y \in \mathcal{M}$ and $a \in R$.

(iv) The map $\widehat{u}_R : \widehat{Spec}_\ell R^{(1)} \setminus Spec Z \to \widehat{Spec}_\ell R$, $p \mapsto p \cap R$, is, by Corollary 1 of Proposition 9, a homeomorphism in the topologies $\mathfrak{T}_o$. In particular, $\widehat{u}_R$ sets a bijective correspondence between $|(Spec_\ell R^{(1)} \setminus Spec Z, \mathfrak{T}_o)|$ and $|(Spec_\ell R, \mathfrak{T}_o)|$. Since by (i) $|(Spec_\ell R^{(1)} \setminus Spec Z, \mathfrak{T}_o)| = Max_\ell R^{(1)} \cap IR^{(1)} \setminus Spec Z$, then the statement follows from the equality

$$Max_\ell^{reg} R = \{\mathcal{M} \cap R \mid \mathcal{M} \in Max_\ell R^{(1)} \setminus Spec Z\},$$

which is, in turn, a corollary of the statement (iii).

2) Clearly, $Max_\ell^{reg} R \cap IR \subset |(Spec_\ell R, \mathfrak{T})|$.

On the other hand, $|(Spec_\ell R, \mathfrak{T})| \subset |(Spec_\ell R, \mathfrak{T}_o)|$, since the topology $\mathfrak{T}_o$ is stronger, then $\mathfrak{T}$. □

B. Proposition. The subset W of $(Spec_\ell R, \mathfrak{T})$ is irreducible if and only if the ideal $\tau(W) = \bigcap\{p \mid p \in W\}$ is prime. In particular, the space $(Spec_\ell R, \mathfrak{T})$ is irreducible iff $\widehat{rad}_\ell(R) \in Spec R$.

Proof. Let $\{\alpha, \beta\} \subset IR$. Since $[W \subset V_\ell(\alpha)] \Leftrightarrow [\alpha \subset \tau(W)]$ and $V_\ell(\alpha) \cup V_\ell(\beta) = V_\ell(\alpha\beta)$, then $[W \subset V_\ell(\alpha) \cup V_\ell(\beta)] \Leftrightarrow [\alpha\beta \subset \tau(W)]$.

These implications imply, obviously, an equivalence between the irreducibility of W and the fact that $\mathfrak{r}(W)$ is prime.

C. Recall that an (algebraic) dimension of a topological space X is the greatest integer d such that there exists a strictly increasing sequence $W_0 \subsetneqq \ldots \subsetneqq W_d$ of non-empty, closed and irreducible subspaces of X. If there is no s ch a sequences, set $\dim X = \infty$.

With the topologies $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$ two notions of the ring dimension are associated:

$$\dim_\ell(R) \overset{\text{def}}{=} \dim(\mathrm{Spec}_\ell R, \mathfrak{T}_0) = \dim(\mathrm{Spec}_\ell R, \mathfrak{T}_1)$$

and $\dim(R) = \dim(\mathrm{Spec}_\ell R, \mathfrak{T})$.

They may be easily defined directly:

$\dim_\ell(R)$ is the greatest integer d, such that there exists a strictly increasing chain $p_0 \subsetneqq p_1 \subsetneqq \ldots \subsetneqq p_d$ of ideals from $\mathrm{Spec}_\ell R$ (or from $\widehat{\mathrm{Spec}}_\ell R$);

$\dim(R)$ is the greatest number k for which there exists a strictly increasing chain $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \ldots \subsetneqq \mathfrak{p}_k$ of prime ideals such that $\mathrm{rad}_\ell(\mathfrak{p}_i) = \mathfrak{p}_i$ for all i, or, equivalently, for every $0 \leqslant i \leqslant k$ the ring $R/\mathfrak{p}_i$ has no non-zero locally nilpotent ideals (see Appendix, § 1).

6. Left affine schemes

1. <u>Spaces of irreducible components.</u> We are interested not only in topological spaces themselves but rather in presheaves and sheaves on them. From this point of view only the category of closed subsets of the space is important and a selection of a "topological representative" of a given category of closed sets is the matter of convenience. For instance, sometimes it is more desirable to deal with the subspace of closed points of an algebraic variety than with the whole spectrum. Similar is the situation with morphisms: if X, X', Y, Y' are topological spaces such that $\mathcal{C}\ell X \simeq \mathcal{C}\ell X'$ and $\mathcal{C}\ell Y \simeq \mathcal{C}\ell Y'$, then any continuous map $f: X' \to Y'$ induces the direct image functor from the category of (pre)sheaves on X into the category of (pre)sheaves on Y. Thus we come to an extension $\widetilde{\mathcal{T}op}$ of the category $\mathcal{T}op$ of topological spaces whose formal definition runs as follows: $\mathcal{O}\ell\,\widetilde{\mathcal{T}op} = \mathcal{O}\ell\,\mathcal{T}op$ and the arrows $X \to Y$ are morphisms of cosity of closed sets $\phi: \underline{X} \to \underline{Y}$ such that for a continuous map $f: X' \to Y'$ there exists a commuting diagram

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\varphi} & \underline{Y} \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle \imath} \\ \underline{X}' & \xrightarrow{f} & \underline{Y}' \end{array}$$

After a superficial glance on the definition of $\widetilde{\mathcal{T}op}$ one might be afraid that the composition $(X \xrightarrow{\varphi} Y,\ Y \xrightarrow{\psi} Z) \mapsto (X \xrightarrow{\psi\circ\varphi} Z)$ might lead out of the category. Thanks to the following statement one should not worry about that.

<u>Proposition.</u> 1) *The composition* $(X \xrightarrow{\varphi} Y, Y \xrightarrow{\psi} Z) \mapsto (X \xrightarrow{\psi\circ\varphi} Z)$ *makes* $\widetilde{\mathcal{T}op}$ *into a category.*

2) *The canonical functor* $\widetilde{\mathcal{C}\ell}: \mathcal{T}op \to \widetilde{\mathcal{T}op}$ *sending* $X \xrightarrow{f} Y$

into $X \xrightarrow{f} Y$ possesses a faithfully strict right adjoint

Proof. (i) Let X be a topological space. Denote irrX the space whose set of points is the set of irreducible closed subsets of X. There is a canonical map $c_X : X \to \mathrm{irr}\,X$ assigning to each point $x \in X$ its closure $\overline{\{x\}}$. The topology of irrX is the strongest of the topologies with respect to which c_X is continuous. In other words, a set $W \subset \mathrm{irr}\,X$ is closed if and only if $c_X^{-1}(W) = \bigcup\{Y \mid Y \in W\}$ is a closed subset of X. It is not difficult to see that c_X induces a cosity isomorphism and for all Y such that there exists an isomorphism $\varphi : \mathcal{cl}\,Y \simeq \mathcal{cl}\,X$, there exists a unique homeomorphism $\mathrm{irr}\varphi : \mathrm{irr}X \xrightarrow{\sim} \mathrm{irr}Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ c_X \downarrow\wr & \mathrm{irr}\varphi & \wr\downarrow c_Y \\ \mathrm{irr}X & \longrightarrow & \mathrm{irr}Y \end{array} \qquad (1)$$

Besides, for any continuous map $X \xrightarrow{f} Y$ the correspondence $V \mapsto \overline{f(V)}$ correctly determines a unique continuous map $\mathrm{irr}f : \mathrm{irr}X \to \mathrm{irr}\,Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ c_X \downarrow & \mathrm{irr}f & \downarrow c_Y \\ \mathrm{irr}X & \longrightarrow & \mathrm{irr}Y \end{array} \qquad (2)$$

commutes. In fact, let V be a closed subset of X and $\overline{f(V)} \subset W_1 \cup W_2$ where W_1 and W_2 are closed subsets of Y. Then $V \subset f^{-1}(W_1) \cup f^{-1}(W_2)$ and, if V is irreducible, this implies $V \subset f^{-1}(W_i)$ for $i = 1$ or 2. It follows $\overline{f(V)} \subset W_i$. Therefore we have demonstrated that the correspondence $V \mapsto \overline{f(V)}$ determines a map $\mathrm{irr}X \to \mathrm{irr}Y$. From the (obvious) commutativity of the diagram (2) and the definition of topology on irrX the continuity of irr f follows. The uniqueness of irrf for which (2) commutes, follows from the fact that $\underline{c}_X$ is isomorphism.

(iii) Now let $X \xrightarrow{\phi} Y$ be an arbitrary morphism of $\widetilde{Top}$, while $\varphi : cl X \simeq cl X'$ and $\psi : cl Y \simeq l Y'$ be isomorphisms such that for a continuous map $f : X' \to Y'$ the following diagram commutes:

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\phi} & \underline{Y} \\ \varphi \; \wr\uparrow & & \uparrow\wr \; \psi \\ \underline{X}' & \xrightarrow{\underline{f}} & \underline{Y}' \end{array} \qquad (3)$$

Determine a map $irr\,\varphi : irr\,X \to irr\,Y$ setting $irr\,\phi =$ $= irr\,\psi^{-1} \circ irr\,\underline{f} \circ irr\,\varphi$. It is subject to a standard verification that $irr\,\phi$ does not depend on arbitrariness in the choice of (3); more exactly, for any morphism $\varphi : X \to Y$ of $\widetilde{Top}$ there exists a unique continuous map $irr\,\varphi : irr X \to irr\,Y$ for which the following diagram commutes

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\varphi} & \underline{Y} \\ \underline{c}_X \; \wr\| & & \downarrow\wr \; \underline{c}_Y \\ irr\,X & \xrightarrow{irr\,\varphi} & irr\,Y \end{array} \qquad (4)$$

In other words, the arrows $X \to Y$ of $\widetilde{Top}$ are all continuous maps $irr\,X \to irr\,Y$. This immediately implies the closedness of $\widetilde{Top}$ with respect to the natural composition.

(iv) Clearly $c = \{c_X \mid X \in Ob\,Top\}$ is the morphism of the identity functor Id_{Top} into $irr \circ \widetilde{cl}$. The family of canonical isomorphisms $\widetilde{cl} \circ irr\,X \to X$, $X \in Ob\,\widetilde{Top}$, assigning to a closed subset $V \subset$ irr X the closed (by definition of the topology on irr X) set $\bigcup\{Y \mid Y \in V\}$ is a morphism of functors $h : \widetilde{cl} \circ irr \xrightarrow{\sim} Id_{\widetilde{Top}}$. It is easy to see that the following relations hold

$$irr\,h \circ c\,irr = id_{irr}\,, \quad h\,\widetilde{cl} \circ \widetilde{cl}\,c = id_{\widetilde{cl}}\,.$$

In other words, $\widetilde{cl} \dashv irr$ and (c, h) are conjunction morphisms. Since h is an isomorphism, then irr is full and faithful.

Corollary 1. *The following properties of a topological space X are equivalent:*

1) X *satisfies*

a) *any irreducible closed subset of* X *contains a generic point (i.e. is the closure of a point);*

b) for any $(x, y) \in X \times X$ $[\overline{\{x\}} = \overline{\{y\}}] \Longrightarrow [x = y]$.

2) *Any quasihomeomorphism* $X \longrightarrow Y$ *is a homeomorphism.*

3) *The canonical map* $c_X \colon X \longrightarrow irrX$ *is a homeomorphism.*

Proof. The condition (a) is equivalent to surjectiveness of c_X and (b) to its injectiveness. Therefore $1) \Longrightarrow 3)$. Clearly, $2) \Longrightarrow 3)$. It remains to show that $3) \Longrightarrow 2)$.

Let $f\colon X \longrightarrow Y$ be a quasihomeomorphism satisfying (b), i.e. c_Y is injective. The bijectiveness of c_X and $irr\underline{f}$ in the commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "c_X"'] & Y \arrow[d, "c_Y"] \\ irrX \arrow[r, "irr\underline{f}"] & irrY \end{tikzcd}$$

implies surjectiveness and, hence, bijectiveness of c_Y. It follows that f is also bijective and, therefore, homeomorphism.

Let $\widehat{\widehat{\mathcal{T}op}}$ denotes the full subcategory of $\mathcal{T}op$ formed by all the spaces X, satisfying the equivalent conditions of Corollary 1.

Corollary 2. 1) *The embedding* $\widehat{\widehat{\mathcal{T}op}} \hookrightarrow \mathcal{T}op$ *possesses a left adjoint functor* Irr *assigning* irrX *to a space* X *and* irrf *to a continuous map* f.

2) $\widehat{\widehat{\mathcal{T}op}}$ *is equivalent to the residue category of* $\mathcal{T}op$

modulo all the quasihomeomorphisms.

3) dim(X)=dim(irrX) for any topological space X (where dim is algebraic dimension; see 5.18.C). The dimesion of a space $Y \in \mathcal{O}b\,\hat{\hat{\mathcal{T}}}op$ coincides with the upper bound d of the lengths of the chains. (Recall that a sequence $y_0,\dots,y_k$ of points of the space Y is a chain of length k, starting in y_0 and ending in y_k, if $y_i \neq y_{i+1}$ and y_{i+1} y_i for all $0 \leq i < k$.)

Proof. 1), 2). It directly follows from Proposition 1 that $\widetilde{\mathcal{T}op}$ is isomorphic to the residue category of $\mathcal{T}op$ modulo all the quasihomeomorphisms; see [12], Ch.I, § 1.

It is not difficult to see that the functor irr: $\widetilde{\mathcal{T}op} \to \mathcal{T}op$ takes values in $\widehat{\widehat{\mathcal{T}op}}$, and its corestriction on $\widehat{\widehat{\mathcal{T}op}}$ is an equivalence of categories, since both conjuction morphisms $\{h.\}$ and $\{c.\}$ are isomorphisms, when we confine ourself to $\widehat{\widehat{\mathcal{T}op}}$.

3) It is clear that dimension is invariant with respect to quasihomeomorphisms. The second part of the statement follows directly from the definition of dimension (see 5.18.C) and properties a) and b) of the spaces from $\widehat{\widehat{\mathcal{T}op}}$ (see Corollary 1).□

Therefore the functor $\widehat{\widehat{\mathrm{irr}}} \overset{\text{def}}{=} \mathrm{irr}\,|\,\widehat{\widehat{\mathcal{T}op}}$ performs a natural for the study of "geometric objects", i.e. the spaces ringed with presheaves, decrease of the number of objects and increase of the number of morphisms. So, if there is nothing special against it, it is advisable to pass from ringed spaces $(X, \mathcal{O})$ to their quasiisomorphic ringed spaces $(\mathrm{irr}X, c_{X*}\mathcal{O})$.

Example. Let A be a commutative associative ring with unit, $Max A$ the space of its maximal ideals with the Jacobson-Zariski topology; $\mathfrak{J}Spec A$ the subspace of SpecA formed by all the simple ideals that coincide with their Jacobson radical, i.e. are intersections of maximal ideals. It is easy to verify that the map assigning to an irreducible closed subset W the ideal $r(W) = \bigcap\{\mu \mid \mu \in W\}$ performs a homeomorphism of $irr\, Max A$ onto $\mathfrak{J}Spec A$.

Now let B be a (commutative and unitary) Jacobson ring, i.e. $\mathfrak{J}Spec B = Spec B$; for instance, B is a finitely generated algebra over a field or over the ring of integers. Not every ring morphism $B \to A$ induces a map $Max A \to Max B$, but every morphism induces the map $\mathfrak{J}Spec A \longrightarrow \mathfrak{J}Spec B$. $\square$

In what follows we will meet a noncommutative analogue of the situation described in this example.

2. Spaces of irreducible components of the left spectrum. Set $\overline{Spec}\, R = \{p \in Spec R \mid p = rad_\ell(p)\}$.

Proposition. 1) There is a canonical homeomorphism

$$\tilde{c} : (\widetilde{Spec}_\ell R, \tilde{\mathfrak{T}}_0) \xrightarrow{\sim} irr(Spec_\ell R, \mathfrak{T}_0)$$

where $\tilde{\mathfrak{T}}_0$ is the quotient topology of the topology $\mathfrak{T}_0$.

2) There is a canonical homeomorphism

$$irr(Spec_\ell R, \mathfrak{T}) \xrightarrow{\sim} \overline{Spec}\, R$$

assigning to an irreducible closed set W the ideal $r(W) = \bigcap\{p \mid p \in W\}$.

Proof. 1) Since the closure of any subset $X \subset Spec_\ell R$ equals $\overline{\mathfrak{T}_0 X} \overset{def}{=} \bigcup\{V_\ell(\mu) \mid \mu \in X\}$, then the irreducible closed subsets of $(Spec_\ell R, \mathfrak{T}_0)$ are exactly all the $V_\ell(p)$, $p \in \widehat{Spec}_\ell R$. In other words every irreducible closed subset $W \subset Spec_\ell R$ possesses a generic point, and therefore (see Corollary 1 of

Proposition 1) the canonical map $c : \mathrm{Spec}_\ell R \to \mathrm{irr}(\mathrm{Spec}_\ell R, \mathfrak{T}_0)$ is surjective. Since $[V_\ell(p) = V_\ell(p')] \Leftrightarrow [p \simeq p']$, then c induces an injective and therefore bijective map $\tilde{c} : ({}^\sim\mathrm{Spec}_\ell R, {}^\sim\mathfrak{T}_0) \to \mathrm{irr}(\mathrm{Spec}_\ell R, \mathfrak{T}_0)$. Clearly, c is a homeomorphism.

2) This is a corollary of Proposition 5.14.B. □

3. <u>Main homeomorphisms. Closed points of $\overline{\mathrm{Spec}}\, R$.</u>

For any subset $x \subset R$ let
$$\overline{V}(x) = \overline{V}^R(x) \overset{\text{def}}{=} \{p \in \overline{\mathrm{Spec}}\, R \mid x \subset p\}, \quad \overline{U}(x) = \overline{U}^R(x) \overset{\text{def}}{=} \overline{\mathrm{Spec}}\, R \smallsetminus \overline{V}(x).$$

<u>Proposition</u>. 1) *Let α be a two-sided ideal of R.*

i) *The map $p \mapsto p/\alpha$ determines a homeomorphism of the closed subspace $\overline{V}(\alpha)$ of $\overline{\mathrm{Spec}}\, R$ onto $\overline{\mathrm{Spec}}\, R/\alpha$.*

ii) *The map $p \mapsto p \cap \alpha$ determines a homeomorphism $\overline{u}_\alpha$ of the open subspace $\overline{U}(\alpha)$ onto $\overline{\mathrm{Spec}}\, \alpha$.*

2) *Let e be a nonzero idempotent in R. The map $\mu \mapsto \mu \cap eRe$ determines a homeomorphism $\overline{u}_{eRe} : \overline{U}(e) \xrightarrow{\sim} \overline{\mathrm{Spec}}\, eRe$.*

<u>Proof</u>. 1) These statements follow from Propositions 5.9, 5.11 and Proposition 2, of the existence of a natural homeomorphism $\mathrm{irr}(\mathrm{Spec}_\ell R, \mathfrak{T}) \xrightarrow{\sim} \overline{\mathrm{Spec}}\, R$. The following commuting diagrams serve as "justifying documents":

(i)
$$\begin{tikzcd}
\mu \arrow[rrrr, mapsto] \arrow[ddd, mapsto] & & & & \mu_s \arrow[dd, mapsto] \\
& V_\ell(\alpha) \arrow[r] \arrow[d, "\wr"'] & \mathrm{irr}\, V_\ell(\alpha) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \overline{V}(\alpha) \arrow[d] & \\
& \mathrm{Spec}_\ell R/\alpha \arrow[r] & \mathrm{irr}\, \mathrm{Spec}_\ell R/\alpha \arrow[r, "\sim"] & \overline{\mathrm{Spec}}\, R/\alpha & \mu_{s/\alpha} \arrow[d, equal] \\
\mu/\alpha \arrow[rrrr, mapsto] & & & & (\mu/\alpha)_s
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
p \arrow[rrrr, mapsto] \arrow[ddd, mapsto] & & & & p_s \arrow[dd, mapsto] \\
& U_\ell(\alpha) \arrow[r] \arrow[d] & \mathrm{irr}\, U_\ell(\alpha) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \overline{U}(\alpha) \arrow[d] & \\
& \mathrm{Spec}_\ell \alpha \arrow[r] & \mathrm{irr}\, \mathrm{Spec}_\ell \alpha \arrow[r, "\sim"] & \overline{\mathrm{Spec}}\, \alpha & p_s \cap \alpha \arrow[d, equal] \\
p \cap \alpha \arrow[rrrr, mapsto] & & & & (p \cap \alpha)_s
\end{tikzcd}$$

2)
$$\begin{array}{ccccccc}
\mathcal{M} & \longmapsto & & & & & \mathcal{M}_s \\
\Big\downarrow & \widehat{U}_e(eRe) & \longrightarrow & irr\,\widehat{U}_e(eRe) & \xrightarrow{\sim} & \overline{U}(eRe)=\overline{U}(e) & \Big\downarrow \\
 & \wr\downarrow & & \wr\downarrow & & \downarrow & \\
 & \widehat{Spec}_e eRe & \longrightarrow & irr\,\widehat{Spec}_e eRe & \xrightarrow{\sim} & \overline{Spec}\, eRe & \\
\mathcal{M}\cap eRe = e\mathcal{M}e & \longmapsto & & & & (e\mathcal{M}e)_s = e\mathcal{M}_s e &
\end{array}$$

In the last diagram we have made use of the fact that the inclusion $\widehat{Spec}_e R' \hookrightarrow Spec_e R'$ is a quasihomeomorphism for any associative ring R' (Corollary 2 of Proposition 1.6). □

Corollary 1. *Let α be a two-sided ideal, e a nonzero idempotent of R. Then the map $\mathcal{M} \mapsto \mathcal{M}\cap e\alpha e$ induces a homeomorphism* $\overline{U}(e)\cap\overline{U}(\alpha) \xrightarrow{\sim} \overline{Spec}\, e\alpha e$.

Corollary 2. *Let f be an idempotent in R different from unit, α a two-sided ideal. The map $\mathcal{M} \mapsto \mathcal{M}\cap(1-f)\alpha(1-f)$ performs a homeomorphism* $\overline{U}(e)\cap\overline{U}(\alpha) \xrightarrow{\sim} \overline{Spec}\, e\alpha e$.

These statements are proved as are Corollaries 2 and 3 of Proposition 5.11. □

Corollary 3. *The map $\mathcal{M} \mapsto \mathcal{M}\cap R$ performs a homeomorphism of $\overline{Spec}\, R^{(1)} \setminus Spec\, Z$ onto $\overline{Spec}\, R$. (Here as always $R^{(1)}$ is the ring obtained from R by incorporating the unit.)*

Proof. $\overline{Spec}\, R^{(1)} \setminus Spec\, Z = \overline{U}^{R^{(1)}}(R)$. □

Corollary 4. *The set $|\overline{Spec}\, R|$ of the closed points of $\overline{Spec}\, R$ coincides with the set $Max^{reg} R$ of two-sided maximal regular ideals of R.*

Proof. 1) Let R be a ring with right unit. Then $Max\, R \subset \{\mathcal{M}_s \mid \mathcal{M} \in Max_e R\}$ and therefore $Max\, R \subset \overline{Spec}\, R$. Clearly, $Max\, R \subset |\overline{Spec}\, R|$ and, since any proper two-sided ideal is contained in an ideal from $Max\, R$, we have the converse inclusion.

2) In general case the homeomorphism $\overline{Spec}\, R^{(1)} \setminus Spec\, Z \xrightarrow{\sim} \overline{Spec}\, R$

induces a bijection $|\overline{\mathrm{Spec}}\,R^{(1)} \setminus \mathrm{Spec}\,Z| \xrightarrow{\sim} |\overline{\mathrm{Spec}}\,R|$. Clearly, $|\overline{\mathrm{Spec}}\,R^{(1)} \setminus \mathrm{Spec}\,Z| = |\overline{\mathrm{Spec}}\,R^{(1)}| \setminus \mathrm{Spec}\,Z = \mathrm{Max}\,R^{(1)} \setminus \mathrm{Spec}\,Z$. Thus, $|\overline{\mathrm{Spec}}\,R| = \{\mu \cap R \mid \mu \in \mathrm{Max}\,R^{(1)}$ and $R \not\subset \mu\}$. But, if $\mu \in \mathrm{Max}\,R^{(1)}$ and $R \not\subset \mu$, then $\mu + R = R^{(1)}$ and, therefore, $1 = x + a$ for some $x \in \mu$ and $a \in R$. As follows from the regularity criterion provided by Lemma 5.18 (step (ii) of the proof of Proposition 5.18.A), this implies the regularity of $\mu \cap R$.

Let us show that $\mu \cap R \in \mathrm{Max}\,R$. If m is a proper two-sided ideal of R, containing $\mu \cap R$, then $m \subset (m : Ra) \subset (m : a) = m$ and, hence, $m = (m : Ra) = (m : Ra)_{R^{(1)}} \cap R$; in particular, $(m : Ra)_{R^{(1)}}$ is a proper two-sided ideal of $R^{(1)}$. Since $(\mu \cap R : Ra) \underset{R^{(1)}}{=} (\mu : Ra)_{R^{(1)}} \supset \mu$, then the inclusion $\mu \cap R \subset m$ implies the inclusion $\mu \subset$ $\subset (m : Ra)_{R^{(1)}}$; the latter can be substituted by equality thanks to the maximality of μ. Therefore $\mu \cap R = (m : Ra)_{R^{(1)}} \cap R = (m : Ra) = m$.

So, we have proved that $|\overline{\mathrm{Spec}}\,R| \subset \mathrm{Max}^{reg}\,R$.

Now let $m \in \mathrm{Max}^{reg}\,R$. Then m, as any regular left ideal, is contained in some left maximal regular ideal μ (see [16], Lemma 1.2.1, or prove this simple fact yourself). Since m is maximal, it coincides with μ_s; therefore $m \in \overline{\mathrm{Spec}}\,R$. □

Comparison with Proposition 5.18.A shows how many closed points did we acquire, in general case, passing from $(\mathrm{Spec}_\ell R, \mathfrak{T})$ to its quasihomeomorphic $\overline{\mathrm{Spec}}\,R$.

4. <u>Canonical open embeddings</u>. We have spent quite a time discussing topological spaces and ignoring the structure (pre) sheaves. Now we will restore the equilibrium. To not lose the objects like presheaves of rings $\mathcal{O}_R$ and presheaves of $\mathcal{O}_R$-modules $\mathcal{O}_M$, $M \in \mathrm{Ob}\,R\text{-mod}$, we will broaden the

traditional frameworks and consider the preringed spaces and presheaves of modules over them. Just in case let us elucidate what were speaking about: a preringed space is a pair $(X, \mathcal{O})$ where X is a topological space, $\mathcal{O}$ is a presheave of associative rings over X and a morphism of preringed spaces $(X, \mathcal{O}) \to (X', \mathcal{O}')$ is a pair $(\varphi, \varphi^{\diamond})$ consisting of a continuous map $\varphi : X \longrightarrow X'$ and a morphism $\varphi^{\diamond}$ of the presheave $\mathcal{O}'$ into the direct image $\varphi_{*}\mathcal{O}$ of $\mathcal{O}$.

The category of preringed spaces thus defined will be denoted $SpRings$. Denote $S'h_{\omega}Rings$ and $ShRings$ the full subcategories of $SpRings$ formed by ω-ringed spaces (the $(X, \mathcal{O})$ such that $\mathcal{O}$ is a ω-sheaf) and ringed spaces respectively.

Further for preringed spaces of a particular form several general notions will be used:

A preringed space $(X, \mathcal{O})$ is connected if its basic topological space X is connected and irreducible if so is X.

A preringed space $(X, \mathcal{O})$ is reduced or rad_{ℓ}-reduced if $\mathcal{O}$ is a presheave of rad_{ℓ}-semisimple rings.

For any radical r in the category of associative rings we similarly define r-reduced preringed spaces. The canonical ringed space $(Spec\, R, \widetilde{R})$ of a left semiprimary Noetherian ring R (see 4.17, 4.18) is a good example of a β-reduced ringed space where β is the low Bair radical. In what follows we will deal with J-reduced ω-ringed spaces where J is as usual the Jac obson radical.

An open subspace of a preringed space $(X, \mathcal{O})$ is a preringed space $(U, \mathcal{O}|_U)$ where U is an open subset of X. A preringed space morphism $(\varphi, \varphi^\circ): (X, \mathcal{O}) \to (X', \mathcal{O}')$ is an open embedding if it induces an isomorphism of $(X, \mathcal{O})$ with an open subspace $(\varphi(X), \mathcal{O}'|\varphi(X))$.

The open embeddings--the main tools for constructing global objects from local ones--are the main characters of this section.

Proposition. *For any twosided ideal* α *of a ring* R *the map* $\mu \mapsto \mu \cap \alpha$ *induces preringed space isomorphisms:*

$$(U_\ell(\alpha), {}^0\mathcal{O}_R|U_\ell(\alpha)) \xrightarrow{\sim} (Spec_\ell \alpha, {}^0\mathcal{O}_\alpha)$$

$$(U_\ell(\alpha), {}^1\mathcal{O}_R|U_\ell(\alpha)) \xrightarrow{\sim} (Spec_\ell \alpha, {}^1\mathcal{O}_\alpha)$$

$$(U_\ell(\alpha), \mathcal{O}_R|U_\ell(\alpha)) \xrightarrow{\sim} (Spec_\ell \alpha, \mathcal{O}_\alpha)$$

Proof. One half of the statement, on homeomorphicy of the map $u_\alpha: U_\ell(\alpha) \to Spec_\ell \alpha$, $\mu \mapsto \mu \cap \alpha$ in topologies $\mathfrak{T}$, $\mathfrak{T}_1$ and $\mathfrak{T}_0$, is proved above (proposition 5.9) . Let us prove the second half-the existence of the natural isomorphisms of the corresponding presheaves.

The presheave ${}^0\mathcal{O}_R|U_\ell(\alpha)$ assigns to a closed subset $W \cap U_\ell(\alpha)$ of $U_\ell(\alpha)$ (where W is a closed subset of $(Spec_\ell R, \mathfrak{T}_0)$) the ring $G_{\mathcal{F}_{V_\ell(\alpha) \cup W}} R$ Clearly $\alpha \in \mathcal{F}_{V_\ell(\alpha) \cup W}$ for any W since $\alpha \in \mathcal{F}_{V_\ell(\alpha)}$ and $\mathcal{F}_{V_\ell(\alpha)} \subset \mathcal{F}_{V_\ell(\alpha) \cup W}$.

Lemma. *Let* $\mathcal{F}$ *be a radical filter of left ideals of* R *and* α *a twosided ideal from* $\mathcal{F}$. *Then* $\mathcal{F} \cap \alpha \overset{def}{=} \{\nu \cap \alpha \mid \nu \in \mathcal{F}\}$

is a radical filter of left ideals of α and there exists a unique isomorphism $G^{\alpha}_{\mathcal{F}\cap\alpha}\,\alpha \xrightarrow{\sim} G^{R}_{\mathcal{F}}\,R$ such that the following diagram commutes

$$\begin{tikzcd}
G^{\alpha}_{\mathcal{F}\cap\alpha}\,\alpha \arrow[r, "\sim"] & G^{R}_{\mathcal{F}}\,R \\
\alpha \arrow[u, "j_{\mathcal{F}\cap\alpha,\alpha}"] \arrow[r, hook] & R \arrow[u, "j_{\mathcal{F},R}"']
\end{tikzcd} \qquad (1)$$

Proof. The radicality of $\mathcal{F}\cap\alpha$ is subject to a straightforward verification. Clearly $\mathcal{F}\cap\alpha$ is a cofinal subset of ideals in $\mathcal{F}$. Therefore $\mathcal{F}^{1}M = (\mathcal{F}\cap\alpha)^{1}M$ for any R-module M and $G_{\mathcal{F}}M = G^{R}_{\mathcal{F}}M = \varinjlim\left(Hom_R(m, (\mathcal{F}\cap\alpha)^{1}M) \mid m\in\mathcal{F}\cap\alpha\right)$. Now notice that for any pair R-modules M and N we have $Hom_R(M,N) = Hom_{\alpha}(M,N)$ if the $\{\alpha\}$-torsion of N is zero.

In fact for any $f\in Hom_{\alpha}(M,N)$, $x\in R$ and $\lambda\in\alpha$ we have $\lambda f(x-) = f(\lambda x-) = \lambda x f(-)$. I.e. $\alpha\left(f(x-) - x f(-)\right) = 0$. Since N has no $\{\alpha\}$-torsion, then $f(x-) = x f(-)$ meaning (since $x\in R$ is arbitrary) that $f\in Hom_R(M,N)$.

In particular, $Hom_R(-,(\mathcal{F}\cap\alpha)^{1}M) = Hom_{\alpha}(-,(\mathcal{F}\cap\alpha)^{1}M)$ and therefore

$$G^{R}_{\mathcal{F}}M \simeq \varinjlim\left(Hom_{\alpha}(m,(\mathcal{F}\cap\alpha)^{1}M) \mid m\in\mathcal{F}\cap\alpha\right) \simeq G^{\alpha}_{\mathcal{F}\cap\alpha}M.$$

Applying the established isomorphism $G^{R}_{\mathcal{F}}M \simeq G^{\alpha}_{\mathcal{F}\cap\alpha}M$ to the R-modules α and R we get the commuting diagram

$$\begin{tikzcd}
R \arrow[r] & G^{R}_{\mathcal{F}}R \arrow[r, "\sim"] & G^{\alpha}_{\mathcal{F}\cap\alpha}R \\
\alpha \arrow[u, hook, "i_{\alpha}"] \arrow[r] \arrow[ur, phantom, "1"] & G^{R}_{\mathcal{F}}\alpha \arrow[u, "\wr"'] \arrow[r, "\sim"] \arrow[ur, phantom, "2"] & R \arrow[u, "\wr"'] \arrow[ul]
\end{tikzcd} \qquad (2)$$

in which $G_{\mathcal{F}}\, i^{\alpha}$ is an isomorphism since $\alpha \in \mathcal{F}$. Therefore all the arrows of the subdiagram 2 in particular $\widetilde{i^{\alpha}}$ are isomorphisms. Since $G_{\mathcal{F}}\, i^{\alpha}$ is uniquely determined by the commutativity of the subdiagram 1, then $\widetilde{i}^{\alpha}$ is uniquely determined by the commutativity (1).

Obviously lemma and remark just before it implies the existence of a unique isomorphism ${}^{o}\mathcal{O}_{\alpha} \xrightarrow{\sim} u_{\alpha *}{}^{o}\mathcal{O}_{R} \,|\, U_{\ell}(\alpha)$ such that the following diagram commutes:

$$\begin{array}{ccc} {}^{o}\mathcal{O}_{\alpha} & \xrightarrow{\quad\sim\quad} & u_{\alpha *}{}^{o}\mathcal{O}_{R} \,|\, U_{\ell}(\alpha) \\ \uparrow & & \uparrow \\ \alpha & \hookrightarrow & R \end{array} \qquad (3)$$

Here α and R are considered as constant presheaves on $(Spec_{\ell}\alpha, \mathfrak{T}_{o})$; and the vertical arrows are sets of morphisms $\{ j_{\mathcal{F}^{\alpha}_{W}, \alpha} \mid W \in \mathfrak{T}_{o} \}$ and $\{ j_{\mathcal{F}^{\alpha}_{W}, R} = u_{\alpha *}\, j_{\mathcal{F}^{R}_{u_{\alpha}^{-1}W}} \}$.

Thus, we have constructed an isomorphism

$(u_{\alpha}, {}^{o}u_{\alpha}^{\diamond}) : (U_{\ell}(\alpha), {}^{o}\mathcal{O}_{R} \,|\, U_{\ell}(\alpha)) \xrightarrow{\sim} (Spec_{\ell}\alpha, {}^{o}\mathcal{O}_{\alpha})$.

It induces the other two isomorphisms, mentioned in the formulation of proposition. $\square$

Remark. The canonical isomorphism $G^{R}_{\mathcal{F}} M \xrightarrow{\sim} G^{\alpha}_{\mathcal{F} \cap \alpha}$ for any radical filter $\mathcal{F} \subset I_{\ell} R$, ideal $\alpha \in \mathcal{F} \cap IR$ and any R-module M obtained in the process of the demonstration of Lemma presheave isomorphisms ${}^{o}u_{\alpha, M} : {}^{o}\mathcal{O}_{i^{\alpha}_{*} M} \xrightarrow{\sim} u_{\alpha *}{}^{o}\mathcal{O}_{M} \,|\, U_{\ell}(\alpha)$ follow (here i^{α}_{*} is the restriction of the scalar ring functor corresponding to the embedding $i^{\alpha} : \alpha \hookrightarrow R$). The restrictions of morphisms ${}^{o}u_{\alpha, M}$ on $\mathfrak{T}_{1}$ and $\mathfrak{T}$ determine isomorphisms

${}^{1}u_{\alpha, M} : {}^{1}\mathcal{O}_{i^{\alpha}_{*} M} \xrightarrow{\sim} u_{\alpha *}{}^{1}\mathcal{O}_{M} \,|\, U_{\ell}(\alpha)$, $u_{\alpha, M} : \mathcal{O}_{i^{\alpha}_{*} M} \xrightarrow{\sim} u_{\alpha *}\mathcal{O}_{M} \,|\, U_{\ell}(\alpha)$.

For any presheave F on $(Spec_{\ell} R, \mathfrak{T})$ denote by $\overline{F}$ the corresponding presheave on $\overline{Spec}\, R$ the direct image of F with respect to the canonical quasihomeomorphism $Spec_{\ell} R \longrightarrow \overline{Spec}\, R$

Corollary. For any twosided ideal α of R and any R-module M there are canonical isomorphisms

a) of preringed spaces

$$(\bar{u}_{\alpha}, \bar{u}^{\diamond}_{\alpha}) : (\overline{U}(\alpha), \overline{\mathcal{O}}_R|\overline{U}(\alpha)) \xrightarrow{\sim} (\overline{Spec}\,\alpha, \overline{\mathcal{O}}_{\alpha});$$

b) of presheaves of $\overline{\mathcal{O}}_{\alpha}$ -modules $\overline{\mathcal{O}}_{i^{\alpha}_{*}M} \xrightarrow{\sim} \bar{u}_{\alpha *}\overline{\mathcal{O}}_M|\overline{U}(\alpha)$.

Proof. The statement follows directly from Proposition 4 subsequent Remark and Proposition 3. $\square$

If $(X, \mathcal{O})$ is a preringed space then $(X, \mathcal{O}^a)$ will be called the ringed space associated with $(X, \mathcal{O})$. Since $\mathcal{G}^a|_U = (\mathcal{G}|_U)^a$ for any presheave $\mathcal{G}$ an X and any open subset $U \subset X$ then the open embeddings of preringed spaces are uniquely extendible till open embeddings of associated ringed spaces. In particularly in all the isomorphisms of Proposition 4 its corollary and remark we can replace presheaves by associated sheaves.

5. Left schemes and quasischemes. Now we have enough arguments to give the following definitions:

A ringed space $(X, \mathcal{O})$ will be called

a left affine quasischeme, if $(X, \mathcal{O}) \simeq (\overline{Spec}\, R, \overline{\mathcal{O}}^a_R)$ (for an associated ring R ;

a left affine scheme, if $(X, \mathcal{O}) \simeq (\overline{Spec}\, R, \overline{\mathcal{O}}^a_R)$ for an associated unitary ring R

a left quasischeme (left scheme) if there exists an open covering $\{U_{\alpha} \mid \alpha \in \mathfrak{A}\}$ of X such that $(U_{\alpha}, \mathcal{O}|U_{\alpha})$ is a left affine quasischeme (left affine scheme) for any $\alpha \in \mathfrak{A}$.

A few words on the interrelations of these notions.

1) By Corollary of Proposition 4 an open subspace of a left affine quasischeme is a left affine quasischeme. This statement may be refined:

$$\begin{tikzcd}
(X, \mathcal{O}) \arrow[r, "(\varphi, \varphi^{\diamond})", "\sim"'] & (\overline{\mathrm{Spec}}\, R, \overline{\mathcal{O}}^a_R) & \\
& & (\overline{\mathrm{Spec}}\, \alpha, \overline{\mathcal{O}}^a_\alpha) \arrow[ul] \\
(U, \mathcal{O}|_U) \arrow[uu] \arrow[r, "\sim"] & (\varphi(U), \overline{\mathcal{O}}^a_R|_{\varphi(U)}) \arrow[ur] &
\end{tikzcd}$$

Here $\alpha = \tau(\varphi(U)^\perp)$; $\varphi(U)^\perp$ is the comlement in $\overline{\mathrm{Spec}}\, R$ to the image of an open subset $U \subset X$.

This implies that an arbitrary open subspace of a left quasischeme is a left quasischeme.

2) Specialization of the same corollary of Proposition 4 shows that any left affine quasischeme is isomorphic to a left affine scheme with a truncated $\mathrm{Spec}\, \mathbb{Z}$, since for any R we have a canonical isomorphism

$$(\overline{\mathrm{Spec}}\, R, \overline{\mathcal{O}}^a_R) \simeq (\overline{\mathrm{Spec}}\, R^{(1)} \setminus \mathrm{Spec}\, \mathbb{Z}, \overline{\mathcal{O}}^a_{R^{(1)}}|_{\overline{\mathrm{Spec}}\, R^{(1)} \setminus \mathrm{Spec}\, \mathbb{Z}}).$$

3) Call a (pre)ringed space $(X, \mathcal{O})$ <u>commutative</u> if $\mathcal{O}$ is a (pre)sheave of commutative rings. Clearly a commutative left affine scheme is an affine scheme in the usual sense.

A commutative affine quasischeme is isomorphic due to 2) to an affine scheme with a truncated closed subspace $\mathrm{Spec}\, \mathbb{Z}$ and, therefore, is a scheme. Besides, a commutative affine quasischeme is an affine scheme if and only if its basic topological space is quasicompact.

Recall that by Proposition 5.7 $\overline{Spec}\, R$ is quasi-compact for any associative ring with unit. Conversely let R be a ring (not necessarily commutative yet) such that $\overline{Spec}\, R$ is quasicompact. Then there exists a finitely generated ideal $\alpha \in I R$ such that $\overline{Spec}\, R = \overline{U}(\alpha) \simeq$ $\simeq \overline{Spec}\,\alpha$ (see Corollary 1 of Proposition 5.7) We will consider α as an ideal of the unitary ring $R^{(1)}$. Let $\{t_1, \dots, t_k\}$ be a finite set of generators of α (as a twosided ideal) and $t = t_1 \cdot \ldots \cdot t_k$. If R (and $R^{(1)}$) is a commutative ring, then $\bar{\mathcal{O}}^a_{R^{(1)}}(\overline{U}(\alpha)) \simeq (t)^{-1} R^{(1)}$ and $Spec(t)^{-1}R^{(1)} = \overline{U}(\alpha) = Spec\, R$.

4) Any commutative quasischeme is a scheme since it is glued together of affine quasischemes which are canonically isomorphic to open subschemes of affine schemes. In the non-commutative case this is not so; there exist left affine quasi-schemes which are not schemes.

5) However any left quasischeme is isomorphic to an open subspace (subquasischeme) of the left scheme.

Indeed, let $(X, \mathcal{O})$ be a left quasischeme $\{U_i \mid i \in J\}$ its quasiaffine covering i.e. there are isomorphisms $(U_i, \mathcal{O}|_{U_i}) \simeq (\overline{Spec}\, R_i, \bar{\mathcal{O}}^a_{R_i})$. which determine the the gluing conditions

$$(U_{ij}, \bar{\mathcal{O}}^a_{R_i}|_{U_{ij}}) \simeq (U_{ji}, \bar{\mathcal{O}}^a_{R_j}|_{U_{ji}}),\ \{i, j\} \subset J.$$

The passage of affine quasischemes to their compactification $(\overline{Spec}\, R_i^{(1)}, \bar{\mathcal{O}}^a_{R_i^{(1)}})$ does not affect the glued data with the help of which we glue a left scheme of left affine schemes $(\overline{Spec}\, R_i^{(1)}, \bar{\mathcal{O}}^a_{R_i^{(1)}})$. In the thus constructed left scheme one naturally single out an open subspace isomorphic to the initial quasischeme.

6) Let R be a ring; $Ann_r R$ is its twosided ideal $\{x\in R \mid Rx=0\}$. The canonical epimorphism $R \twoheadrightarrow R/Ann_r R = \check{R}$ induces the isomorphism $(\widehat{Spec}\, R, \bar{\mathcal{O}}_R) \xrightarrow{\sim} (\widehat{Spec}\, \check{R}, \bar{\mathcal{O}}_{\check{R}})$.

In fact, $Ann_r R \subset \widehat{rad}_\ell(R)$ and therefore the map $\mathcal{M} \mapsto \mathcal{M}/Ann_r R$ performs the homeomorphism $\widehat{Spec}\, R \xrightarrow{\sim} \widehat{Spec}\, \check{R}$. Moreover, $Ann_r R$ is exactly the kernel of the canonical ring morphism $R \to Hom_R(R, R)$, $x \mapsto r_x$ (see 2.10) and $G_{\{R\}} R = G_{\{R\}} \check{R} \simeq G_{\{\check{R}\}} \check{R}$.

7) If R is a ring with right unit, then $(\widehat{Spec}\, R, \bar{\mathcal{O}}^a_R)$ is a left affine scheme, since in this case $\check{R} \simeq Hom_R(R,R)^o$.

6.V. Reduced left schemes and $\widehat{rad}_\ell$-semisimple modules.

Denote by $\widehat{I_\ell R}$ the family of all left ideals ν of R such that $rad_\ell(\nu) = \nu_s \overset{def}{=} \nu \cap (\nu : R)$. And R-module M will be called $\widehat{rad}_\ell$-semisimple if $Ann\, \xi \in \widehat{I_\ell R}$ for any $\xi \in \mathcal{M}$. A full subcategory of $R\text{-}mod$ formed by $\widehat{rad}_\ell$-semisimple module we will denote by $R\text{-}\widehat{mod}$. These new characters enjoy the following properties:

1) The intersection of an arbitrary family of ideals of $\widehat{I_\ell R}$ belong to $\widehat{I_\ell R}$.

2) $[\nu \in \widehat{I_\ell R} \setminus \{R\}] \Leftrightarrow [R/\nu \in Ob\, R\text{-}\widehat{mod}, (\nu:R) \subset \nu]$.

In fact, let $(Ann\,\xi)_s = rad_\ell(Ann\,\xi)$ for any $\xi \in R/\nu$. This means that $(\nu : t)_s = rad_\ell((\nu:t))$ for any $t \in R \setminus \nu$. But $\nu_s = \bigcap\{(\nu:t)_s \mid t \in R \setminus \nu\}$ and therefore $\nu_s = rad_\ell(\nu)$.

Conversely suppose that $\nu \in \widehat{I_\ell R}$; i.e. $\nu_s = rad_\ell(\nu) = \bigcap\{p \in \widehat{Spec}_\ell R \mid \nu \to p\}$. This implies that $(\nu:t)_s \subset rad_\ell((\nu:t)) \subset$ $p:t) \mid p \in \widehat{V}_\ell(\nu)\} = (\nu_s : t)$. On the other hand, $(\nu_s : t) \subset (\nu : t)$ and therefore $rad_\ell((\nu:t)) \subset (\nu_s:t)_s \subset (\nu:t)_s$. Hence $(\nu:t)_s = rad_\ell((\nu:t))$.

3) The left R-module R is $\widehat{rad_\ell}$-semisimple if and only if R is $\widehat{rad_\ell}$-semisimple, i.e. $\widehat{rad_\ell}(R)=0$.

In fact by (2) R-module R is $\widehat{rad_\ell}$-semisimple if and only if $0 \in \widehat{I_\ell R}$. But clearly
$[0 \in \widehat{I_\ell R}] \Longleftrightarrow [\widehat{rad_\ell}(R)=0]$.

4) It is no difficult to see that the category $R\text{-}\widehat{mod}$ of $\widehat{rad_\ell}$-semisimple modules is closed with respect to arbitrary direct products (in $R\text{-}mod$) and contains together with every module all its submodules. In particular $R\text{-}\widehat{mod}$ is closed with respect to direct sum.

5) Any R-module with zero Jackobson radical is $\widehat{rad_\ell}$-semisimple.

In fact, every simple R-module clearly belongs to $R\text{-}\widehat{mod}$. It is known ([2], Proposition 18.0.2) that every R-module with $J=0$ is a submodule in a product of simple modules.

Proposition. *For any $\widehat{rad_\ell}$-semisimple R-module M the structure presheave $\mathcal{O}_M$ is a ω-sheave.*

Proof. By Proposition 4.5 we are to verify the validity of the implication

$$[\nu \in \mathcal{F}_{V_\ell(\alpha)} \amalg \mathcal{F}_{V_\ell(\beta)} = \mathcal{F}_{V_\ell(\alpha\cap\beta)}] \Rightarrow [\nu \in \mathcal{F}_{V_\ell(\alpha)} \circ \mathcal{F}_{V_\ell(\beta)}]$$

for any $\nu \in \widehat{I_\ell R}$ and any pair $\{\alpha, \beta\}$ of twosided ideal.

Since $\mathcal{F}_{V_\ell(\alpha\cap\beta)} = \{n \in I_\ell R \mid \alpha\cap\beta \subset rad_\ell(n)\}$ and $\widehat{I_\ell R} = \{m \in I_\ell R \mid rad_\ell(m) \subset m\}$ the ideal ν from $\widehat{I_\ell R}$ belongs to $\mathcal{F}_{V_\ell(\alpha\cap\beta)}$ if and only if $\alpha\cap\beta \subset \nu$.
But $\alpha\cap\beta \in \mathcal{F}_{V_\ell(\alpha)} \circ \mathcal{F}_{V_\ell(\beta)}$ since $\alpha \in \mathcal{F}_{V_\ell(\alpha)}$ and $\beta \in \mathcal{F}_{V_\ell(\beta)}$; and therefore $\nu \in \mathcal{F}_{V_\ell(\alpha)} \circ \mathcal{F}_{V_\ell(\beta)}$. □

Corollary 1. (a) If R is a $\widehat{rad}_\ell$-semisimple ring then $\mathcal{O}_R$ is an ω-sheave.

(b) If R is a ring with right unit then for any unitary R-module M from R-$\widehat{mod}$ the canonical R-module morphism $M \to \Gamma\mathcal{O}^a_M$ is an isomorphism. In particular if R is a $\widehat{rad}_\ell$-semisimple ring with unit then $R \longrightarrow \Gamma\mathcal{O}^a_R$ is an isomorphism.

Proof. The statement (a) is a direct corollary of Proposition and a corollary of $\widehat{rad}_\ell$-semisimplicity of R as a ring and as a module (see (3)).

(b) If R is a ring with right unit then the space $(Spec_\ell R, \mathfrak{T})$ is quasicompact by Proposition 5.7 and for any M from ${}_R\mathcal{M}od$ the canonical arrow $\{R\}^1 M \to \Gamma\mathcal{O}^a_M$ is a R-module isomorphism. In particular this is the case for module from R-$\widehat{mod}$ by Proposition 6. Now notice that $M = \{R\}^1 M$ if M is $\widehat{rad}_\ell$-semisimple. $\square$

Lemma. For any radical filter $\mathfrak{F}$ and a subset $W \subset Spec_\ell R$ we have

$$G_{\mathfrak{F}}(r(W \setminus \mathfrak{F})) = \bigcap\{G_{\mathfrak{F}}P \mid P \in W \setminus \mathfrak{F}\}. \qquad (1)$$

If $r(W \cap \mathfrak{F}) \in \mathfrak{F}$ then

$$G_{\mathfrak{F}}(r(W)) = r(G_{\mathfrak{F}}(W)). \qquad (2)$$

If $\mathfrak{F} = \mathfrak{F}_{V_\ell(\alpha)}$ for some $\alpha \in IR$ then for any $n \in I_\ell R$

$$G_{\mathfrak{F}}(rad_\ell(n)) = r(G_{\mathfrak{F}}(\hat{V}_\ell(n))) \supset rad_\ell(G_{\mathfrak{F}} n). \qquad (3)$$

Proof. 1) (i) Clearly,

$$G_{\mathfrak{F}}(r(W \setminus \mathfrak{F})) \subset \bigcap\{G_{\mathfrak{F}}P \mid P \in W \setminus \mathfrak{F}\} = r(G_{\mathfrak{F}}(W \setminus \mathfrak{F})). \quad (4)$$

ii) Notice that for any $P \in \widehat{Spec}_\ell R \setminus \mathfrak{F}$ we have $P = j_{\mathfrak{F}}^{-1}(G_{\mathfrak{F}_P} P)$. In fact $P = j_P^{-1}(G_{\mathfrak{F}_P} P)$ by Proposition 5.1 and the including $\mathfrak{F} \subset \mathfrak{F}_P$ implies

$$P \subset j_{\mathfrak{F}}^{-1}(G_{\mathfrak{F}} P) \subset j_P^{-1}(G_{\mathfrak{F}_P} P).$$

(iii) Therefore we have

$$j_{\mathcal{F}}^{-1}(\bigcap\{G_{\mathcal{F}}p \mid p\in W\}) = \bigcap\{j_{\mathcal{F}}^{-1}(G_{\mathcal{F}}p) \mid p\in W\} =$$
$$= \bigcap\{p \mid p \in W\setminus\mathcal{F}\} \stackrel{\text{def}}{=} r(W\setminus\mathcal{F})$$

implying the inclusion convert to (4)

$$\bigcap\{G_{\mathcal{F}}p \mid p\in W\setminus\mathcal{F}\} \subset G_{\mathcal{F}}(j_{\mathcal{F}}^{-1}(\bigcap\{G_{\mathcal{F}}p \mid p\in W\setminus\mathcal{F}\}))=$$
$$= G_{\mathcal{F}}(r(W\setminus\mathcal{F})).$$

2) If $r(W\cap\mathcal{F})\in\mathcal{F}$ then $G_{\mathcal{F}}(r(W\cap\mathcal{F})) = G_{\mathcal{F}}R$ and

$$G_{\mathcal{F}}(r(W)) = G_{\mathcal{F}}(r(W\setminus\mathcal{F})\cap r(W\cap\mathcal{F})) = G_{\mathcal{F}}(r(W\setminus\mathcal{F}))\cap G_{\mathcal{F}}(r(W\cap\mathcal{F})) =$$

(since $G_{\mathcal{F}}$ is left exact and therefore "preserves" the intersections of pairs of ideals) $= G_{\mathcal{F}}(r(W\setminus\mathcal{F})) = \bigcap\{G_{\mathcal{F}}p \mid p\in W\setminus\mathcal{F}\} =$
$= \bigcap\{G_{\mathcal{F}}p \mid p\in W\} \stackrel{\text{def}}{=} r(G_{\mathcal{F}}(W)).$

3) If $\mathcal{F} = \mathcal{F}_{V_\ell(\alpha)}$ for some $\alpha\in IR$ then $r(W\cap\mathcal{F})\in\mathcal{F}$ for any $W\subset \mathrm{Spec}_\ell R$ and therefore (2) holds for any $W\subset \mathrm{Spec}_\ell R$; if $W = \hat{V}_\ell(n)$, $n\in I_\ell R$ then (2) takes the form

$$G_{\mathcal{F}}(\mathrm{rad}_\ell(n)) = r(G_{\mathcal{F}}(\hat{V}_\ell(n))) = r(G_{\mathcal{F}}(V_\ell(n))).$$

By Corollary 2 of Proposition 2.9 (for any radical filter $\mathcal{F}$) the functor $G_{\mathcal{F}}$ sends $\mathrm{Spec}_\ell R\setminus\mathcal{F}$ into $\mathrm{Spec}_\ell G_{\mathcal{F}}R$. Besides as states Corollary 1 of Proposition 2.9 $m\mapsto G_{\mathcal{F}}m$ is a functor from $I_\ell^{\succ} R$ into $I_\ell^{\succ} G_{\mathcal{F}}R$. Therefore, for any $n\in I_\ell R$ the functor $G_{\mathcal{F}}$ ~~and~~ transfers the ideals of $V_\ell(n)\setminus\mathcal{F}$ into the ideals of $V_\ell(G_{\mathcal{F}}n)$. This implies that $\mathrm{rad}_\ell(G_{\mathcal{F}}n)\subset r(G_{\mathcal{F}}(\hat{V}_\ell(n)))$. □

Corollary 2. *The following properties of the unitary ring* R *are equivalent:*

1) R *is a* $\widehat{\mathrm{rad}}_\ell$-*semisimple ring*;

2) $(\widehat{\mathrm{Spec}}\, R, \bar{\mathcal{O}}_R)$ is a reduced preringed space ;

3) The left affine scheme $(\overline{Spec}\, R, \bar{\mathcal{O}}^a_R)$ is reduced and $R \simeq \Gamma \bar{\mathcal{O}}^a_R$

4) $(\overline{Spec}\, R, \bar{\mathcal{O}}^a_R)$ is reduced and $\bar{\mathcal{O}}_R$ is a ω-sheave.

Proof. 1) $\Rightarrow$ 2). By Lemma $G_{\mathcal{F}_{V_\ell(\alpha)}} R$ is $\widehat{rad}_\ell$-semisimple for any $\alpha \in IR$ if R is $\widehat{rad}_\ell$-semisimple since $\widehat{rad}_\ell\, G_{\mathcal{F}_{V_\ell(\alpha)}} R \subset G_{\mathcal{F}_{V_\ell(\alpha)}}(\widehat{rad}_\ell(R))$.

2) $\Rightarrow$ 1) Since R is unitary, then $\mathcal{O}_R(\emptyset) \simeq R$.

3) $\Rightarrow$ 4) Follows from Corollary 1.

3) $\Rightarrow$ 1) is trivial.

1) $\Rightarrow$ 3) (i) Let us show that the $\widehat{rad}_\ell$-semisimplicity of R implies $\widehat{rad}_\ell$-semisimplicity of all the fibres of the structural sheaf $\bar{\mathcal{O}}^a_R$.

In fact for any $\mathcal{M} \in Spec_\ell R$ the ideal $\bar{\mathcal{O}}^a_{\mathcal{M},P}$ of $\bar{\mathcal{O}}^a_{R,P} = \varinjlim \{G_{\mathcal{F}_{V_\ell(\alpha)}} R \mid \alpha \in IR \cap \mathcal{F}_P\}$ either coincides with $\bar{\mathcal{O}}^a_{R,P}$ (if $\mathcal{M}_s \subset P$) or belongs to $Spec_\ell \bar{\mathcal{O}}^a_{R,P}$ (see Corollary 4 of Proposition 12). Lemma 6 implies that $[\widehat{rad}_\ell(R) = 0] \Rightarrow [\cap \{G_{\mathcal{F}_{V_\ell(\alpha)}} \mathcal{M} \mid \mathcal{M} \in Spec_\ell R\} = 0$ for any $\alpha \in IR] \Rightarrow [\cap \{\bar{\mathcal{O}}^a_{\mathcal{M},P} \mid \mathcal{M} \in Spec_\ell R\} = 0]$.

(ii) Following one of the conventional view points on sheaves let us identify for any closed subset $W \subset \overline{Spec}\, R$ the ring of "sections" $\bar{\mathcal{O}}^a_R(W)$ (over the compliment to W) with the corresponding subring in $\prod_{P \in W^\perp} \bar{\mathcal{O}}^a_{R,P}$ where $W^\perp \overset{def}{=} \overline{Spec}\, R \setminus W$. It is not difficult to see that for any closed subset $W \subset \overline{Spec}\, R$ and any $\nu \in I_\ell R$ the ideal $\bar{\mathcal{O}}^a_\nu(W)$ of $\bar{\mathcal{O}}^a_R(W)$ equals $(\prod_{P \in W^\perp} \bar{\mathcal{O}}^a_{\nu,P}) \cap \bar{\mathcal{O}}^a_R(W)$. From here we easily deduce that for any $\mathcal{M} \in Spec_\ell R$ and any closed subset

(a) $\bar{\mathcal{O}}^a_\mu(W)$ is either improper ideal or belongs to the left spectrum of $\bar{\mathcal{O}}^a_R(W)$;

b) $\cap\{\bar{\mathcal{O}}^a_\mu(W) \mid \mu \in Spec_\ell R\} = 0$. □

7. Maximal left spectrum and ringed structural spaces

1. Maximal left spectrum. We will call thus the set $Max^{reg}_\ell R$ of maximal left regular ideals of a ring R.

Proposition. *Let $\mu \in Max^{reg}_\ell R$ and ν be a regular left ideal. Then the following conditions are equivalent:*

1) $\mu \to \nu$;

2) $\nu = (\mu : x)$ *for some* $x \in \mathcal{P}(R) \setminus \mathcal{P}(\mu)$.

Proof. (a) Let $m \in I^w_\ell R$ and $\nu \in I_\ell R$. Then the relation $m \to \nu$ means that there exists an element $\xi \in \bigoplus_\omega R/m$ such that the canonical epimorphism $R \twoheadrightarrow R/\nu$ factors through the epimorphism $R \twoheadrightarrow R\xi$, $\lambda \mapsto \lambda\cdot\xi$; i.e. there is a commutative diagram

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr, two heads] & \\ R\xi \arrow[rr, two heads] & & R/\nu \end{tikzcd}$$

In fact since $m \in I^w_\ell R$ then $m \to \nu$ if and only if $(m : x) \subset \nu$ for a finite subset $x = \{x_1, \dots, x_k\} \subset R$. Let $\bar{x}_i$ be the image of x_i in R/m. Then $(m : x) = Ann\,\xi$ where $\xi = \bar{x}_1 \oplus \dots \oplus \bar{x}_k$; i.e. ξ is the desired vector.

(b) Now let $\mu \in Max^{reg}_\ell R$ and $\mu \to \nu$. Due to (a) there exists a cyclic R-submodule $R\xi$ of $\bigoplus_\omega R/\mu$ (where $\bigoplus_\omega R/m$ is the direct sum of countably many copies of R/m) such that there exists an epimorphism $\varphi : R\xi \twoheadrightarrow R/\nu$. Since submodules of semisimple modules are semisimple (see e.g. [13], Ch. III, § 1) then $R\xi$ and $\ker\varphi$ are semisimple.

This implies that $R\xi \simeq \ker\varphi \oplus R/\nu$; i.e. R/ν is semisimple and can be embedded into $R\xi$. This means that $R/\nu \simeq \overset{k}{\underset{1}{\oplus}} R/\mu$ for an finite k.

Now suppose that ν is regular, a the right unit modulo ν, i.e. $x - xa \in \nu$ for any $x \in R$, and a_ν the image of a in R/ν. Obviously $Ra_\nu = R/\nu$ and $\operatorname{Ann} a_\nu = \nu$. Let ξ_ν be the image of a_ν with respect to the isomorphism $\varphi: R/\nu \xrightarrow{\sim} \oplus R/\mu$ (k copies). Clearly $\operatorname{Ann}\xi_\nu = \operatorname{Ann} a_\nu = \nu$. On the other hand $\xi_\nu = \bar{x}_1 \oplus \dots \oplus \bar{x}_k$ for a set $x = \{x_1, \dots, x_k\} \subset R$ ($\bar{x}_i$ is the image of x_i in R/μ) and therefore $\operatorname{Ann}\xi_\nu = (\mu : x)$.

Thus we have proved the implication 1) $\Rightarrow$ 2). The converse implication is obvious. □

Corollary 1. *If* $\{\mu, \mu'\} \subset \operatorname{Max}_\ell^{reg} R$ *and* $\mu \to \mu'$ *then* $\mu' = (\mu : t)$ *for some* $t \in R \setminus \mu$ *and therefore* $\mu \simeq \mu'$.

Proof. By Proposition 1 the relation $\mu \to \mu'$ means that $\mu' = (\mu : x)$ for a set $x = \{x_1, \dots, x_k\} \subset R \setminus \mu$. The maximality of μ' implies $\mu' = (\mu : x_i)$ for all i, $1 \le i \le k$. □

Corollary 2. *Let* τ *be a topology on* $\operatorname{Spec}_\ell R$ *such that the closure of any point* $p \in \operatorname{Spec}_\ell R$ *coincides with* $V_\ell(p)$ (*e.g.* $\tau = \mathfrak{T}_1$ *or* $\tau = \mathfrak{T}_0$, *the strongest topology with this property*); $^\sim\tau$ *the quotient topology of* τ *on* $^\sim\operatorname{Spec}_\ell R$. *Then* $|(^\sim\operatorname{Spec}_\ell R, {}^\sim\tau)| = {}^\sim\operatorname{Max}_\ell^{reg} R$.

More exactly the map $\operatorname{Spec}_\ell R \twoheadrightarrow {}^\sim\operatorname{Spec}_\ell R$ *induces a surjection* ℓ_R *of the maximal left spectrum on to the set of closed points of* $(^\sim\operatorname{Spec}_\ell R, {}^\sim\tau)$ *and* $\ell_R(\mu) = \ell_R(\mu')$ *if and only if* $\mu' = (\mu : t)$ *for some* $t \in R \setminus \mu$.

Proof. Since $|(\sim\mathrm{Spec}_\ell R, \sim\tau)| = |(\sim\mathrm{Spec}_\ell R, \sim\mathfrak{T}_0)|$ it suffices to verify the validity of the statement for $\mathfrak{T}_0$. Let us make use of the homeomorphism

$$(\sim\mathrm{Spec}_\ell R^{(1)} \setminus \mathrm{Spec} Z, \widetilde{\mathfrak{T}}_0) \xrightarrow{\sim} (\sim\mathrm{Spec}_\ell R, \sim\mathfrak{T}_0)$$

induced by the homeomorhpism (see proof of Proposition 5.14.A and 1.6):

$$\mathrm{Spec}_\ell R^{(1)} \setminus \mathrm{Spec} Z \xrightarrow{\sim} \mathrm{Spec}_\ell R, \quad \mathcal{M} \mapsto \mathcal{M} \cap R,$$

which in its turn induces the bijection

$$|(\sim\mathrm{Spec}_\ell R^{(1)}, \sim\mathfrak{T}_0)| \setminus \mathrm{Spec} Z \xrightarrow{\sim} |(\sim\mathrm{Spec}_\ell R, \sim\mathfrak{T}_0)|. \qquad (2)$$

Since $V_\ell(p) \cap \mathrm{Max}_\ell R' \neq \emptyset$ for any ring R with (right) unit and any point $p \in \mathrm{Spec}_\ell R'$, then (2) enters the commutative diagram

$$\begin{tikzcd}
|(\sim\mathrm{Spec}_\ell R^{(1)}, \sim\mathfrak{T}_0)| \setminus \mathrm{Spec} Z \arrow[r, "\sim"] & |(\sim\mathrm{Spec}_\ell R, \sim\mathfrak{T}_0)| \\
\mathrm{Max}_\ell R^{(1)} \setminus \mathrm{Spec} Z \arrow[u, "\eta"] \arrow[ur, "\eta'"] \arrow[r, "\sim"] & \mathrm{Max}_\ell^{reg} R \arrow[u, "\ell_R"']
\end{tikzcd}$$

in which η and therefore η' are surjections. This implies that ℓ_R is a surjection. It remains to make use of Corollary 1 thanks to which $\ell_R(\mathcal{M}) = \ell_R(\mathcal{M}')$ if and only if $\mathcal{M}' = (\mathcal{M} : t)$ for some $t \in R \setminus \mathcal{M}$. $\square$

The topologies $\mathfrak{T}_0$, $\mathfrak{T}_1$ and $\mathfrak{T}$ induce topologies on $\mathrm{Max}_\ell^{reg} R$ that will be denoted by $\widehat{\mathfrak{T}}_0$, $\widehat{\mathfrak{T}}_1$ and $\widehat{\mathfrak{T}}$ respectively.

2. Structure presheaves and sheaves on $\mathrm{Max}_\ell^{reg} R$. To each R-module M a structure presheave ${}^{\circ}\widehat{\mathcal{O}}_M$ on $(\mathrm{Max}_\ell^{reg} R, \widehat{\mathfrak{T}}_0)$ corresponds. It sends a closed set W into the R-module $G\,\dot{\mathfrak{F}}_W M$. The restrictions of ${}^{\circ}\widehat{\mathcal{O}}_M$ onto $\widehat{\mathfrak{T}}_1$ and $\widehat{\mathfrak{T}}_0$ will be denoted by ${}^{1}\widehat{\mathcal{O}}_M$ and $\widehat{\mathcal{O}}_M$ respectively. Let us discuss

the local behaviour of the associated sheaves ${}^{o}\hat{\mathcal{O}}^{a}_{M}$, ${}^{1}\hat{\mathcal{O}}^{a}_{M}$ and $\hat{\mathcal{O}}^{a}_{M}$ in the same terms as we perform this for ${}^{o}\mathcal{O}^{a}_{M}$, ${}^{1}\mathcal{O}^{a}_{M}$ and $\mathcal{O}^{a}_{M}$ in 5.5.

A) The fibre of ${}^{o}\hat{\mathcal{O}}^{a}_{M}$ at had any point $\mu \in Max_{\ell}^{reg} R$ is isomorphic to the R-module $G_{\mathcal{F}_\mu} M$:

$${}^{o}\mathcal{O}^{a}_{M,\mu} = \varinjlim (G_{\dot{\mathcal{F}}_{V_\ell(\mathcal{Y})}} M \mid \mu \in \mathcal{Y}) = \varinjlim (G_{\dot{\mathcal{F}}_{V_\ell(\mathcal{Y})}} M \mid \mathcal{Y} \subset \mathcal{F}_\mu) = G_{\mathcal{F}_\mu} M$$

since $\dot{\mathcal{F}}_{V_\ell(\mathcal{F}_\mu)} = \mathcal{F}_\mu$; here $V_\ell(\mathcal{Y}) = \mathcal{Y} \cap Spec_\ell R$.

For any $\tilde{\mu} \in {}^{\sim}Spec_\ell R$ set $\mathcal{F}_{\tilde{\mu}} = \mathcal{F}_\mu$ where μ is a representative of $\tilde{\mu}$. Obviously, $\mathcal{F}_{\tilde{\mu}}$ is well-defined.

Proposition. ${}^{o}\hat{\mathcal{O}}^{a}_{M}$ *is isomorphic to the sheave, sending an arbitrary closed set* W *into the* R-*module* $\prod \{G_{\mathcal{F}_{\tilde{\mu}}} M \mid \tilde{\mu} \in {}^{\sim}\dot{W}^{\perp}\}$ *where* ${}^{\sim}\dot{W}^{\perp}$ *is the image of* $\dot{W}^{\perp} = Max_\ell^{reg} R \setminus W$ *in* ${}^{\sim}Spec_\ell R$.

Proof. The map $(Max_\ell^{reg} R, \mathcal{T}_o) \xrightarrow{\pi} ({}^{\sim}Max_\ell^{reg} R, {}^{\sim}\mathcal{T}_o)$ is a quasi-homeomorphism and as is clear from Corollary 1.1 the space $({}^{\sim}Max_\ell^{reg} R, {}^{\sim}\mathcal{T}_o)$ is discrete (recall that the closure of any subset W of $(Spec_\ell R, \mathcal{T}_o)$ coincide with $\bigcup_{p \in W} V_\ell(p)$). Any sheave on the discrete space sends any its subset into the product of fibres on the points of this subset. It follows from A) that the fibre of $\pi_* {}^{o}\hat{\mathcal{O}}^{a}_{M}$ over an arbitrary point $\tilde{\mu}$ is isomorphic to $G_{\mathcal{F}_{\tilde{\mu}}} M$. $\square$

Let us add a few words on the sheave of rings ${}^{o}\hat{\mathcal{O}}^{a}_{R}$.

Lemma. *Let* $\mathcal{F}$ *be the radical filter and* $\mu \in Max_\ell^{reg} R \setminus \mathcal{F}$. *Then* $G_{\mathcal{F}} \mu$ *is a maximal left ideal of* $G_{\mathcal{F}} R$.

Proof. 1) If ν is a regular ideal from $I_\ell R \setminus \mathcal{F}$ and a right unit modulo ν then $1 - j_{\mathcal{F},R}(a) \in G_{\mathcal{F}} \nu$ where 1 is the unit of $G_{\mathcal{F}} R$ since $R(1 - j_{\mathcal{F},R}(a)) \subset G_{\mathcal{F}} \nu$.

This means that $j_{\mathfrak{F},R}(a)$ is right unit modulo $G_{\mathfrak{F}}\nu$ and in particular $j_{\mathfrak{F},R}(a) \notin \mathfrak{n}$ if $\mathfrak{n}$ is a proper ideal of $G_{\mathfrak{F}}R$ containing $G_{\mathfrak{F}}\nu$.

2) Now let $\mathcal{M} \in Max_\ell^{reg} R \setminus \mathfrak{F}$ and a be right unit modulo $\mathcal{M}$; $\mathfrak{n}$ a proper left ideal of $G_{\mathfrak{F}}R$ containing $G_{\mathfrak{F}}\mathcal{M}$. Then $\mathcal{M} \subset j^{-1}_{\mathfrak{F},R}(\mathfrak{n})$ and $a \notin j^{-1}_{\mathfrak{F},R}(\mathfrak{n})$ as we have just found out. Thus $j^{-1}_{\mathfrak{F},R}(\mathfrak{n})$ is a proper ideal and the maximality of $\mathcal{M}$ implies $j^{-1}_{\mathfrak{F},R}(\mathfrak{n}) = \mathcal{M}$. Since $\mathfrak{n} \subset G_{\mathfrak{F}} j^{-1}_{\mathfrak{F},R}(\mathfrak{n})$ then $\mathfrak{n}$ coincide with $G_{\mathfrak{F}}\mathcal{M}$. $\square$

Corollary. *For any* $\mathcal{M} \in Max_\ell^{reg} R$ *the quasifinal ideal* $G_{\mathfrak{F}_\mathcal{M}}\mathcal{M}$ *of* $G_{\mathfrak{F}_\mathcal{M}}R$ *is a maximal left ideal.*

B) The fibres of ${}^1\hat{\mathcal{O}}^a_{\mathcal{M}}$. The presentation of $\mathfrak{F}_\mathcal{M}$ in the form $\bigcup \{ \dot{\mathfrak{F}}_{V_\ell(n)} \mid n \in \mathfrak{F}_\mathcal{M} \} = \mathfrak{F}_\mathcal{M}$ implies (see 5.5.B) ${}^1\hat{\mathcal{O}}^a_{M,\mathcal{M}} = \varinjlim (G_{\dot{\mathfrak{F}}_{V_\ell(n)}} M \mid n \in \mathfrak{F}_\mathcal{M})$ and the existence of the canonical morphism

$$ {}^1\varphi_{M,\mathcal{M}} : {}^1\hat{\mathcal{O}}^a_{M,\mathcal{M}} \longrightarrow G_{\mathfrak{F}_\mathcal{M}} M . $$

Proposition ~~Proof~~. 1) *The canonical arrow* ${}^1\varphi_{M,\mathcal{M}} : {}^1\hat{\mathcal{O}}^a_{M,\mathcal{M}} \to G_{\mathfrak{F}_\mathcal{M}} M$ *is monomorphism. It is isomorphism if* $H_{\mathfrak{F}_\mathcal{M}}(\dot{\mathfrak{F}}_{V_\ell(n)})^1 M \longrightarrow G_{\mathfrak{F}_\mathcal{M}} M$ *is an epimorphism for some* $n \in \mathfrak{F}_\mathcal{M}$.

2) ${}^1\hat{\mathcal{O}}^a_{M,\mathcal{M}} \simeq G_{\mathfrak{F}_\mathcal{M}} M$ *if one of the following conditions holds:*

a) *ordered with respect to inclusion family of torsion submodules* $\{ \dot{\mathfrak{F}}_{V_\ell(n)} M \mid n \in \mathfrak{F}_\mathcal{M} \}$ *of* M *stabilyzes*;

b) M *is Noetherian*;

c) R *is left Noetherian.*

3) *For every* $\mathcal{M} \in Max_\ell^{reg} R$ *the left ideal* ${}^1\hat{\mathcal{O}}^a_{\mathcal{M},\mathcal{M}}$ *of the ring* ${}^1\hat{\mathcal{O}}^a_{R,\mathcal{M}}$ *is maximal.*

Proof. The heading 1), 2) are specializations of Propositions 2.12 and its first two corollaries. The heading 3) follows from Lemma A (see also heading 1) of Corollary 4 of Proposition 2.12).

C) Fibres of $\widehat{\mathcal{O}}^a_M$. For any $m \in I_\ell R$ set

$$\langle m\rangle \overset{\mathrm{def}}{=} \{\mu' \in Max_\ell^{reg} R \mid \mu'_s \subset m\}.$$

Proposition. Let $\mu \in Max_\ell^{reg} R$.

1) There exists a canonical mono morphism $\varphi_{M,\mu} : \widehat{\mathcal{O}}^a_{M,\mu} \longrightarrow G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} M$ for every $M \in Ob\, R\text{-mod}$.

2) $\varphi_{M,\mu}$ is an isomorphism if one of the following conditions holds:

a) for some $\alpha \in IR \cap \mathfrak{F}_\mu$ the natural arrow $H_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} \dot{\mathfrak{F}}^1_{V_\ell(\alpha)} M \longrightarrow G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} M$ is epimorphism;

b) an ordered by inclusion family of submodules $\{\dot{\mathfrak{F}}_{V_\ell(\alpha)} M \mid \alpha \in \mathfrak{F}_\mu \cap IR\}$ of M stabilizes;

c) M is Noetherian;

d) $(Max_\ell^{reg} R, \hat{\mathfrak{I}})$ is a Noetherian space e.g. R is a ring with the ascending chain condition for twosided ideals.

3) Let $\mu' \in Max_\ell^{reg} R$. Then

e) $G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} \mu' \in Max_\ell G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} R$ and $\widehat{\mathcal{O}}^a_{\mu',\mu} \in Max_\ell \widehat{\mathcal{O}}^a_{R,\mu}$ if $\mu'_s \subset \mu$;

f) $G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} \mu' = G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} R$ and $\widehat{\mathcal{O}}^a_{\mu',\mu} = \widehat{\mathcal{O}}^a_{R,\mu}$ if $\mu'_s \not\subset \mu$.

Proof. 1) As in 5.5.C we establish that for any $m \in I_\ell R$ we have

$$\bigcup\{\dot{\mathfrak{F}}_{V_\ell(\alpha)} \mid \alpha \in IR,\ \alpha \not\subset m\} = \{n \in I_\ell R \mid \text{if } \mu' \in Max_\ell^{reg} R \text{ and } n \to \mu' \text{ then } \alpha \subset \mu' \text{ for some } \alpha \in \mathfrak{F}_m \cap IR\} =$$
$$= \{n \in I_\ell R \mid \text{if } \mu' \in Max_\ell^{reg} R \text{ and } n \to \mu' \text{ then } \mu'_s \not\subset m\} = \dot{\mathfrak{F}}_{\langle m\rangle} = \dot{\mathfrak{F}}_{\langle m_s\rangle}.$$

Proposition 2.12 implies the existence of canonical morphisms $\varphi_{M,\mu} : \hat{\mathcal{O}}^a_{M,\mu} \longrightarrow G_{\dot{\mathfrak{F}}_{\langle\mu\rangle}} M$, $M \in Ob\, R\text{-mod}$.

2) A specialization of Proposition 2.12 and its corollaries implies that $\varphi_{M,\mu}$ is an isomorphism provided (a)-(d) hold.

3) It is not difficult to see (if one looks at the equalities in the proof of 1)) that $[\mu' \in \dot{\mathfrak{F}}_{\langle\mu\rangle}] \Leftrightarrow [\mu'_s \not\subset \mu]$ for any $\mu' \in Max_\ell^{reg} R$. Thus 3) follows from Lemma A. $\square$

Note that for any $\alpha \in IR$ the filter $\dot{\mathfrak{F}}_{V_\ell(\alpha)}$ consists of all $n \in I_\ell R$ such that $\alpha \subset J_\ell(n)$ where $J_\ell(n) \overset{def}{=} \bigcap \{\mu \in Max_\ell^{reg} R \mid n \to \mu\}$ is the "left continuation" of the Jacobson radical (see 5.13).

3. <u>Spaces of irreducible components</u>. The space of irreducible components $irr(Max_\ell^{reg} R, \hat{S}_o)$ is isomorphic to the discrete space $\sim Max_\ell^{reg} R$.

This fact follows from Corollary 1 of Proposition 1; essentially it has been already established during the proof of Proposition 2.A.

Denote $\mathcal{P}Spec R$ the subspace of the prim spectrum of R formed by all the prime ideals which are intersections of the families of the primitive ideals.

<u>Proposition</u>. 1) The map $W \mapsto \imath(W)$ performs a homeomorphism $irr(Max_\ell^{reg} R, \hat{\underline{S}})$ onto $\mathcal{P}Spec R$

2) $\mathcal{P}Spec R$ is quasicompact if R possesses right or left unit.

<u>Proof.</u> 1) Follows directly from Proposition 5.14.B.

2) $(Max_\ell R, \hat{\underline{S}})$ is quasicompact if R possesses a right unit. This is actually proved during the proof of Propo-

sition 5.7 (see also proof of Corollary 1 of Proposition 11). Since $\mathcal{P}\mathrm{Spec}\,R$ is quasihomeomorphic to $(\mathrm{Max}_\ell R, \hat{\mathfrak{S}})$ it is also quasicompact.

Now notice that the picture is symmetric (almost the first time since the beginning of this work)—one could arrive to $\mathcal{P}\mathrm{Spec}\,R$ starting from the space of $(\mathrm{Max}_\ell^{reg} R, \hat{\mathfrak{I}})$ of right regular ideals. Therefore $(\mathrm{Max}_\ell^{reg} R, \hat{\mathfrak{S}})$, $\mathcal{P}\mathrm{Spec}\,R$ and $(\mathrm{Max}_\ell^{reg} R, \hat{\mathfrak{S}})$ could be quasicompact (or possess any other property invariant with respect to quasihomeomorphisms) only simultaneously and therefore R having a left unit implies quasicompactness of $\mathcal{P}\mathrm{Spec}\,R$. □

Corollary. *The natural embedding* $\mathcal{P}\mathrm{rim}\,R \hookrightarrow \mathcal{P}\mathrm{Spec}\,R$ *is a quasihomeomorphism. In particular* $\mathcal{P}\mathrm{Spec}\,R \simeq \mathrm{irr}\,\mathcal{P}\mathrm{rim}\,R$.

Proof. $\mathcal{P}\mathrm{rim}\,R$ is the image of the canonical quasihomeomorphism $\mathrm{Max}_\ell^{reg} R \longrightarrow \mathcal{P}\mathrm{Spec}\,R$, $\mu \mapsto \mu_s$. □

4. Main homeomorphisms. On main homeomorphisms of maximal left spectrums see statements in 5.13. Here we will derive a corollary from Proposition 5.13.

Proposition. 1) *Let* $\alpha \in \mathrm{I}R$. *The map* $\mu \mapsto \mu \cap \alpha$ *performs the homeomorphism of* $\mathcal{P}\mathcal{U}(\alpha) \stackrel{def}{=} \{p \in \mathcal{P}\mathrm{Spec}\,R \mid \alpha \not\subset p\}$ *onto* $\mathcal{P}\mathrm{Spec}\,\alpha$.

2) *For any non-zero idempotent* **e** *of* R *the map* $p \mapsto p \cap eRe$ *determines a homeomorphism of* $\mathcal{P}\mathcal{U}(eRe) = \{p \in \mathcal{P}\mathrm{Spec}\,R \mid eRe \not\subset p\}$ *onto* $\mathcal{P}\mathrm{Spec}\,eRe$.

Proof is almost identical to that Proposition 6.3; the only difference being that homeomorphisms of Propositions 5.9 and 5.11 are used instead of homeomorphisms from 5.13. □

Corollary 1. Let $\alpha \in I R$, e a non-zero idempotent, f a non-unit idempotent.

1) The map $\mathcal{M} \longmapsto \mathcal{M} \cap e\alpha e$ determines a homeomorphism

$$\mathcal{P}\mathcal{U}(e) \cap \mathcal{P}\mathcal{U}(\alpha) \xrightarrow{\sim} \mathcal{P}Spec\, e\alpha e .$$

2) The map $\mathcal{M} \mapsto \mathcal{M} \cap (1-f)\alpha(1-f)$ determines a homeomorphism $\mathcal{P}\mathcal{U}((1-f)R(1-f)) \cap \mathcal{P}\mathcal{U}(\alpha) \xrightarrow{\sim} \mathcal{P}Spec\,(1-f)\alpha(1-f)$.

Proof is similar to that of Corollaries 2 and 3 of Proposition 5.11. □

Corollary 2. The map $\mathcal{M} \mapsto \mathcal{M} \cap R$ performs a homeomorphism $\mathcal{P}Spec\, R^{(1)} \setminus Spec\, Z \xrightarrow{\sim} \mathcal{P}Spec\, R$.

Corollary 3. The set of closed points of $\mathcal{P}Spec\, R$ coincides with the set $Max^{reg}\, R$ of maximal twosided regular ideals (and so does $|\overline{Spec}\; R|$, see Corollary 4 of Proposition 6.3).

Proof is the same as that of Corollary of Proposition 6.3: a) if R is a ring with (right) unit, then $|\mathcal{P}Spec R| = Max\, R$ as is easy to verify; b) for an arbitrary R the homeomorphism $\mathcal{P}Spec\, R^{(1)} \setminus Spec Z \simeq \mathcal{P}Spec\, R$ induces a bijection of $|\mathcal{P}Spec\, R^{(1)} \setminus Spec Z| = Max\, R^{(1)} \setminus Spec\, \underline{Z}$ with $Max^{reg} R$. □

Remark. The statements of this section may be deduced with the help of approximately the same argument from Jacobson's theorems ([5], Ch. IX, § 2) making use of the canonical homeomorphism $\mathrm{irr}\,\mathcal{P}rim\, R \xrightarrow{\sim} \mathcal{P}Spec\, R$, $W \mapsto r(W)$. In their turn the statements of §§ 2 and 3 of Ch. IX in [5] are corollaries of the above facts. □

5. Canonical open embeddings. Denote by $\mathcal{P}F$ the direct image of a presheave F on $(Max^{reg}_{e} R, \dot{\underline{\jmath}})$ with respect to

the quasihomeomorphism

Proposition. For any twosided ideal α of R the map $\mathcal{M} \mapsto \mathcal{M} \cap \alpha$ induces isomorphism of preringed spaces:

$$(\mathcal{U}_\ell(\alpha), {}^{o}\hat{\mathcal{O}}_R|_{\mathcal{U}_\ell(\alpha)}) \xrightarrow{\sim} (Max_\ell^{reg}\alpha, {}^{o}\hat{\mathcal{O}}_\alpha),$$
$$(\mathcal{U}_\ell(\alpha), {}^{1}\hat{\mathcal{O}}_R|_{\mathcal{U}_\ell(\alpha)}) \xrightarrow{\sim} (Max_\ell^{reg}\alpha, {}^{1}\hat{\mathcal{O}}_\alpha),$$
$$(\mathcal{U}_\ell(\alpha), \hat{\mathcal{O}}_R|_{\mathcal{U}_\ell(\alpha)}) \xrightarrow{\sim} (Max_\ell^{reg}\alpha, \hat{\mathcal{O}}_\alpha),$$
$$(\mathcal{P}\mathcal{U}(\alpha), \mathcal{P}\mathcal{O}_R|_{\mathcal{P}\mathcal{U}(\alpha)}) \xrightarrow{\sim} (\mathcal{P}Spec\,\alpha, \mathcal{P}\mathcal{O}_\alpha).$$

Proof. The statements follows from Propositions 4 and 5.13 and Lemma 6.4 (see proof of Proposition 6.4).

6. Semisimple rings and J_ℓ-semisimple modules. Denote by $I_\ell^J R$ the family of all proper left ideals n of R such that $J_\ell(n) \subset n$ or equivalently $n_s = J_\ell(n)$. A R-module M is J_ℓ -semisimple if $Ann\,\xi \in I_\ell^J R$ for every $\xi \in M \setminus \{0\}$. The full subcategory of R-mod formed by J_ℓ-semisimple modules will be denoted by $R\text{-}^{J_\ell}mod$.

Proposition. 1) The subcategory $R\text{-}^{J_\ell}mod$ is closed with respect to products (in R-mod) and contains together with every module or its submodules.

2) Every R-module with zero Jackobson radical is J_ℓ-semisimple.

3) Let $\nu \in I_\ell R$ and $(\nu : R) \subset \nu$. Then R/ν is J -semisimple if and only if $\nu \in I_\ell^J R$.

4) A left R-module R is J_ℓ-semisimple if and only if R is a semisimple ring, i.e. J(R)=0.

Proof. 1) It is subject to a straightforward verification that $I_\ell^J R$ is closed with respect to intersections of arbitrary family of ideals, since $J_\ell(\bigcap_{i\in I} n_i) \subset \bigcap_{i\in I} J_\ell(n_i)$

and, therefore, the fact that $J_\ell(n_i) \subset n_i$ for all $i \in I$ implies $J_\ell(\bigcap_{i\in I} n_i) \subset \bigcap_{i\in I} n_i$. It follows that together with any family of modules $\{M_i \mid i \in I\}$ the category $R\text{-}{}^{J_\ell}\mathrm{mod}$ contains $\prod_{i\in I} M_i$. The statement on submodules is trivial.

2) Recall that the Jac obson radical $J(M)$ of M is the intersection of kernels of all the morphisms of M into irreducible R-modules. Therefore $J(M)=0$ means exactly that M is isomorphic to a submodule in the product of a family of irreducible modules. Clearly all irreducible modules are J_ℓ-semisimple. Therefore 2) follows from 1).

3) The fact that $[(\nu : R) \subset \nu,\ R/\nu \in R\text{-}{}^{J_\ell}\mathrm{mod}] \Leftrightarrow [\nu \in I_\ell^J R]$ is verified as the fact

$$[(\nu : R) \subset \nu,\ R/\nu \in R\text{-}\widehat{\mathrm{mod}}] \Leftrightarrow [\nu \in I_\ell^J R]$$

in 6.6.

4) By 3) the R-module R is J_ℓ-semisimple if and only if $0 \in I_\ell^J R$. It is easy to see that $[0 \in I_\ell^J R] \Leftrightarrow [J(R) = 0]$. □

Note that any J_ℓ-semisimple module is $\widehat{\mathrm{rad}}_\ell$-semisimple.

7. Structure presheaves of J_ℓ-semisimple modules. There holds a statement similar to Proposition 6.6:

Proposition. *For any J_ℓ-semisimple module M the structure presheave $\hat{\mathcal{O}}_M$ is a ω-sheave.*

Proof. As was noted at the end of n.2 for any $\alpha \in IR$ the filter $\dot{\mathcal{F}}_{V_\ell(\alpha)}$ equals $\{n \in I_\ell R \mid \alpha \subset J_\ell(n)\}$. Therefore for any pair $\{\alpha, \beta\} \subset IR$

$$(\dot{\mathcal{F}}_{V_\ell(\alpha)} \amalg \dot{\mathcal{F}}_{V_\ell(\beta)}) \cap I_\ell^J R = \dot{\mathcal{F}}_{V_\ell(\alpha\cap\beta)} \cap I_\ell^J R =$$
$$= \{ n \in I_\ell^J R \mid \alpha\cap\beta \subset n \} = ({}^{\alpha}\mathcal{F} \circ {}^{\beta}\mathcal{F}) \cap I_\ell^J R \subset$$
$$\subset (\dot{\mathcal{F}}_{V_\ell(\alpha)} \circ \dot{\mathcal{F}}_{V_\ell(\beta)}) \cap I_\ell^J R$$

and therefore $\dot{\mathcal{F}}_{V_\ell(\alpha\cap\beta)} \cap I_\ell^J R = (\dot{\mathcal{F}}_{V_\ell(\alpha)} \circ \dot{\mathcal{F}}_{V_\ell(\beta)}) \cap I_\ell^J R$.

Thus the statement follows from Proposition 4.5. □

Corollary 1. (i) *If R is a semisimple ring then* $\hat{\mathcal{O}}_R$ *is a* ω*-sheave.*

(ii) *If R is a ring with right unit then for any* J_ℓ*-semisimple R-module M the canonical arrow* $M \to \Gamma\hat{\mathcal{O}}_M^a$ *is an R-module isomorphism.*

Proof. (i) The statement follows from heading 4) of Proposition 6 and Proposition 7.

(ii) If R possesses right unit then by Proposition 3 $(Max_\ell^{reg} R, \mathfrak{J})$ is quasicompact and for any R-module M such that $\hat{\mathcal{O}}_M$ is an ω-sheave the canonical injection $\{R\}^1 M \longrightarrow \Gamma\hat{\mathcal{O}}_M^a$ is an isomorphism (we have made use of the fact that $\dot{\mathcal{F}}_\varnothing = \{R\}$ and $G_{\{R\}} M \simeq \{R\}^1 M$ where R is a ring with right unit. Now notice that $\{R\}^1 M \simeq M$ for any J_ℓ-semisimple R-module M. □

A preringed space $(X, \mathcal{O})$ will be called semisimple if $\mathcal{O}$ is a presheave of semisimple rings.

Corollary 2. *The following properties of a unitary ring R are equivalent:*

1) R *is semisimple*

2) *the preringed space* $(Max_\ell^{reg} R, \hat{\mathcal{O}}_R)$ *is semisimple*

3) the ringed space $(Max_\ell^{reg} R, \hat{\mathcal{O}}_R^a)$ is semisimple and $\Gamma \hat{\mathcal{O}}_R^a \simeq R$.

Proof is based on an analogue of one of the statements of Lemma 6.6:

Lemma. For any twosided ideal α of R

$$G_{\dot{\mathcal{F}}_{V_\ell(\alpha)}}(J_\ell(n)) = \cap\{G_{\dot{\mathcal{F}}_{V_\ell(\alpha)}}\mu \mid \mu \in \dot{V}_\ell(n)\},\ n \in I_\ell R;$$

in particular

Proof. If $\mathcal{F} = \dot{\mathcal{F}}_{V_\ell(\alpha)}$ for some α of IR then $z(W \cap \mathcal{F}) \in \mathcal{F}$ for any $W \subset Max_\ell^{reg} R$, and therefore by Lemma 6.6 $G_{\mathcal{F}}(z(W)) = z(G_{\mathcal{F}}(W))$.

When $W = \dot{V}_\ell(n)$, $n \in I_\ell R$, this identity takes the form

$$G_{\mathcal{F}}(J_\ell(n)) = \cap\{G_{\mathcal{F}}\mu \mid \mu \in \dot{V}_\ell(n)\}$$

In particular by Lemma 2.A the functor $G_{\mathcal{F}}$ sends ideals of $Max_\ell^{reg} R \setminus \mathcal{F}$ into the maximal left ideals of $G_{\mathcal{F}} R$. Therefore $G_{\mathcal{F}}(J(R)) = \cap\{G_{\mathcal{F}}\mu \mid \mu \in Max_\ell^{reg} R \setminus \mathcal{F}\} \supset J(G_{\mathcal{F}} R)$. $\square$

Lemma implies that 1) $\Rightarrow$ 2). Clearly 2) $\Rightarrow$ 1) $\Rightarrow$ 3). We verify that 1) $\Rightarrow$ 3) in approximately the same way as we verify the corresponding implication in the proof of Carollary 2 of Proposition 6.6:

from semisimplicity of R we derive with the help of Lemma and heading 3) of Proposition 2.C the semisimplicity of all the fibres of $\hat{\mathcal{O}}_R^a$;

it follows from the identity $\hat{\mathcal{O}}_m^a(W) = \hat{\mathcal{O}}_R^a(W) \cap \prod_{\mu \in Max_\ell^{reg} R \setminus W} \hat{\mathcal{O}}_{m,\mu}^a$ for any $m \in I_\ell R$ and any closed subset W of $(Max_\ell^{reg} R, \hat{\mathfrak{z}})$ that for any $\mu \in Max_\ell^{reg} R$ the ideal $\hat{\mathcal{O}}_\mu^a(W)$ is either non-proper or belongs to $Max_\ell \hat{\mathcal{O}}_R^a(W)$, and $\cap\{\hat{\mathcal{O}}_\mu^a(W) \mid \mu \in Max_\ell^{reg} R\} = 0$. $\square$

Section 8. The category $\mathrm{I}_\ell^{\succ} R$ and non-commutative algebra.

Let as above R be an associative ring. For any set $\mathcal{F}$ of left ideals of R denote by $\mathrm{I}_\ell^{\succ} R \setminus \mathcal{F}$ the full subcategory of $\mathrm{I}_\ell^{>} R$ whose $\mathrm{I}_\ell R \sim \mathcal{F}$.

1. $\underline{\mathrm{I}_\ell^{\succ} R}$ <u>and the structure of radical filters</u>. The following statement is similar to Proposition 2.8.

Proposition. $Max(\mathrm{I}_\ell^{\succ} R \sim \mathcal{F}) \subset Spec_\ell R$ *for any radical filter* $\mathcal{F}$.

Proof. Let $\mathcal{M} \in Max(\mathrm{I}_\ell^{\succ} R \sim \mathcal{F})$. Since $\mathcal{M} \to (\mathcal{M} : x)$ for any $x \in R$, then the maximality of $\mathcal{M}$ implies $[x \in R, (\mathcal{M} : x) \nrightarrow \mathcal{M}] \Longleftrightarrow [(\mathcal{M} : x) \in \mathcal{F}]$. Therefore this implies that

$\mathcal{M} \in \mathrm{I}_\ell^{*} R$ (see 1.6) and therefore $\widehat{\mathcal{M}} \overset{def}{=} = \{x \in R \mid (\mathcal{M} : x) \nrightarrow \mathcal{M}\}$ is, by Proposition 1.6, an ideal from $\widehat{Spec}_\ell R$; $\widehat{\mathcal{M}}$ coincides with $\mathcal{M}_{\mathcal{F}} \overset{def}{=} \{\lambda \in R \mid (\mathcal{M} : \lambda) \in \mathcal{F}\}$.

For any $n \in \mathrm{I}_\ell R$ we have $[n \in \mathrm{I}_\ell R \setminus \mathcal{F}] \Leftrightarrow [n_{\mathcal{F}} \in \mathrm{I}_\ell R \setminus \mathcal{F}]$. Therefore the maximality of $\mathcal{M}$ (and the inclusion $\mathcal{M} \subset \mathcal{M}_{\mathcal{F}}$) imply that $\mathcal{M}$ and $\mathcal{M}_{\mathcal{F}} = \widehat{\mathcal{M}}$ are isomorphic. □

Corollary 1. *Let* $\mathfrak{T}$ *be a family of uniformed filters of left ideals of* R *such that*

(a) *all the filters from* $\mathfrak{T}$ *are of finite type,*

(b) $\mathcal{F} \circ \mathcal{G} \subset \bigcup\{\mathcal{F}' \mid \mathcal{F}' \in \mathfrak{T}\}$ *for any* $\{\mathcal{F}, \mathcal{G}\} \subset \mathfrak{T}$

Then $Max(\mathrm{I}_\ell^{\succ} R \sim \bigcup\{\mathcal{F}' \mid \mathcal{F}' \in \mathfrak{T}\}) \subset Spec_\ell R$.

In fact, it follows from (a) and (b) that $\sum \mathfrak{T} \overset{def}{=} = \bigcup\{\mathcal{F}' \mid \mathcal{F}' \in \mathfrak{T}\}$ is a radical filter. □

Corollary 2. Let $\mathcal{F}$ be a radical filter and for any $n \in I_\ell R \setminus \mathcal{F}$ there exists $\mathcal{M} \in Max(I_\ell^{\succ} R \setminus \mathcal{F})$ such that $n \to \mathcal{M}$. Then

(1) $\mathcal{F} = \bigcap\{\mathcal{F}_{\mathcal{M}} \mid \mathcal{M} \in Max(I_\ell^{\succ} R \setminus \mathcal{F})\} = \mathcal{F}_{V_\ell(\mathcal{F})} \overset{def}{=}$
$= \bigcap\{\mathcal{F}_p \mid$

(2) $Max(IR \setminus \mathcal{F}) \subset Spec_\ell^s R \overset{def}{=} \{p_s \mid p \in Spec_\ell R\}$.

Proof. 1) Clearly $\mathcal{F} \subset \mathcal{F}_{V_\ell(\mathcal{F})} \subset \bigcap\{\mathcal{F}_{\mathcal{M}} \mid \mathcal{M} \in Max(I_\ell^{\succ} R \setminus \mathcal{F})\}$ since $Max(I_\ell^{\succ} R \setminus \mathcal{F}) \subset Spec_\ell R \setminus \mathcal{F}$. On the other hand by hypothesis a left ideal n such that $n \not\to \mathcal{M}$ for all $\mathcal{M} \in Max(I_\ell^{\succ} R \setminus \mathcal{F})$ necessarily belongs to $\mathcal{F}$.

2) Let $m \in Max(IR \setminus \mathcal{F})$. By hypothesis $m \subset \mathcal{M}$ for some $\mathcal{M} \in Max(I_\ell^{\succ} R \setminus \mathcal{F})$ and therefore $m \subset \mathcal{M}_s$. Since $\mathcal{M}_s \notin \mathcal{F}$ and m is a maximal ideal from $IR \setminus \mathcal{F}$ then $\mathcal{M}_s = m$. $\square$

2. $\sup^{\succ}$ and $\sup$.

Here after we will assume that R possesses right unit. The reformulations for the general case are left to the reader.

For any family of left ideals $\{n^i \mid i \in I\}$ denote by $\sup^{\succ}\{n^i \mid i \in I\}$ its exact upper boundary in $I_\ell^{\succ} R$ if it exists. In other words $\sup^{\succ}\{n^i \mid i \in I\}$ is the colimit of the diagram in $I_\ell^{\succ} R$ generated by $\{n^i \mid i \in I\}$.

Lemma. The following properties of $\{n^i \mid i \in I\} \subset I_\ell R$ are equivalent:

1) $\sup^{\succ}\{n^i \mid i \in I\}$ exists.

2) There exists the family $\{t_i \mid i \in I\} \subset \mathcal{P}(R)$ such that

$$\sup\{(n^i : t_i) \mid i \in I\} \to \sup\{(n^i : x_i) \mid i \in I\}$$

for any other family $\{x_i \mid i\in I\}\subset \mathcal{P}(R)$;

and $\sup^{\succ}\{n^i \mid i\in I\}\simeq \sup\{(n^i:t_i)\mid i\in I\}$.

Proof. 1) $\Rightarrow$ 2). Clearly, $n^i\to(n^i:x_i)\subset \sup\{(n^k:x_k)\mid k\in I\}$ for any $\{x_j\mid j\in I\}$ and $i\in I$; therefore $\sup^{\succ}\{n^i\mid i\in I\}\to \sup\{(n^i:x_i)\mid i\in I\}$. On the other hand, $n^i\to \sup^{\succ}\{n^k\mid k\in I\}$ means exactly that $(n^i:t_i)\subset \sup^{\succ}\{n^k\mid k\in I\}$ for some $t_i\in\mathcal{P}(R)$. Select such t_i for every $i\in I$. Then $\sup\{(n^i:t_i)\mid i\in I\}\subset \sup^{\succ}\{n^i\mid i\in I\}$ and therefore $\sup\{(n^i:t_i)\mid i\in I\}\simeq$ $\simeq \sup^{\succ}\{n^i\mid i\in I\}$.

2) $\Longrightarrow$ 1). Let $\{t_i\mid i\in I\}$ be a family satisfying (2) and $\{n^i\to m\mid i\in I\}$ a cone in $I_\ell^{\succ}R$. This means that there exists a family $\{x_i\mid i\in I\}\subset\mathcal{P}(R)$ such that $(n^i:x_i)\subset m$ for every $i\in I$. By hypothesis $\sup\{(n^i:t_i)\mid i\in I\}\to \sup\{(n^i:x_i\mid i\in I\}\subset m$. Therefore the cone $\{n^j\to \sup\{(n^i:t_i)\mid i\in I\}\mid j\in I\}$ is initial. □

3. Symmetric radical filters. Clearly the assignment $n\longmapsto n_s$ is a functor $I_\ell^{\succ}R\longrightarrow IR$ (right conjugate to the embedding $IR\hookrightarrow I_\ell^{\succ}R$). In particular,

If $\{n^i\mid i\in I\}$ is an directed family of ideals in $I_\ell^{\succ}R$, then the family of twosided ideals $\{n_s^i\mid i\in I\}$ is directed with respect to inclusion;

$$\sup\{n_s^i\mid i\in I\}\subset(\sup\{(n^i:x_i)\mid i\in I\})_s$$

for any $\{n^i\mid i\in I\}\subset I_\ell R$ and an arbitrary $\{x_i\mid i\in I\}\subset\mathcal{P}(R)$.

We are interested in the following property of directed in $I_\ell^{\succ}R$ families of ideals $\{n^i\mid i\in I\}\subset I_\ell R$:

($\mathcal{L}_s$) There exists a subset $\{t_i \mid i \in I\} \subset \mathcal{P}(R)$ such that

$$\sup\{n^i_s \mid i \in I\} = (\sup\{(n^i : t_i) \mid i \in I\})_s .$$

Proposition. Let $\mathcal{F}$ be a symmetric radical filter of bifinite type such that for every linearly ordered (with respect to $\rightarrow$) family $\{n^i \mid i \in I\} \subset I_\ell R \setminus \mathcal{F}$ ($\mathcal{L}_s$) holds. Then

1) $\mathcal{F} = \bigcap\{\mathcal{F}_\mu \mid \mu \in Max(I^{\flat}_\ell R \setminus \mathcal{F})\} = \mathcal{F}_{V_\ell(\mathcal{F})}$;

2) $Max(IR \setminus \mathcal{F}) \subset Spec^s_\ell R \overset{def}{=} \{P_s \mid P \in Spec_\ell R\}$.

Proof. Let $\{n^i \mid i \in I\}$ be a linearly ordered with respect to $\longrightarrow$ family from $I_\ell R \setminus \mathcal{F}$. By hypothesis $\sup\{n^i_s \mid i \in I\} = (\sup\{(n^i : t_i) \mid i \in I\})_s$ for some $\{t_i \mid i \in I\} \subset \mathcal{P}(R)$.

Suppose $\sup\{(n^i : t_i) \mid i \in I\} \in \mathcal{F}$. Since $\mathcal{F}$ is symmetric, then $\sup\{n^i_s \mid i \in I\} = (\sup_i (n^i : t_i))_s$ belongs to $\mathcal{F}$. Thanks to bifiniteness of $\mathcal{F}$ and the fact that $\{n^i_s \mid i \in I\}$ is linearey ordered then ideal n^j_s (and therefore n^j) belong to $\mathcal{F}$ for some $j \in I$, This contradicts with the assumption; i.e. $\sup\{(n^i : t_i) \mid i \in I\} \notin \mathcal{F}$.

Since $n^j \longrightarrow \sup\{(n^i : t_i) \mid i \in I\}$ for every $j \in I$, we may apply Zorn's lemma and deduce that every ideal from $I^{\flat}_\ell R \setminus \mathcal{F}$ is majorated by an ideal from $Max(I^{\flat}_\ell R \setminus \mathcal{F})$. It remains to refer to Corollary 2.2. $\square$

Corollary. Let every linearly ordered chain of ideals from $I^{\flat}_\ell R$ satisfy ($\mathcal{L}_s$) . Then any radical symmetric filter $\mathcal{F}$ of bifinite type coincides with $\bigcap\{\mathcal{F}_\mu \mid \mu \in Max(I^{\flat}_\ell R \setminus \mathcal{F})\} = \mathcal{F}_{V_\ell(\mathcal{F})}$.

The full subcategory of Rings, consisting of all the rings satisfying to the conditions of Corollary, will be denoted by $\mathcal{L}_s$Rings.

4. The prime spectrum and the left spectrum.

Proposition 1. Let R be a ring such that

(η) For every primary ideal p the set of left ideals $\{(p:x) \mid x \in R \setminus p\}$ possesses maximal (with respect to $\to$) elements.

Then

1) the map $\mathcal{M} \mapsto \mathcal{M}_s$ performs the surjection $\widehat{Spec}_\ell R \to Spec\, R$; in particular, $Spec\, R = \overline{Spec}\, R$;

2) for any symmetric radical filter $\mathcal{F}$ of bifinite type

$$\mathcal{F} \cap IR = \bigcap\{\mathcal{F}_{\mathcal{M}} \mid \mathcal{M} \in \widehat{Spec}_\ell R \setminus \mathcal{F}\} \cap IR \overset{def}{=} \mathcal{F}_{V_\ell(\mathcal{F})} \cap IR$$

Proof. 1) Let $p \in Spec\, R$ and $(p:x_0)$ be a maximal (with respect to $\to$) element of $\{(p:x) \mid x \in R \setminus p\}$. Let us show that $(p:x_0) \in \widehat{Spec}_\ell R$.

Let $y \in R$ and $((p:x_0):y) \not\to (p:yx_0)$. The maximality of $(p:x_0)$ and the identity $((p:x_0):y) = (p:yx_0)$ imply that $yx_0 \in p$; i.e. $y \in (p:x_0)$ (clearly, the primariness of p is not used here).

Since $x_0 \in R \setminus p$ and p is primary, then $(p:x_0)_s = (p:(R,x_0)) = p$ and therefore p belongs to the image of the map $\widehat{Spec}_\ell R \to Spec\, R$, $\mathcal{M} \to \mathcal{M}_s$.

2) Let $\mathcal{F}$ be a symmetric radical filter of bifinite type, $\alpha \in IR \setminus \mathcal{F}$. By corollary of Proposition 2.8 there exists $p \in Spec\, R \setminus \mathcal{F}$ such that $\alpha \subset p$. Let $(p:x_0)$ be a maximal element of $\{(p:x) \mid x \in R \setminus p\}$ with

respect to $\longrightarrow$. Notice that $(p : x_0) \notin \mathcal{F}$.

In fact, since $\mathcal{F}$ is symmetric, then $[(p:x_0) \in \mathcal{F}] \Leftrightarrow$ $\Leftrightarrow [(p:x_0)_s \in \mathcal{F}]$; but $(p:x_0)_s = p$ as we have just verified, and by hypothesis $p \notin \mathcal{F}$.

Thus $\alpha \subset (p:x_0)$, $(p:x_0) \in \hat{Spec}_\ell R \setminus \mathcal{F}$ and $\mathcal{F} \subset \mathcal{F}_{(p:x_0)}$.

Since $\alpha \in IR \setminus \mathcal{F}$ is arbitrary, then

$$IR \cap \mathcal{F} = IR \cap (\cap \{\mathcal{F}_\mu \mid \mu \in \hat{Spec}_\ell R \setminus \mathcal{F}\}). \square$$

Since $\mathcal{F}$ is symmetric, then

$$\mathcal{F} = \left(\overline{\cap \{\mathcal{F}_\mu \mid \mu \in \hat{Spec}_\ell R \setminus \mathcal{F}\} \cap IR}\right)$$

Corollary. Let R satisfy $(\mathfrak{h})$. Then

1) $rad_\ell(\alpha) = \mathfrak{p}(\alpha)$ for any twosided ideal α of R.

In particular, the left radical R coincides with its low Barr radical.

2) If α is twosided ideal of R finitely generated as a left ideal then $^\alpha\hat{\mathcal{F}} = \overline{(\mathcal{F}_{V_\ell(\alpha)} \cap IR)}$

Proof. 1) By definition $\mathfrak{p}^R(\alpha) = \cap \{p \in Spec R \mid \alpha \subset p\}$ for any $\alpha \in IR$. On the other hand $rad_\ell^R(\alpha) = \cap \{p' \mid p' \in \hat{Spec} R \mid \alpha \subset p'\}$. By Proposition 4 $\hat{Spec}\, R$ coincides with the primary spectrum.

2) If α is a twosided ideal of R finitely generated as left ideal then the "radical closure" $^\alpha\hat{\mathcal{F}}$ of is a symmetric filter of finite (and therefore bifinite) type (see Example 4.17). Thus we may make use of the second statement of Proposition 4 which states that

$$IR \cap {}^\alpha\hat{\mathcal{F}} = \cap \{\mathcal{F}_\mu \mid \mu \in \hat{Spec}_\ell R \setminus \mathcal{F}\} \cap IR$$

Denote by $S_\ell Rings$ the full subcategory of the category of rings distinguished by heading property $(\mathfrak{h})$.

The definition of $\mathcal{S}_\ell$ Rings might look even less constructive than the definition of $\mathcal{L}_s$ Rings. However, I know practically nothing on $\mathcal{L}_s$ Rings whereas the following section contains a rather satisfactory estimation of $\mathcal{S}_\ell$ Rings "from the low".

5. Semiprime Goldi rings and $\mathcal{S}_\ell$ Rings. A left ideal ν of R is called a left annihilator if $\nu = (0:x)$ for a non-empty subset $x \subset R$. Usually they write $\ell(x)$ instead of (0:x) especially when one have to deal simultaneously with right annihilators denoted by $r(x)\,(=\{\lambda \in R \mid x\lambda = 0\})$.

R is called a left Goldi ring if

(1) R satisfies the descending chain condition for left annihilators;

(2) R doesn't contain any infinite direct sums of left non-zero ideals.

Clearly, any left Noetherian ring is a Goldi ring. The converse is false: a classical example is the polynomial ring in countable many commuting variables; it doesn't have zero divises nor direct sums of left ideals, though it is not, obviously, Noetherian.

Recall that R is called semiprime if it does not have non-zero nilpotent ideals or, equivalently, $\mathfrak{P}(R) = 0$.

The following fact (Lemma 7.2.1 in [16]) plays an important role in the study of Goldi rings.

Lemma 1. Let R be semiprime ring, satisfying maximality condition for left annihilators. If n and m are left ideals of R, $m \subset n$ and $r(n) \neq r(m)$, then there exists $a \in n$ such that $n \cdot a \neq 0$ and $na \cap m = 0$.

Proof see in [16]. □

Corollary 1. Any semiprime left Goldi ring satisfies the minimality condition for its left annihilators.

Proof. Let $\{m_i \mid 1 \leq i < \infty\}$ be a strictly descending chain of left annihilators. The relation $m_{i+1} \subsetneqq m_i$ implies $r(m_i) \neq r(m_{i+1})$ for all $i \geqslant 1$. Making use of Lemma 1 select in each m_i a non-zero left ideal (of R) such that $\nu_i \cap m_{i+1} = 0$. Therefore $\sum_{i=1}^{\infty} \nu_i$ is an infinite direct sum of non-zero left ideals which contradicts the definition of a Goldi ring. □

Corollary 2. Let R be a semiprime left Goldi ring. Then for every left annihilator m of R there exists $x_m \in \mathcal{P}(R)$ such that

Proof. Clearly if m is a left annihilator then (m:y) is the left annihilator for any subset $y \subset R$. By Corollary 1 the set of left annihilators $\{(m:x) \mid x \in \mathcal{P}(R)\}$ possesses a minimal with respect to inclusion element $(m:x_m)$.

If $(m:x_m) \neq (m:R)$ then there exists $y \in \mathcal{P}(R)$ such that $(m:x_m) \not\subset (m:y)$. But then $(m:x_m+y) = (m:x_m) \cap (m:y) \subsetneq (m:x_m)$ contradicting to the minimality of $(m:x_m)$. $\square$

Proposition 1. 1) Let R be a prime left Goldi ring. Then any proper left annihilator in R is isomorphic to the zero ideal.

2) Any semiprime left Goldi ring with unit satisfies the maximality and minimality conditions for left annihilators with respect to preorder $\to$.

Proof. 1) Recall that R is called prime if its zero ideal is prim . Let m be a left annihilator different from R and R be prim . In other words $m = (0:x)$ where x is a non-zero subset of R. Since 0 is prime and $Rx \neq 0$, then $(m:R) = ((0:x):R) = (0:Rx) = 0$. By Corollary 2 $(m:R) = (m:x_m)$ for some $x_m \in \mathcal{P}(R)$ and therefore $m \longrightarrow (m:R)$. Since $(m:R) = 0 \subset m$, this arrow is an isomorphism.

2) Let X be a subset of left annihilators of a semiprime unitary left Goldi ring R. Consider the set $X_R \overset{\text{def}}{=} \{(\nu:R) \mid \nu \in X\}$ also consisting of left annihilators. Let $(\nu_0:R)$ be a maximal element of X_R, $\nu_0 \in X$, whose existence is guaranteed by Corollary 1. Let $\nu \in X$ and $\nu_0 \longrightarrow \nu$. The relation $\nu_0 \to \nu$ implies as is easy to see $(\nu_0:R) \subset (\nu:R)$ (without any assumptions on R and its ideals: if $(\nu_0:x) \subset \nu$, $x \subset R$, then $(\nu_0:R) \subset \nu$, and since $(\nu_0:R)$ is a two-sided ideal this implies $(\nu_0:R) \subset ((\nu_0:R):R) \subset (\nu:R)$). Since

$(\mathcal{V}_0 : R)$ is a maximal element of X_R, then $(\mathcal{V}_0 : R) = (\mathcal{V} : R)$. By Corrollary 2 of Lemma 1 for any left annihilator m of R there exists an arrow $m \rightarrow (m : R)$. Therefore we've got a diagram

$$\begin{array}{ccc} \mathcal{V}_0 & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ (\mathcal{V}_0 : R) & = & (\mathcal{V} : R) \end{array} \qquad (1)$$

Since R is a ring with unit (this is the first time that we make use of this), then $\mathcal{V}'_s = (\mathcal{V}' : R)$ for any left ideal $\mathcal{V}'$; thus, the vertical arrows in (1) are isomorphisms and therefore so is the horizontal arrow.

The existence of a minimal with respect to $\rightarrow$ element in X is verified similarly. □

Corrolary. Let R be a ring such that for any prime ideal p the ring R/p is a left Goldi ring. Then R belongs to the category $\mathcal{S}_\ell \mathrm{Rings}$ (see n. 4) and $\mathcal{S}pec R \subset \widehat{\mathcal{S}pec}_\ell R$.

Proof. By the first statement of Proposition 1 any proper left annihilator of R/p is isomorphic to a zero ideal. This implies that any proper left annihilator of R/p belongs to $\widehat{\mathcal{S}pec}_\ell R/p$.

In fact, let $m = (0 : w)$ for some non-zero subset w of R/p, $a \in R/p$ and $(m : a) \not\rightarrow m$. This means that $(m : a) = (0 : aw) = R/p$ and therefore $aw = \{0\}$; i.e. $a \in (0 : w) = m$ as required.

In particular, $0 = (0 : R/p) \in \widehat{\mathcal{S}pec}_\ell R/p$. But this means that $p \in \widehat{\mathcal{S}pec}_\ell R$ (Proposition 5.9) and for any subset $x \subset R \setminus p$ the left ideal $(p : x)$ belongs to $\widehat{\mathcal{S}pec}_\ell R$ and is isomorphic to p. □

We have proved even more than promised in the formulation.

Examples. 1) Let R be a left Noetherian ring. For any two-sided ideal μ of R and in particular for $\mu \in \operatorname{Spec} R$ the ring R/μ is also left Noetherian and therefore is a left Goldi ring.

2) Let A be a commutative ring with unit, R an A-algebra. The algebra R is called a PI-algebra if for some $d > 0$ there exists a non-zero polynomial $f \in A[x_1, \dots, x_d]$ in non-commuting variables $x_1, \dots, x_d$ such that $f(\lambda_1, \dots, \lambda_d) = 0$ for all $\{\lambda_1, \dots, \lambda_d\} \subset R$.

Clearly, any commutative algebra is a PI-algebra with the polynomial identity $x_1 x_2 - x_2 x_1 = 0$;

if R is an algebra of dimension b over a field k then R satisfies the so-called standard polynomial identity

$$[x_1, \dots, x_{b+1}] = \sum_{\sigma \in S_{b+1}} \operatorname{sgn}(\sigma) x_{\sigma(1)} \dots x_{\sigma(b+1)}$$

where S_{b+1} is the symmetric group ([16], Lemma 6.2.2). This fact is easily generalized replacing k by a commutative ring;

in particular the full matrix ring A_m over a commutative ring A satisfies the standard identity $[x_1, \dots, x_{m^2+1}] = 0$ (Amitzur and Levitzky showed that A_m satisfies $[x_1, \dots, x_{2m}] = 0$; see a short proof with the help of superalgebras in the appendix to this section);

clearly, a subalgebra of a PI-algebra is a PI-algebra and so is a quotient-algebra;

if R is a PI-algebra without non-zero nill-ideals then R is isomorphic to a subalgebra of A_m where A is a commutative semisimple ring ([16] , Theorem 6.3.2).

After this short introduction addressed to those who had not been acquainted with PI-algebras (for the first reading we recommend Chapter VI of the remarkable book [16]) turn our attention to the following fact which is essential to us at this moment ([16] Lemma 7.3.2):

If R is a prime ring satisfying a polynomial identity over its centroid then R is a two-sided Goldi ring.

Recall that the centroid of R is the ring of all the endomorphisms of the (R,R)-bimodule R. If $R = R^2$ or the left annihilator (O:R) of R is zero, then the centroid of R is commutative ([5], Ch.5, 4, Proposition 1); in particular, centroids of prime rings are commutative.

Therefore if R is a ring such that $R/\mathfrak{B}(R)$ is a PI-algebra, then R/p is a two-sided Goldi ring for any prime ideal p of R. □

Thus, left Noetherian rings and rings, whose quotient modulo the low Bair radical is a PI-algebra (in particular, all the PI-algebras), satisfy the conditions of Corollary of Proposition 1 and, therefore, any ring R belonging to one of these two classes belongs also to $\mathcal{S}_\ell Rings$; and, besides, $\mathcal{S}pec R \subset \widehat{\mathcal{S}pec}_\ell R$

Proposition 2. For any left Noetherian ring R the canonical geometrization of the prime spectre $(\mathcal{S}pec R, \widetilde{R}^a)$ coincides with the left affine quasischeme $(\overline{\mathcal{S}pec}\, R, \overline{\mathcal{O}}_R^a)$. In particular, the Van-Oystaeyen and Vershoren affine schemes (the canonical geometrizations of left Noetherian rings with unit) coincides with left affine schemes of the corresponding rings.

Proof. Recall that the preshief $\tilde{R}$ assigns to a closed set $V(\alpha)$ of the primary spectrum the ring $G_{T_{\alpha}\hat{\mathcal{F}}} R$. Since R is a ring from $\mathcal{S}_\ell\,\mathrm{Rings}$ and the two-sided ideal α is finitely generated as a left ideal, then ${}^{\alpha}\hat{\mathcal{F}}$ coincides with $\mathcal{F}_{V_\ell(\alpha)}$ (Corollary of Proposition 4). Besides, $\overline{\mathcal{S}pec}\,R = \mathcal{S}pec\,R$ by Proposition 4. Thus $(\overline{\mathcal{S}pec}\,R, \bar{\mathcal{O}}_R) = (\mathcal{S}pec\,R, \tilde{R})$ and therefore $(\overline{\mathcal{S}pec}\,R, \bar{\mathcal{O}}_R^a) = (\mathcal{S}pec\,R, \tilde{R}^a)$. $\square$

We will consider two more subcategories of $\mathcal{S}_\ell\,\mathrm{Rings}$.

6. Rings the uniform filters of left ideals of which are symmetric. Denote by $\mathcal{LS}\mathrm{Rings}$ the full subcategory of Rings formed by such rings.

Proposition. The following properties of a ring R are equivalent:

(a) for any left ideal n there exists $x \in \mathcal{P}(R)$ such that $n_s = (n : x)$;

(b) the embedding $IR \hookrightarrow I_\ell^{\mathfrak{t}} R$ is an equivalence of categories;

(c) the filters ${}^m\mathcal{F} = \{n \in I_\ell R \mid m \to n\}$ are symmetric for all $m \in I_\ell R$;

(d) R is a ring from $\mathcal{LS}\mathrm{Rings}$;

(e) for any family $\{n^i \mid i \in I\}$ of left ideals there exists $\sup^{\mathfrak{t}}\{n^i \mid i \in I\}$ and is isomorphic to the two-sided ideal $\sup\{n_s^i \mid i \in I\}$.

Proof is straightforward and is left to the reader.

Corollary. 1) $\mathcal{LS}\mathrm{Rings}$ is a subcategory of $\mathcal{L}_s\mathrm{Rings}$ and $\mathcal{S}_\ell\,\mathrm{Rings}$.

2) Let R be a ring from $\mathcal{L}\mathcal{S}Rings$. Then for any radical filter $\mathcal{F}$ of left ideals of R and an arbitrary $m \in I_\ell R$ the following identities hold:

$$\mathcal{F} = \bigcap\{\mathcal{F}_{\mu} \mid \mu \in Max(I_\ell^{\succ} R \setminus \mathcal{F})\} = \mathcal{F}_{V_\ell(\mathcal{F})};\ {}^{m}\widehat{\mathcal{F}} = \mathcal{F}_{V_\ell(m_s)}.$$

3) If R is a ring from $\mathcal{L}\mathcal{S}Rings$ then $\overline{Spec}\, R = Spec\, R \subset \widehat{Spec}_\ell R$ and any ideal from the left spectrum is isomorphic to a primary ideal.

Proof. 1) The inclusion $\mathcal{L}\mathcal{S}Rings \subset \mathcal{L}_s Rings$ obviously follows from (e) of Proposition 6.

Let R be a ring from $\mathcal{L}\mathcal{S}Rings$ and $p \in Spec\, R$. Then $(p:x)_s = (p:(R,x)) = p$ for any $x \in R \setminus p$; and at the same time $(p:x) \simeq (p:x)_s$ thanks to (a) of Proposition 6.

3) Therefore, $Spec\, R \subset \widehat{Spec}_\ell R$ and any ideal μ from $Spec_\ell R$ is isomorphic to a primary ideal μ_s (heading (a) of Proposition 6).

2) Follows from the symmetricity of radical filters, corollary of Proposition 4 and the fact that ${}^{m}\mathcal{F} = {}^{m_s}\mathcal{F}$ for all $m \in I_\ell R$. $\square$

7. Uniformly left Noetherian rings. A ring R, such that preorder $I_\ell^{\succ} R$ is Noetherian, is called a uniformly left Noetherian (or $I_\ell^{\succ}$-Noetherian) ring. The full subcategory of Rings, formed by $I_\ell^{\succ}$-Noetherian rings, will be denoted by $I_\ell^{\succ}Rings$. It is easy to see that

- all the rings from $\mathcal{L}S\ Rings$ with a Noetherian preordering of two-sided ideals belong to $I_\ell^{\succ}\ Rings$;

- if R is a $I_\ell^{\succ}$-Noetherian ring then

$$\mathcal{F} = \bigcap\{\mathcal{F}_\mu \mid \mu \in Max\,(I_\ell^{\succ}R \setminus \mathcal{F})\} = \mathcal{F}_{V_\ell(\mathcal{F})},\ {}^m\widehat{\mathcal{F}} = \mathcal{F}_{V_\ell(m)}$$

for any radical filter $\mathcal{F}$ and any $m \in I_\ell R$;
and in addition any closed set in the topology $\mathcal{T}_1$ (see 5.4) is of the form $V_\ell(m) = \{p \mid m \to p\}$ for some $m \in I_\ell R$.

8. The support of a module. The support of an R-module M is the set $Supp(M)$ of all the ideals $p \in Spec_\ell R$ such that $\mathcal{F}_p M \neq M$ or equivalently $G_{\mathcal{F}_p} M \neq 0$.

If R is a ring with right unit then $[\ RM = 0] \Leftrightarrow$ $\Leftrightarrow [Supp(M) = \emptyset]$ for any R-module M. In particular, if M is a unitary R-module, then $[Supp(M) = \emptyset] \Leftrightarrow [M = 0]$.

Proposition. 1) If N is a submodule of an R-module M, then

$$Supp(M) = Supp(N) \cup Supp(M/N).$$

2) If M is the sum of a family of its submodules $\{N_i \mid i \in I\}$ then $Supp(M) = \bigcup_{i \in I} Supp(N_i)$.

Proof. 1) Clearly, $Supp(N) \subset Supp(M)$.

Let $\xi \in M$, $\bar{\xi}$ be the image of ξ in M/N and $Ann\,\bar{\xi} \notin \mathcal{F}_p$, $p \in Spec_\ell R$. Then clearly $Ann\,\xi \notin \mathcal{F}_p$, i.e. $p \in Supp(M)$; therefore $Supp(M/N) \subset Supp(M)$.

Since $G_{\mathcal{F}_p}$ is left exact then the sequence

$$0 \to G_{\mathcal{F}_p} N \to G_{\mathcal{F}_p} M \to G_{\mathcal{F}_p}(M/N)$$

is exact. Therefore, if $G_{\mathcal{F}_p} M \neq 0$ then either $G_{\mathcal{F}_p} N \neq 0$ (i.e. $p \in Supp(N)$) or $G_{\mathcal{F}_p}(M/N) \neq 0$.

2) The inclusion $\bigcup_{i\in I} Supp(N_i) \subset Supp(M)$ and the implication $[\mathcal{F}_p M \neq M] \Rightarrow [\mathcal{F}_p N_i \neq N_i$ for some $i \in I]$ are equally obvious. $\square$

Corollary 1. Let $\{\xi_i \mid i \in I\}$ be a family of generators of an R-module M. Then $Supp(M) = \bigcup_{i\in I} V_\ell(Ann\, \xi_i)$.

In particular, for any R-module M its support is a closed subset of $(Spec_\ell R, \mathfrak{T}_0)$.

Corollary 2. If M is an R-module with a finite family of generators $\xi = \{\xi_i \mid i \in I\}$, then

$$Supp(M) = V_\ell(Ann\, \xi) = V_\ell(\bigcap_{i\in I} Ann\, \xi_i).$$

Proof. By Corollary 1 $Supp(M) = \bigcup_{i\in I} V_\ell(Ann\, \xi_i)$. It is shown in 5.4 that

$$\bigcup_{i\in I} V_\ell(n_i) = V_\ell(\bigcap_{i\in I} n_i)$$

for any finite family $\{n_i \mid i \in I\}$ of left ideals. $\square$

Therefore, the support of an R-module of finite type is a closed subset of $(Spec_\ell R, \mathfrak{T}_1)$.

9. Associated ideals. For any R-module M denote by $\widehat{Ass}(M)$ or $\widehat{Ass}_R(M)$ when the indication to the ring of "scalars" is needed the family of all the ideals $\mathbf{p}$ from $\widehat{Spec}_\ell R$ such that $\mathbf{p} = Ann\, \xi$ for some $\xi \in M$. Denote by $Ass(M)$ (or $Ass_R(M)$) the set of all $\mathbf{p} \in Spec_\ell R$ such that $\mathbf{p} \simeq \mathbf{p}'$ for some $\mathbf{p}' \in \widehat{Ass}(M)$. The ideals from $Ass(M)$ will be called associated and those from $\widehat{Ass}(M)$ strictly associated with M.

If an R-module M is a union of a family of its submodules $\{M_i \mid i \in I\}$ then obviously

$$\widehat{Ass}(M) = \bigcup_{i\in I} \widehat{Ass}(M_i)$$

and therefore $Ass(M) = \bigcup_{i\in I} Ass(M_i)$.

Proposition 1. For any $p \in \widehat{Spec}_\ell R$ and any non-zero submodule M of R/p the set consists of ideals isomorphic to p.

Proof. $Ann\,\xi = (p : x_\xi)$ for any $\xi \in R/p$ where x_ξ is a preimage of ξ in R. Therefore, if $\xi \neq 0$ then $Ann\,\xi \simeq p$. □

In what follows for convenience we will confine ourselves to the study of unitary rings and modules. In the non-unitary case all the formulations hold if we pass (also in the definition of associated ideals) to the "extended" left spectrum $Spec_\ell R \cup \{R\}$, i.e. to the left spectrum of R[1].

Proposition 2. Let M be an R-module. Any maximal (with respect to the ordering in $I_\ell^{\succ} R$) element of $\{Ann\,\xi \mid \xi \in M \setminus \{0\}\}$ belongs to $\widehat{Ass}_R(M)$.

Proof. Let $p = Ann\,\xi'$ be such a maximal element; $x \in R$ and $(p:x) \not\to p$. Since $(p:x) = Ann\, x\xi'$ then the maximality of p among the ideals of the form $Ann\,\xi$, $\xi \in M \setminus \{0\}$, implies that $x\xi' = 0$; i.e. $x \in Ann\,\xi' = p$. □

Corollary 1. Let M be a module over $I_\ell^{\succ}$ -Noetherian ring R. Then $[M \neq 0] \Leftrightarrow [Ass(M) \neq \emptyset]$.

Proof. Clearly, $Ass(0) = \emptyset$. If $M \neq 0$ then $\{Ann\,\xi \mid \xi \in M \setminus \{0\}\} \neq \emptyset$ and since R is $I_\ell^{\succ}$-Noetherian, this set possesses a maximal element. □

Corollary 2. Let R be a $I_\ell^{\succ}$-Noetherian ring M an R-module. The following properties of a left ideal M of R are equivalent:

(a) $n \not\to p$ for any $p \in Ass(M)$;

(b) $n \not\to Ann\,\xi$ for any $\xi \in M \setminus \{0\}$.

Proposition 3. Let M be an R-module, N a submodule. Then

$$\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N).$$

Proof. Obviously, $\mathrm{Ass}(N) \subset \mathrm{Ass}(M)$. Let $p \in \widehat{\mathrm{Ass}}(M)$, E a submodule of M isomorphic to R/p ; $F = E \cap N$.

If $F = 0$ then E is isomorphic to a submodule of M/N implying $p \in \mathrm{Ass}(M/N)$. If $F \neq 0$ then the annihilator of every element from $F \setminus \{0\}$ is isomorphic to p by Proposition 1 and therefore $p \in \mathrm{Ass}(N)$. □

Corollary 1. Let an R-module M be a direct sum of modules $\{M_i \mid i \in I\}$. Then

Proof. Clearly, $\bigcup_{i \in I} \mathrm{Ass}(M_i) \subset \mathrm{Ass}(M)$. Let us verify the converse inclusion.

a) Let $\mathrm{Card}(I) = 2$; i.e. $I = \{i, j\}$. Since M/M_i is isomorphic to M_j then by Proposition 3

$$\mathrm{Ass}(M) \subset \mathrm{Ass}(M_i) \cup \mathrm{Ass}(M_j)$$

b) By induction we deduce from here the inclusion $\mathrm{Ass}(\bigoplus_{i \in J} M_i) \subset \bigcup_{i \in J} \mathrm{Ass}(M_i)$ for all finite J. Finally, $M = \bigcup_J \bar{M}_J$ where J runs the set of all finite subsets of I and $\bar{M}_J \overset{\mathrm{def}}{=} \bigoplus_{i \in J} M_i$; therefore (see (1))

$$\mathrm{Ass}(M) = \bigcup_J \mathrm{Ass}(\bar{M}_J) \subset \bigcup_{i \in I} \mathrm{Ass}(M_i). \quad □$$

Corollary 2. Let $\{Q_i \mid i \in I\}$ be a finite family of submodules of an R-module M such that $\bigcap_{i \in I} Q_i = 0$. Then

$$\mathrm{Ass}(M) \subset \bigcup_{i \in I} \mathrm{Ass}(M/Q_i)$$

In fact, the canonical map $M \to \bigoplus_{i \in I} M/Q_i$ is injective so it suffices to apply Corollary 1. □

Proposition 4. Let M be an R-module and Φ a subset in $\tilde{\ }\mathrm{Ass}(M)$. There exists a submodule $N \subset M$ such that $\tilde{\ }\mathrm{Ass}(N) = \tilde{\ }\mathrm{Ass}(M) \setminus \Phi$ and $\tilde{\ }\mathrm{Ass}(M/N) = \Phi$.

Proof. Let $\mathfrak{S}_\Phi$ be a family of submodules P of M such that $\tilde{\ }Ass(P) \subset \tilde{\ }Ass(M) \setminus \Phi$. It follows from (1) that $\mathfrak{S}_\Phi$ is ordered with respect to inclusion and inductive; besides it is non-empty since $0 \in \mathfrak{S}_\Phi$. Let N be a maximal element of $\mathfrak{S}_\Phi$. By hypothesis $\tilde{\ }Ass(N) \subset$ $\subset \tilde{\ }Ass(M) \setminus \Phi$. To complete the proof it suffices (by Proposition 3) to show that $\tilde{\ }Ass(M/N) \subset \Phi$.

Let $p \in \widehat{Ass}(M/N)$. Then M/N contains a submodule F/N isomorphic to R/p. It follows from Propositions 1 and 3 that $Ass(F) \subset Ass(N) \cup \{p' \mid p' \simeq p\}$. Since N is maximal in $\mathfrak{S}_\Phi$ then $F \in \mathfrak{S}_\Phi$ and therefore $\tilde{p} \in \Phi$. □

Remark. The inductiveness of $\mathfrak{S}_\Phi$ (see Proof of Proposition 4) implies that for any $P \in \mathfrak{S}_\Phi$ there exists a containing P maximal element **N** of $\mathfrak{S}_\Phi$ such that (as we have just verified) $\tilde{\ }Ass(N) = \tilde{\ }Ass(M) \setminus \Phi$ and $\tilde{\ }Ass(M/N) = \Phi$. □

Example. Let $\Phi = \tilde{\ }(Ass(M) \setminus \mathcal{F})$, where $\mathcal{F}$ is a radical filter of left ideals of R. Obviously $\mathcal{F}M \in \mathfrak{S}_\Phi$ and therefore there exists a submodule N of M such that

$$\mathcal{F}M \subset N,\ N \in Max\,\mathfrak{S}_\Phi;\ Ass(N) = Ass(M) \cap \mathcal{F},\ Ass(M/N) = Ass(M) \setminus \mathcal{F}.$$

If R is a commutative Noetherian ring and $\mathcal{F} = F_S$ for a multiplicative subset $S \subset R$ only one submodule of M namely $\mathcal{F}M$ satisfies

$$Ass(N) = Ass(M) \cap \mathcal{F},\ Ass(M/N) = Ass(M) \setminus \mathcal{F}$$

(see [3], Ch. IV, § 1, No.2, Proposition 6). Is this statement true for non-commutative rings and modules over them? We will answer this question in subsection 11 (Proposition 2).

10. One more variety of a spectrum. Recall (See 1.6) that $I_\ell^* R$ denotes the subset of all left ideals n of R such that $[\{z_1, z_2\} \subset R,\ (n:z_i) \not\rightarrowtail n,\ i=1,2] \Rightarrow [(n:\{z_1,z_2\}) \not\rightarrowtail n]$. By Proposition 1.6 the set $\hat{n} \overset{\text{def}}{=} \{\lambda \in R \mid (n:\lambda) \not\rightarrowtail n\}$ is an ideal from $\widehat{Spec}_\ell R$ for any $n \in I_\ell^* R$.

Lemma 1. The following properties of $n \in I_\ell R$ are equivalent:

(i) $n \in I_\ell^* R$ and

(4) $[z \in R, (n:z) \not\rightarrowtail n] \Rightarrow [(n:z) \not\rightarrowtail \hat{n}]$

(i.e. $(n:z) \not\rightarrowtail n$ for all $z \in \hat{n}$), $\hat{n} \neq R$;

(ii) $\hat{n} = n_{\mathcal{F}} \overset{\text{def}}{=} \{\lambda \in R \mid (n:\lambda) \in \mathcal{F}\} \neq R$ for some radical filter $\mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii). Condition (4) means exactly that $\hat{n} = n_{\mathcal{F}_{\hat{n}}}$. The filter $\mathcal{F}_{\hat{n}}$ is radical since $\hat{n} \in \widehat{Spec}_\ell R$.

(ii) $\Rightarrow$ (i). It follows from the equality $\hat{n} = n_{\mathcal{F}}$ for a topologizing filter $\mathcal{F}$ that $n \in I_\ell^* R$. Since $n_{\mathcal{F}} \neq R$ then $n \notin \mathcal{F}$ and therefore $[(n:z) \not\rightarrowtail n] \Rightarrow [(n:z) \in \mathcal{F}$ (by hypothesis)$] \Rightarrow [(n:z) \not\rightarrowtail n_{\mathcal{F}}$ (since $n_{\mathcal{F}} \notin \mathcal{F}$ whenever $n \notin \mathcal{F}$)]. □

Denote by $Spec_\ell^* R$ a family of left ideals n satisfying the equivalent conditions of Lemma 1. For any radical filter $\mathcal{F}$ of left ideals of R denote by $Spec_\ell^{\mathcal{F}} R$ the set of all $n \in I_\ell R$ such that $\hat{n} = n_{\mathcal{F}} \neq R$. As is clear from Lemma 1 $Spec_\ell^* R = \cup\{Spec_\ell^{\mathcal{F}} R \mid \mathcal{F}$ is a radical filter$\} = \cup\{Spec_\ell^{\mathcal{F}_p} R \mid p \in \widehat{Spec}_\ell R\}$.

The obvious properties of $Spec_\ell^* R$ and its subsets $Spec_\ell^{\mathcal{F}} R$ are listed in the following

Proposition 1. 1) $\mathrm{Spec}_\ell R \subset \mathrm{Spec}_\ell^* R$ and $\widehat{\mathrm{Spec}}_\ell R \cap \mathrm{Spec}_\ell^{\mathcal{F}} R = \widehat{\mathrm{Spec}}_\ell R \setminus \mathcal{F}$.

2) The map $\mu \mapsto \hat{\mu}$, $\mu \in \mathrm{Spec}_\ell^* R$ is left inverse to the embedding $\widehat{\mathrm{Spec}}_\ell R \hookrightarrow \mathrm{Spec}_\ell^* R$.

3) For any radical filter $\mathcal{F}$ and any ideal $n \in \mathrm{Spec}_\ell^{\mathcal{F}} R$ the ideal $G_{\mathcal{F}} n$ of $G_{\mathcal{F}} R$ belongs to $\widehat{\mathrm{Spec}}_\ell G_{\mathcal{F}} R$

4) If **R** is commutative then $\mathrm{Spec}_\ell^* R = \mathrm{Spec}_\ell R$.

Proof. 1) Since the embedding $p \subset \hat{p}$ is an isomorphism in $I_\ell^{\vdash} R$ for any $p \in \mathrm{Spec}_\ell R$ by Proposition 1.6, then $[m \not\mapsto p] \Leftrightarrow [m \not\mapsto \hat{p}]$; in particular, $\mathrm{Spec}_\ell R \subset \mathrm{Spec}_\ell^* R$.

Clearly, $p = \hat{p} = p_{\mathcal{F}}$ for any $p \in \widehat{\mathrm{Spec}}_\ell R \setminus \mathcal{F}$.

2) Follows from the above phrase.

3) $G_{\mathcal{F}} n = G_{\mathcal{F}} n_{\mathcal{F}}$ for any n from $I_\ell R \setminus \mathcal{F}$. If $n \in \mathrm{Spec}_\ell^{\mathcal{F}} R$, then $n_{\mathcal{F}} = \hat{n} \in \widehat{\mathrm{Spec}}_\ell R \setminus \mathcal{F}$ and, therefore, by Proposition 2.9 Corollary 2 of

$$G_{\mathcal{F}} n \in \widehat{\mathrm{Spec}}_\ell G_{\mathcal{F}} R .$$

4) Now let R be commutative. Then condition (4) of Lemma 1 takes the form

$$[(n:z) \not\subset n,\ z \in R] \Rightarrow [(n:z) \not\subset \hat{n}];$$

in other words, if $z \in R$ and there exists $y \in R \setminus n$ such that $yz \in n$, then there exists $x \in R$ such that $xz \in n$ and $(n:x) \subset n$. But $xz = zx$ and, therefore, $z \in (n:x) \subset n$. Hence an ideal n of a commutative ring, satisfying (4), is simple. $\square$

Proposition 2. Let $\mathcal{F}$ be a radical filter of left ideals of R, M an R-module and $M \neq \mathcal{F}M$. Then any maximal (with respect to the ordering in $I_\ell^{\vdash} R$) element of $\{\mathrm{Ann}\,\xi \mid \xi \in M \setminus \mathcal{F}M\}$ belongs to $\mathrm{Spec}_\ell^{\mathcal{F}} R$.

Proof. Let $p = \mathrm{Ann}\,\xi$ be a maximal element of $\{\mathrm{Ann}\,\xi' \mid \xi' \in M \setminus \mathcal{F}M\}$, $z \in R$ and $(p:z) \not\to p$. Since $(p:z) = \mathrm{Ann}\, z\xi$ then the maximality of p implies $z\xi \in \mathcal{F}M$; i.e. $z \in p_{\mathcal{F}}$. Therefore $\hat{p} \subset p_{\mathcal{F}}$. The inclusion $n_{\mathcal{F}} \subset \hat{n}$ takes place for any $n \in I_\ell R \setminus \mathcal{F}$. □

Set

$$\hat{\mathrm{Ass}}{}_R^{\mathcal{F}}(M) = \mathrm{Spec}_\ell^{\mathcal{F}} R \cap \{\mathrm{Ann}\,\xi \mid \xi \in M\};$$

$$\hat{\mathrm{Ass}}{}_R^{*}(M) = \mathrm{Spec}_\ell^{*} R \cap \{\mathrm{Ann}\,\xi \mid \xi \in M\};$$

$$\mathrm{Ass}_R^{*}(M) = \{p \in \mathrm{Spec}_\ell^{*} R \mid p \simeq p' \text{ for some } p' \in \hat{\mathrm{Ass}}{}_R^{*}(M)\};$$

$$\mathrm{Ass}_R^{\mathcal{F}}(M) = \mathrm{Ass}_R^{*}(M) \cap \mathrm{Spec}_\ell^{\mathcal{F}} R.$$

The elements of $\mathrm{Ass}_R^{*}(M)$ will be called ideals $*$-associated with M.

Corollary 1. Let R be a $I_\ell^{\curlywedge}$-Noetherian ring, $\mathcal{F}$ a radical filter of its left ideals. The following properties of an R-module M are equivalent:

(a) $M \neq \mathcal{F}M$;

(b) $\mathrm{Ass}_R^{*}(M) \setminus \mathcal{F} \neq \varnothing$;

(c) $\mathrm{Ass}_R^{\mathcal{F}}(M) \neq \varnothing$.

Proof. Clearly, $(c) \Rightarrow (b) \Rightarrow (a)$.

(a) $\Longrightarrow$ (c). If $M \neq \mathcal{F}M$ then $\mathcal{I}_M^{\mathcal{F}} \overset{\mathrm{def}}{=} \{\mathrm{Ann}\,\xi \mid \xi \in M \setminus \mathcal{F}M\} \neq \varnothing$. Since R is $I_\ell^{\curlywedge}$-Noetherian then $\mathcal{I}_M^{\mathcal{F}}$ is inductive and therefore possesses maximal (with respect to the preordering in $I_\ell^{\curlywedge} R$) elements each of which belongs to $\mathrm{Spec}_\ell^{\mathcal{F}} R$ by Proposition 2. □

Corollary 2. Let $\mathcal{F}$ be a radical filter of left ideals of a $I_\ell^{\curlywedge}$-Noetherian ring R. Then for any $p' \in \hat{\mathrm{Spec}}_\ell\, G_{\mathcal{F}} R$ such that $j_{\mathcal{F}}^{-1}(p') \notin \mathcal{F}$ there exists $p_0 \in \hat{\mathrm{Spec}}_\ell R \setminus \mathcal{F}$ such that $p' \simeq G_{\mathcal{F}}\, p_0$.

Proof. Let $\mathfrak{p}' \in \widehat{\mathcal{S}pec}_\ell G_\mathcal{F} R$ and $\mathfrak{p} \overset{def}{=} j_\mathcal{F}^{-1} \mathfrak{p}'$. Consider the set $\{(\mathfrak{p}:x) \mid x \in R \setminus \mathfrak{p}_\mathcal{F}\}$ and let $\mathfrak{p}_0 = (\mathfrak{p}:\lambda)$ be a maximal (in $\mathcal{I}_\ell^\mathfrak{f} R$) element of this set. By Lemma 2.9 $\mathfrak{p}' = G_\mathcal{F}\mathfrak{p}$. Since $\mathfrak{p}_0 \notin \mathcal{F}$, then $G_\mathcal{F}\mathfrak{p}_0$ is a proper ideal: besides $G_\mathcal{F}\mathfrak{p}_0 = G_\mathcal{F}(\mathfrak{p}:\lambda) = (\mathfrak{p}' : j_\mathcal{F}(\lambda))$. This implies that $\mathfrak{p}' \simeq G_\mathcal{F}\mathfrak{p}_0$. Since $\{(\mathfrak{p}:x) \mid x \in R \setminus \mathfrak{p}_\mathcal{F}\}$ is nothing else but $\{Ann\,\xi \mid \xi \in R/\mathfrak{p} \setminus \mathcal{F}(R/\mathfrak{p})\}$, then $\mathfrak{p}_0$ belongs to $\widehat{Ass}^\mathcal{F}(R/\mathfrak{p})$ by Proposition 2 and therefore $\mathfrak{p}_{0\mathcal{F}} = \widehat{\mathfrak{p}_0} \in \widehat{\mathcal{S}pec}_\ell R \setminus \mathcal{F}$. Now the statement follows from the identity $G_\mathcal{F}\mathfrak{p}_0 = G_\mathcal{F}(\mathfrak{p}_{0\mathcal{F}})$. □

11. Localizations of associated and $*$-associated ideals.

Proposition 1. Let $\mathcal{F}$ be a radical filter of left ideals of R, M an R-module.

1) The map $\mathcal{M} \mapsto G_\mathcal{F}\mathcal{M}$ induces the map $\pi_\mathcal{F} : \widehat{Ass}_R^\mathcal{F}(M) \to$
$\longrightarrow \widehat{Ass}_{G_\mathcal{F}R}(G_\mathcal{F}M)$ such that

$$[\pi_\mathcal{F}(\mathfrak{p}) = \pi_\mathcal{F}(\mathfrak{p}')] \Leftrightarrow [\mathfrak{p}_\mathcal{F} = \mathfrak{p}'_\mathcal{F}].$$

2) If R is $\mathcal{I}_\ell^\mathfrak{f}$-Noetherian then for any $\mathfrak{p}'$ from $\widehat{Ass}_{G_\mathcal{F}R}(G_\mathcal{F}M)$ there exists an ideal $\mathfrak{p} \in \widehat{Ass}_R^\mathcal{F}(M)$ such that $G_\mathcal{F}\mathfrak{p} = (\mathfrak{p}' : x)$ for some $x \in j_\mathcal{F}(R) \setminus \mathfrak{p}'$ in particular, $\mathfrak{p}' \simeq G_\mathcal{F}\mathfrak{p}$.

Proof. 1) For any $\xi \in \mathcal{M}$ we have

$$G_\mathcal{F} Ann_R \xi = Ann_{G_\mathcal{F}R}(j_{\mathcal{F},\mathcal{M}}(\xi)).$$

In fact, the annihilator of ξ can be defined as a (unique) left ideal R such that the square

$$\begin{tikzcd} R \arrow[r, "r \mapsto r\cdot\xi"] & \mathcal{M} \\ Ann\,\xi \arrow[u, hook] \arrow[r] & 0 \arrow[u] \end{tikzcd} \qquad (1)$$

is Cartesian. $G_{\mathfrak{F}}$ is left exact and in particular, transforms (1) into the Cartesian square

$$\begin{tikzcd} G_{\mathfrak{F}}R \arrow[r, "\lambda \mapsto \lambda\cdot j_{\mathfrak{F},M}(\xi)"] & G_{\mathfrak{F}}M \\ G_{\mathfrak{F}}\operatorname{Ann}\xi \arrow[u, hook] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

Thus $G_{\mathfrak{F}}$ transforms the ideals from $\widehat{\operatorname{Ass}}_R^{\mathfrak{F}}(M)$ (provided $\widehat{\operatorname{Ass}}_R^{\mathfrak{F}}(M)\neq\emptyset$, i.e. $M\neq\mathfrak{F}M$) into the annihilators of non-zero elements of the $G_{\mathfrak{F}}R$ -module $G_{\mathfrak{F}}M$. By heading 3) of Proposition 10.1 these annihilators belong to $\widehat{\operatorname{Ass}}_{G_{\mathfrak{F}}R}(G_{\mathfrak{F}}M)$.

2) Now let R be I_ℓ^{ξ}-Noetherian ring, $\xi\in G_{\mathfrak{F}}M$ and $p'=\operatorname{Ann}_{G_{\mathfrak{F}}R}\xi\in\widehat{\operatorname{Ass}}_{G_{\mathfrak{F}}R}(G_{\mathfrak{F}}M)$. Since $G_{\mathfrak{F}}M$ is a $\mathfrak{F}$ -torsion-free R-module, then $p''\overset{\mathrm{def}}{=\!=}j_{\mathfrak{F}}^{-1}p'$ (which obviously coincides with $\operatorname{Ann}_R\xi$) does not belong to $\mathfrak{F}$. Therefore $p'=G_{\mathfrak{F}}p''$ by Lemma 2.9. Let m be an ideal of $\mathfrak{F}$ such that $m\cdot\xi\subset j_{\mathfrak{F},M}(M)$; z an arbitrary element from $m\setminus p'$; ξ_z an element of M such that $j_{\mathfrak{F},M}(\xi_z)=z\cdot\xi$; $\mu\overset{\mathrm{def}}{=}\operatorname{Ann}\xi_z$. Let p be a maximal (with respect to preordering in $I_\ell^{\xi}R$) element of the set

$$\{\operatorname{Ann}\eta\mid\eta\in R\xi_z\setminus\mathfrak{F}(R\xi_z)\}=\{(\mu:\lambda)\mid\lambda\in R\setminus\mu_{\mathfrak{F}}\}$$

whose existence follows from the I_ℓ^{ξ} -Noetherianness of R; i.e. $p=(\mu:\lambda_0)$ for some $\lambda_0\in R\setminus\mu_{\mathfrak{F}}$. Clearly, $p\in\widehat{\operatorname{Ass}}_R^{\mathfrak{F}}(M)$ and $G_{\mathfrak{F}}p=(G_{\mathfrak{F}}\mu:j_{\mathfrak{F}}(\lambda_0))$. Further, as had been shown in 1),

$$G_{\mathfrak{F}}\mu=\operatorname{Ann}_{G_{\mathfrak{F}}R}(j_{\mathfrak{F}}(\xi_z))=\operatorname{Ann}_{G_{\mathfrak{F}}R}(z\cdot\xi)=(p':j_{\mathfrak{F}}(z)).$$

Therefore

$$G_{\mathfrak{F}} p = (G_{\mathfrak{F}}\mu : j_{\mathfrak{F}}(\lambda_0)) = ((p' : j_{\mathfrak{F}}(z)) : j_{\mathfrak{F}}(\lambda_0)) = (p' : j_{\mathfrak{F}}(\lambda_0 z)).$$

Further, let us notice that this and the identity $p' = G_{\mathfrak{F}} p''$ where $p'' = j_{\mathfrak{F}}^{-1} p'$ implies $p_{\mathfrak{F}} = (p'' : \lambda_0 z)$. □

Proposition 2. Let $\mathfrak{F}$ be a radical filter of left ideals of $I_\ell^{\vdash}$-Noetherian ring R and M an R-module. Then the following properties of submodule $\mathcal{N} \subset \mathcal{M}$ are equivalent

(1) $Ass(\mathcal{M}/\mathcal{N}) \subset Ass(\mathcal{M}) \setminus \mathfrak{F}$ and $Ass^{\mathfrak{F}}(\mathcal{N}) = \emptyset$;

(2) $Ass(\mathcal{M}/\mathcal{N}) \subset Ass(\mathcal{M}) \setminus \mathfrak{F}$ and $Ass^*(\mathcal{N}) \subset \mathfrak{F}$;

(3) $\mathcal{N} = \mathfrak{F}\mathcal{M}$.

Proof. (1)$\Leftrightarrow$(2) follows from the identities $Ass^{\mathfrak{F}}(\mathcal{N}) = Ass^*(\mathcal{N}) \cap Ass^{\mathfrak{F}}(\mathcal{M})$, $Ass^{\mathfrak{F}}(\mathcal{M}) \cap \mathfrak{F} = \emptyset$; these identities clearly hold for arbitrary rings. An R-module $\mathcal{N} = \mathfrak{F}\mathcal{M}$ satisfies (1) (see Example at the end of 9) for any R.

(1) $\Rightarrow$ (3). Now let R be a $I_\ell^{\vdash}$-Noetherian ring. First notice that $Ass(\mathcal{M}/\mathcal{N}) \subset Ass(\mathcal{M}) \setminus \mathfrak{F}$ implies $\mathfrak{F}\mathcal{M} \subset \mathcal{N}$.

In fact, if $\mathfrak{F}\mathcal{M} \not\subset \mathcal{N}$ then $\mathfrak{F}(\mathcal{M}/\mathcal{N}) \neq 0$. By Corollary 1 of Proposition 9.2

$$[Ass(\mathfrak{F}(\mathcal{M}/\mathcal{N})) \neq \emptyset] \Leftrightarrow [\mathfrak{F}(\mathcal{M}/\mathcal{N}) \neq 0]$$

Therefore if $\mathfrak{F}\mathcal{M} \not\subset \mathcal{N}$ then $Ass(\mathcal{M}/\mathcal{N}) \cap \mathfrak{F} \neq 0$ and $Ass(\mathcal{M}/\mathcal{N}) \not\subset Ass(\mathcal{M}) \setminus \mathfrak{F}$.

By Corollary 1 of Proposition 10.2 $[Ass^{\mathfrak{F}}(\mathcal{N}) = \emptyset] \Leftrightarrow [\mathcal{N} = \mathfrak{F}\mathcal{N}]$; and therefore

$$[Ass^{\mathfrak{F}}(\mathcal{N}) = \emptyset] \Leftrightarrow [\mathcal{N} \subset \mathfrak{F}\mathcal{M}].\ \square$$

This proposition is the answer to the question raised at the end of n.9.

12. Relation with support. The following simple fact is a direct generalization of Proposition 3.7 from 1, Ch. IV in [3].

Proposition 1. Let M be an R-module.

(i) If $p \in Spec_\ell R$ and $n \to p$ for some n from $Ass^*(M)$ (more generally, $Ann\,\xi \to p$ for some $\xi \in M \setminus \{0\}$) then $p \in Supp(M)$.

(ii) If R is a $I_\ell^{\succ}$-Noetherian ring then every ideal p from $Supp(M)$ "contains" an ideal $\mu \in Ass^*(M)$ -- $\mu \to p$. More exactly, there exists an ideal $\mu \in Ass^*(M)$ such that $(\hat{\mu} : x) \subset p$ for some $x \in \mathcal{P}(R)$; in particular, $\hat{\mu}$ belongs to $Supp(M)$.

Proof. By definition

$$[Ann\,\xi \to p] \Leftrightarrow [Ann\,\xi \notin \mathfrak{F}_p] \Leftrightarrow [\xi \notin \mathfrak{F}_p M].$$

Now let R be a $I_\ell^{\succ}$-Noetherian ring and $p \in Supp(M)$. The latter means that $M \neq \mathfrak{F}_p M$. By Corollary 1 of Proposition 10.2 $[M \neq \mathfrak{F}_p M] \Leftrightarrow [Ass^{\mathfrak{F}_p}(M) \neq \emptyset]$. Any element from $Ass^{\mathfrak{F}_p}(M)$ satisfies the conditions of heading (ii) of Proposition. □

For any subset $\mathfrak{m} \subset I_\ell R$ set $\hat{\mathfrak{m}} \overset{def}{=} \{\hat{n} \mid n \in \mathfrak{m}\}$.

Corollary 1. Let M be a module over an $I_\ell^{\succ}$-Noetherian ring R. Then

The minimal elements (with respect to $\to$) of $Supp(M)$ and $\widehat{Ass^*(M)}$ are the same up to isomorphism.

Corollary 2. Let R be a $I_\ell^{\succ}$-Noetherian ring. Then $\widehat{rad}_\ell(R) = \bigcap\{\hat{\mu} \mid \mu \in Ass^* R\} = \mathfrak{r}(R)$.

Proof. For any radical filter $\mathcal{F}$ different from $I_\ell R$ the left ideal R is not the $\mathcal{F}$-torsion, i.e. $R \neq \mathcal{F}R$.

In fact, $[R = \mathcal{F}R] \Leftrightarrow [(0:x) \in \mathcal{F}$ for any $x \in R] \Leftrightarrow [0 \in \mathcal{F}] \Leftrightarrow [\mathcal{F} = I_\ell R]$.

This implies that $\mathcal{S}upp(R) = \mathcal{S}pec_\ell R$ and it remains to make use of Proposition 1. □

13. $*$-associated ideals over $I_\ell^{\succ}$-Noetherian rings. In this section we will show that for the sets $\widehat{Ass^{\mathcal{F}}}(M) = \{\hat{\mu} \mid \mu \in Ass^{\mathcal{F}}(M)\}$ and $\widehat{Ass^{*}}(M) = \{\hat{\mu} \mid \mu \in Ass^{*}(M)\}$ there are analogues of the statements of n.9.

First of all notice that $Ass^{\mathcal{F}}(M) = \bigcup_{i\in I} Ass^{\mathcal{F}}(M_i)$ for any radical filter $\mathcal{F}$ if M is the union of a family of submodules $\{M_i \mid i \in I\}$. Therefore

$$\bigcup_{i\in I} \widehat{Ass^{\mathcal{F}}}(M_i) = \widehat{Ass^{\mathcal{F}}}(M) \qquad (1)$$

$$\bigcup_{i\in I} \widehat{Ass^{*}}(M_i) = \widehat{Ass^{*}}(M) \qquad (2)$$

Proposition 1. *Let* $p \in \mathcal{S}pec_\ell^{\mathcal{F}} R$ *and M a submodule of the module R/p. Then*

1) $[M = \mathcal{F}M] \Leftrightarrow [Ass^{\mathcal{F}}(M) = \varnothing$

2) *if* $M \neq \mathcal{F}M$ *then* $Ass^{\mathcal{F}}(M)$ *consists of ideals isomorphic to p.*

Accordingly, $\widehat{Ass^{\mathcal{F}}}(M)$ *is either empty or consists of ideals* $p' \in \widehat{\mathcal{S}pec}_\ell R$ *isomorphic to* $\hat{p}$.

Proof. If $M = \mathcal{F}M$ then $Ass^{\mathcal{F}}(M) = \varnothing$. If $M \neq \mathcal{F}M$ then for any $\xi \in M \setminus \mathcal{F}M$ the annihilator of ξ equals $(p : x_\xi)$ where x_ξ is a preimage of ξ in R, and $(p : x_\xi) \notin \mathcal{F}$. Since by hypothesis

$p_{\mathcal{F}} = \hat{p} \overset{def}{=} \{\lambda \in R \mid (p:\lambda) \not\simeq p\}$, this means that $(p : x_{\xi}) \simeq p$. $\square$

Proposition 2. Let R be a $I_\ell^{\leftarrow}$-Noetherian ring, $\mathcal{F}$ a radical filter of a left ideals of R and M an R-module; N a submodule of $\mathcal{F}M$. Then $\widehat{Ass^{\mathcal{F}}}(M/N) = \widehat{Ass^{\mathcal{F}}}(M)$.

Proof. If $M = \mathcal{F}M$ then $M/N = \mathcal{F}(M/N)$ and $Ass^{\mathcal{F}}(M/N) = \emptyset = Ass^{\mathcal{F}}(M)$. Now suppose that $M \neq \mathcal{F}M$. Let $\xi \in M \setminus \mathcal{F}M$, $p = Ann\,\xi$, $\bar{\xi}$ be the image of ξ in M/N, $p' = Ann\,\bar{\xi} = \{\lambda \in R \mid \lambda\xi \in N\}$. First notice that $p_{\mathcal{F}} = p'_{\mathcal{F}}$.

In fact, since $N \subset \mathcal{F}M$ and $\mathcal{F}$ is radical, then

$$p'_{\mathcal{F}} = \{\lambda \in R \mid \lambda\bar{\xi} \in \mathcal{F}(M/N) = \mathcal{F}M/N\} = \{\lambda \in R \mid \lambda\xi \in \mathcal{F}M\} = p_{\mathcal{F}}.$$

Let $p \in Spec_\ell^{\mathcal{F}} R$ (i.e. $p \in \widehat{Ass}^{\mathcal{F}}(M)$). Since R is $I_\ell^{\leftarrow}$-Noetherian then there exists $\lambda_0 \in R$ such that $(p' : \lambda_0)$ also belongs to $Spec_\ell^{\mathcal{F}} R$ and therefore to $\widehat{Ass}^{\mathcal{F}}(M/N)$. We have

$$(\widehat{p' : \lambda_0}) = (p' : \lambda_0)_{\mathcal{F}} = (p'_{\mathcal{F}} : \lambda_0) = (p_{\mathcal{F}} : \lambda_0) = (\hat{p} : \lambda_0).$$

Since $p_{\mathcal{F}} = \hat{p} \in \widehat{Spec}_\ell R$ and $\lambda_0 \in R \setminus p'_{\mathcal{F}} = R \setminus \hat{p}$, then $\hat{p} \simeq (\hat{p} : \lambda_0)$.

Suppose that $p' = Ann\,\bar{\xi} \in Spec_\ell^{\mathcal{F}} R$ and z an element of R such that $(p : z) \in Spec_\ell^{\mathcal{F}} R$. Then

$$(\widehat{p : z}) = (p : z)_{\mathcal{F}} = (p_{\mathcal{F}} : z) = (p'_{\mathcal{F}} : z) = (\hat{p}' : z) \simeq \hat{p}'. \quad \square$$

Proposition 3. Let M be a module over an $I_\ell^{\leftarrow}$-Noetherian ring R, N a submodule of M. For any radical filter $\mathcal{F}$ of left ideals of R

$$\widehat{Ass^{\mathcal{F}}}(N) \subset \widehat{Ass^{\mathcal{F}}}(M) \subset \widehat{Ass^{\mathcal{F}}}(N) \cup \widehat{Ass^{\mathcal{F}}}(M/N).$$

Proof. The statement is trivial when $M = \mathcal{F}M$ since in this case $Ass^{\mathcal{F}}(N) = Ass^{\mathcal{F}}(M) = Ass^{\mathcal{F}}(M/N) = \emptyset$.

Let $M \neq \mathcal{F}M$, $\xi \in M$, $p = \mathrm{Ann}\,\xi \in \mathrm{Spec}_{\ell}^{\mathcal{F}} R$.
Denote $E = N \cap R\xi$ and $\bar{\xi}$ the image of ξ in M/N. Therefore the submodule $R\bar{\xi}$ of M/N is isomorphic to $R\xi/E$.

There are the following possibilities:

1) $E = \mathcal{F}E$. By Proposition 2 in this case $\widehat{\mathrm{Ass}^{\mathcal{F}}(R\bar{\xi})} = \widehat{\mathrm{Ass}^{\mathcal{F}}(R\xi/E)} = \widehat{\mathrm{Ass}^{\mathcal{F}}(R\xi)}$. Since $R\xi \simeq R/p$ then $\widehat{\mathrm{Ass}^{\mathcal{F}}(R\xi)}$ consists of ideals isomorphic to p.

2) $E \neq \mathcal{F}E$. Then $\mathrm{Ass}^{\mathcal{F}}(E)$ consists of ideals isomorphic to p and all the ideals of $\widehat{\mathrm{Ass}^{\mathcal{F}}(E)}$ are isomorphic to p as states Proposition 1. Therefore $\hat{p} \in \widehat{\mathrm{Ass}^{\mathcal{F}}(N)}$.

Thus we have proved that $\widehat{\mathrm{Ass}^{\mathcal{F}}(M)} \subset \widehat{\mathrm{Ass}^{\mathcal{F}}(N)} \cup \widehat{\mathrm{Ass}^{\mathcal{F}}(M/N)}$.
The inclusion $\widehat{\mathrm{Ass}^{\mathcal{F}}(N)} \subset \widehat{\mathrm{Ass}^{\mathcal{F}}(M)}$ is obvious. □

Corollary 1. Let M be a module over an $I_\ell^{\succ}$-Noetherian ring R and N a submodule of M. Then

$$\widehat{\mathrm{Ass}^{*}(N)} \subset \widehat{\mathrm{Ass}^{*}(M)} \subset \widehat{\mathrm{Ass}^{*}(N)} \cup \widehat{\mathrm{Ass}^{*}(M/N)}.$$

Proof. $\mathrm{Ass}^{*}(M) = \bigcup\{\mathrm{Ass}^{\mathcal{F}_p}(M) \mid p \in \widehat{\mathrm{Spec}}_{\ell} R\}$
for any A-module M and therefore $\widehat{\mathrm{Ass}^{*}(M)} = \bigcup\{\widehat{\mathrm{Ass}^{\mathcal{F}_p} M} \mid p \in \widehat{\mathrm{Spec}}_{\ell} R\}$. □

Corollary 2. Let R-module M be the direct sum of the modules $\{M_i \mid i \in I\}$. If R is an $I_\ell^{\succ}$-Noetherian ring then for any radical filter $\mathcal{F}$

$$\widehat{\mathrm{Ass}^{\mathcal{F}}(M)} = \bigcup_{i \in I} \widehat{\mathrm{Ass}^{\mathcal{F}}(M_i)}$$

and $\widehat{\mathrm{Ass}^{*}(M)} = \bigcup_{i \in I} \widehat{\mathrm{Ass}^{*}(M_i)}$.

Proof repeats word for word the arguments of the proof of Corollary 1 of Proposition 9.3. □

Corollary 3. Let R be a $I_\ell^{\succ}$-Noetherian ring, $\{Q_i \mid i \in I\}$ a finite family of submodules of a R-module M such that $\bigcap\{Q_i \mid i \in I\} = 0$. Then

$$\widehat{\mathrm{Ass}^{*}(M)} \subset \bigcup_{i \in I} \widehat{\mathrm{Ass}^{*}(M/Q_i)}$$

and for any radical filter $\mathcal{F}$ of left ideals of R

$$\widehat{Ass}^{\mathcal{F}}(M) \subset \bigcup_{i \in I} \widehat{Ass}^{\mathcal{F}}(M/Q_i).$$

Proposition 4. Let M be a module over an $I_\ell^{\curlyvee}$-Noetherian ring R, $\mathcal{F}$ a radical filter of left ideals of R and Φ a subset in $\sim\widehat{Ass}^{\mathcal{F}}(M)$. Then there exists a submodule N of M such that

$$\sim\widehat{Ass}^{\mathcal{F}}(M) \setminus \Phi = \sim\widehat{Ass}^{\mathcal{F}}(N), \quad \sim\widehat{Ass}^{\mathcal{F}}(M/N) = \Phi.$$

Proof is almost identical to that of Proposition 9.4. □

Remark. It is not difficult to see that $Spec_\ell^{\{R\}} R = \widehat{Spec}_\ell R$ and therefore $\widehat{Ass}^{\{R\}}(M) = Ass(M) \cap \widehat{Spec}_\ell R$ for any R-module M. Therefore Proposition 1 may be considered as a generalization of Proposition 9.1 and Propositions 3 and 4 as "partial " (because of the requirement of $I_\ell^{\curlyvee}$-Noetherianness of R) generalizations of Propositions 9.3 and 9.4 respectively. The same applies to their corollaries. □

14. Noetherian modules over an $I_\ell^{\curlyvee}$-Noetherian ring.

Proposition 1. Let R be a $I_\ell^{\curlyvee}$-Noetherian ring M a Noetherian R-module. There exists a composition series $(M_i)_{0 \le i \le r}$ of M each quotient M_{i+1}/M_i being isomorphic to R/p_i where $p_i \in \widehat{Spec}_\ell R$ for $0 \le i \le r-1$.

Proof. Let $\mathcal{L}$ be the set of submodules of M with composition series with the above property. Since $\mathcal{L} \ne \emptyset$ (obviously $0 \in \mathcal{L}$) and M is Noetherian, then $\mathcal{L}$ possesses a maximal element N. If $N \ne M$ then $M/N \ne 0$; therefore $Ass(M/N) \ne \emptyset$ (Corollary 1, Proposition 9.2). Therefore M/N contains a submodule N'/M isomoprhic to an R-module of the form R/p where $p \in \widehat{Spec}_\ell R$; then by definition $N' \in \mathcal{L}$ contradicting to the maximality of N. Thus $N = M$. □

Proposition 2. Let M be a Noetherian module over an I_ℓ^ξ-Noetherian ring R and $(M_i)_{0\le i\le r}$ a composition series of M such that M_{i+1}/M_i is isomorphic to R/p_i where $p_i \in \widehat{\mathrm{Spec}}_\ell R$ for $0\le i\le r-1$. Then

$$\sim\mathrm{Ass}(M) \subset \sim\widehat{\mathrm{Ass}^*}(M) \subset \sim\{p_0,\dots,p_{r-1}\} \subset \sim\mathrm{Supp}(M) \quad (1)$$

Besides the minimal elements of $\sim\widehat{\mathrm{Ass}^*}(M)$, $\sim\{p_0,\dots,p_{r-1}\}$ and $\sim\mathrm{Supp}(M)$ are the same.

Proof. The inclusion $\sim\mathrm{Ass}(M) \subset \sim\widehat{\mathrm{Ass}^*}(M)$ follows from the evident inclusion $\hat{\mathrm{Ass}}(M) \subset \widehat{\mathrm{Ass}^*}(M)$.

Since $p_i \in \mathrm{Supp}(R/p_i) = \mathrm{Supp}(M_{i+1}/M_i)$ for $0\le i\le r-1$, then $p_i \in \mathrm{Supp}(M_{i+1}) \subset \mathrm{Supp}(M)$ by Proposition 8. Therefore $\{p_0,\dots,p_{r-1}\} \subset \mathrm{Supp}(M)$.

The inclusion $\sim\widehat{\mathrm{Ass}^*}(M) \subset \sim\{p_0,\dots,p_{r-1}\}$ follows directly from Corollary 1 of Proposition 13.3 and the identity $\mathrm{Ass}(R/p) = \mathrm{Ass}^*(R/p)$ for any $p \in \mathrm{Spec}_\ell R$.

Corollary 1 of Proposition 12.1 shows that $\sim\widehat{\mathrm{Ass}^*}(M)$ and $\sim\mathrm{Supp}(M)$ have the same minimal elements. The inclusions (1) show that these elements are also minimal for $\sim\{p_0,\dots,p_{r-1}\}$. □

Corollary 1. Let M be a Noetherian module over I_ℓ^ξ-Noetherian ring R. Then $\sim\mathrm{Ass}(M)$ and $\sim\widehat{\mathrm{Ass}^*}(M)$ are finite.

Corollary 2. Let R be a left Noetherian and I_ℓ^ξ-Noetherian $(m_i)_{0\le i\le r}$ the composition series of the left module R such that for any $0\le i\le r-1$ the quotient m_{i+1}/m_i is isomorphic to $R/p^{(i)}$ where $p^{(i)} \in \widehat{\mathrm{Spec}}_\ell R$. Then

$$(\mathfrak{B}(R)=)\,\widehat{\mathrm{rad}}_\ell(R) = \bigcap_{0\le i\le r-1} p_s^{(i)} = \Big(\bigcap_{1\le i\le r-1} p^{(i)} : R\Big). \quad (2)$$

Formula (2) follows directly from Proposition 2 and Corollary 2 of Proposition 12.1. □

15. Primary submodules. Let M be an R-module and $p \in \widehat{Spec}_\ell R$. We will say that a submodule N of M is p-primary if $Ass(M/N)$ consists of ideals isomorphic to p. In particular a left ideal m of R will be called p-primary if $\tilde{\ } Ass(R/m) = \tilde{\ }\{p\}$. A submodule N of M (a left ideal m of R) will be called primary if it is p-primary for some $p \in \widehat{Spec}_\ell R$.

We will be interested in these notions when R is an $I_\ell^{\succ}$-Noetherian ring.

Proposition 1. Let R be a $I_\ell^{\succ}$-Noetherian ring p an ideal from $\widehat{Spec}_\ell R$ and N a submodule of an R-module M. The following conditions are equivalent:

a) N is a p-primary submodule in M;

b) for any $\xi \in M/N \setminus \{0\}$ there exists a $x \in R \setminus Ann\,\xi$ such that $Ann\,x\xi \simeq p$ and $Ann\,z\xi \to Ann\,x\xi$ for all $z \in R \setminus Ann\,\xi$. In addition $Ann\,x\xi \in \widehat{Spec}_\ell R$.

Proof. a)$\Rightarrow$b) Since R is an $I_\ell^{\succ}$-Noetherian ring then for any $\xi' \in M/N \setminus \{0\}$ there exists an arrow from $Ann\,\xi'$ into an ideal from $Ass(M/N)$. Since $\tilde{\ }Ass(M/N) = \tilde{\ }\{p\}$ by hypothesis, then $Ann\,\xi' \to p$ for all $\xi' \in M/N \setminus \{0\}$.

Fix $\xi \in M/N \setminus \{0\}$. Since $Ass(R\xi) \neq \emptyset$ by Corollary 1 of Proposition 9.2, then $\tilde{\ }Ass(R\xi) = \tilde{\ }\{p\}$ and therefore for some $x \in R$ $Ann\,x\xi \simeq p$.

Let $y \in R$ and $(Ann\,x\xi : y) \not\to Ann\,x\xi$. Since $(Ann\,x\xi : y) = Ann\,yx\xi$, this means that $yx\xi = 0$; i.e. $y \in Ann\,x\xi$.

b)$\Rightarrow$a) is obvious. □

Corollary. A left ideal m of an $I_\ell^{\succ}$-Noetherian ring R is primary if and only if $(m : x) \in \widehat{Spec}_\ell R$ for some $x \in R$ and $(m : y) \to (m : x)$ for all $y \in R \setminus m$.

Examples. 1) Clearly any ideal from $\widehat{Spec}_\ell R$ is primary.

2) Let $\nu \in I_\ell R$ and there exists a unique ideal μ from $Spec_\ell R$ such that $\nu \subset \mu$ (this implies that μ is a maximal left ideal). If M is an R-module such that $M \neq \nu M$ then the submodule νM is μ-primary in M.

In fact any p from $\widehat{Ass}_R(M/\nu M)$ contains ν and therefore $p = \mu$. $\square$

Proposition 2. Let M be a module over an I_ℓ^γ-Noetherian ring R, $p \in \widehat{Spec}_\ell R$ and $\{Q_i \mid i \in I\}$ a finite family of p-primary submodules of M. Then $\cap\{Q_i \mid i \in I\}$ is also p-primary in M.

Proof. $M/\cap\{Q_i \mid i \in I\}$ is isomorphic to a non-zero submodule of $\bigoplus_{i \in I} M/Q_i$. Corollary 1 of Proposition 9.3 implies

$$\widetilde{\ }Ass(\oplus_{i \in I} M/Q_i) = \bigcup_{i \in I} \widetilde{\ }Ass(M/Q_i) = \widetilde{\ }\{p\}.$$

And therefore $\widetilde{\ }Ass(M/\cap\{Q_i \mid i \in I\}) = \widetilde{\ }\{p\}$. $\square$

16. Existence of a primary decomposition. Let R be a I_ℓ^γ-Noetherian ring, M an R-module, N a submodule of M. A primary decomposition of N in M is a finite set $\{Q_i \mid i \in I\}$ of primary submodules of M such that $N = \cap\{Q_i \mid i \in I\}$.

If $\{Q_i \mid i \in I\}$ is a primary decomposition of N in M then the canonical map $M/N \to \bigoplus_{i \in I} M/Q_i$ is injective. Conversely, let N be a submodule of M, and f a monomorphism of M/N into a finite direct sum $P = \bigoplus_{i \in I} P_i$ where every set $\widetilde{\ }Ass(P_i)$ consists of one element p_i. Let f_i be the composition of f with the projection $P \twoheadrightarrow P_i$, Q_i/N the kernel of f_i; $J \overset{def}{=} \{i \in I \mid Q_i \neq M\}$. Then $\{Q_i \mid i \in J\}$ is a primary decomposition of N in M. Besides $\widetilde{\ }Ass(M/N) = \{p_j \mid j \in J\}$ by Corollary 2 of Proposition 9.3.

Proposition 1. Let M be a Noetherian module over an $I_\ell^{\succ}$-Noetherian ring, N a submodule in M. Then there exists a primary decomposition of N in M of the form

$$\{Q(\mathfrak{p}) \mid \mathfrak{p} \in {}^\sim\mathrm{Ass}(M/N)\}$$

where for every $\mathfrak{p} = \tilde{p} \in {}^\sim\mathrm{Ass}(M/N)$ the submodule $Q(\mathfrak{p})$ is p-primary in M.

Proof. For convenience assume that $N=0$. By Corollary 1 of Proposition 14.2 ${}^\sim\mathrm{Ass}(N)$ is finite. By Proposition 9.4 for any $\mathfrak{p} \in {}^\sim\mathrm{Ass}(M)$ there exists a submodule $Q(\mathfrak{p})$ of M such that ${}^\sim\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$ and ${}^\sim\mathrm{Ass}(Q(\mathfrak{p})) = {}^\sim\mathrm{Ass}(M) \setminus \{\mathfrak{p}\}$. Set $P = \bigcap\{Q(\mathfrak{p}) \mid \mathfrak{p} \in {}^\sim\mathrm{Ass}(M)\}$. For any $\mathfrak{p}$ from ${}^\sim\mathrm{Ass}(M)$ we have ${}^\sim\mathrm{Ass}(P) \subset {}^\sim\mathrm{Ass}(Q(\mathfrak{p}))$. And therefore $\mathrm{Ass}(P) = \emptyset$ implying $P = 0$ by Corollary 1 of Proposition 9.2. $\square$

17. Properties of uniqueness in the primary decomposition. Let M be a module over an $I_\ell^{\succ}$-Noetherian ring and N a submodule in M. The primary decomposition $\{Q_i \mid i \in I\}$ of N in M will be called reduced if the following conditions hold:

a) there is no $i \in I$ such that $\bigcap\{Q_j \mid j \in I \setminus \{i\}\} \subset Q_i$,

b) if ${}^\sim\mathrm{Ass}(M/Q_i) = {}^\sim\{p_i\}$ then the ideals p_i, $i \in I$, are pairwise non-isomorphic.

From any primary decomposition $\{Q_i \mid i \in I\}$ of N in M one can get a reduced primary decomposition as follows: let J be a minimal element in the set of the subsets $I' \subset I$ such that $N = \bigcap_{i \in I'} Q_i$. Clearly $\{Q_i \mid i \in J\}$ satisfies (a). Let ${}^\sim\mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$, $\Phi = \{\mathfrak{p}_i \mid i \in J\}$. For any $\mathfrak{p}$ from Φ set $J_{\mathfrak{p}} = \{i \in J \mid \mathfrak{p} = \mathfrak{p}_i\}$ and $Q(\mathfrak{p}) \overset{\mathrm{def}}{=} \bigcap\{Q_i \mid i \in J_{\mathfrak{p}}\}$. Proposition 15.2 implies that $Q(\mathfrak{p})$ is a primary submodule in M. Besides, $N = \bigcap\{Q(\mathfrak{p}) \mid \mathfrak{p} \in \Phi\}$ and therefore $\{Q(\mathfrak{p}) \mid \mathfrak{p} \in \Phi\}$ is a reduced primary decomposition of N in M.

Proposition 1. Let M be a module over an I_l- netherian ring R, N a submodule in M and $\{Q_i \mid i \in I\}$ a primary decomposition of N in M and let $^{\sim}\mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$ for any $i \in I$. This decomposition is reduced if and only if all $\mathfrak{p}_i$ are pairwise different and belong to $^{\sim}\mathrm{Ass}(M/N)$. In this case $^{\sim}\mathrm{Ass}(M/N) = \{\mathfrak{p}_i \mid i \in I\}$ and $^{\sim}\mathrm{Ass}(Q_i/N) = \{\mathfrak{p}_j \mid j \in I \setminus \{i\}\}$ for every $i \in I$.

Proof. If the formulated condition holds then it is impossible that $N = \{Q_j \mid j \in I \setminus \{i\}\}$ since this would imply that $\mathrm{Ass}(M/N) = \{\mathfrak{p}_j \mid j \in I \setminus \{i\}\}$ by Corollary 2 of Proposition 9.3, contradicting to the hypothesis; and therefore the primary decomposition $\{Q_i \mid i \in I\}$ is reduced.

Conversely, by the same Corollary 2 of Proposition 9.3 we always have $^{\sim}\mathrm{Ass}(M/N) \subset \{\mathfrak{p}_i \mid i \in I\}$. On the other hand, for any $i \in I$ set $P_i = \bigcap\{Q_j \mid j \in I \setminus \{i\}\}$. Then $N = P_i \cap Q_i$ and $P_i \neq N$ if $\{Q_i \mid i \in I\}$ is a reduced decomposition. Therefore, P_i/N is isomorphic to the submodule $P_i + Q_i/Q_i$ of P_i/N. Hence $^{\sim}\mathrm{Ass}(P_i/N) = \{\mathfrak{p}_i\}$ by Corollary 1 of Proposition 9.2 and Proposition 9.3, and $\mathfrak{p}_i \in {}^{\sim}\mathrm{Ass}(M/N)$ since $P_i/N \subset M/N$.

Since $N = \bigcap\{Q_i \cap Q_j \mid j \in I \setminus \{i\}\}$, then $^{\sim}\mathrm{Ass}(Q_i/N) \subset \bigcup\{^{\sim}\mathrm{Ass}(Q_i/Q_i \cap Q_j) \mid j \in I \setminus \{i\}\}$ by Corollary 2 of Proposition 9.3. But $Q_i/Q_i \cap Q_j$ is isomorphic to the submodule $Q_i + Q_j/Q_j$ of M/Q_j. Therefore $^{\sim}\mathrm{Ass}(Q_i/Q_i \cap Q_j) \subset \{\mathfrak{p}_j\}$ and $^{\sim}\mathrm{Ass}(Q_i/N) \subset \{\mathfrak{p}_j \mid j \in I \setminus \{i\}\}$. Thus (1) and Proposition 9.3 imply (2). □

Corollary 1. The primary decomposition determined in Proposition 16.1 is reduced.

Corollary 2. Let R be a $\mathcal{I}_\ell^{\succ}$-Noetherian ring, M an R-module, N a submodule of M and $\{Q_i \mid i \in I\}$ the primary decomposition of N in M. Then $Card(I) \geqslant Card(\tilde{\ }Ass(M/N))$. The decomposition $\{Q_i \mid i \in I\}$ is reduced primary decomposition if and only if $Card(I) = Card(\tilde{\ }Ass(M/N))$.

Proof. The constructions preceding Proposition 1 imply that there exists a reduced primary decomposition $\{Q'_j \mid j \in J\}$ of N in M such that $Card(J) \leqslant Card(I)$. Therefore the first statement follows from the second one and the second one is a corollary of Proposition 1. □

Recall that for any (radical) filtre $\mathcal{F}$ of left ideals of R and arbitrary submodule N of an R-module M an $\mathcal{F}$-saturation of N in M is the submodule $N_{\mathcal{F}} = \{\xi \in M \mid m \cdot \xi \subset N$ for some $m \in \mathcal{F}\}$.

Proposition 2. Let R be an $\mathcal{I}_\ell^{\succ}$-Noetherian ring M an R-module, N a submodule in M, $\{Q_i \mid i \in I\}$ a primary decomposition of N in M; $\tilde{\ }Ass(M/Q_i) = \tilde{\ }\{p_i\}$. Then for any $i \in I$ the submodule Q_i contains an $\mathcal{F}_{p_i}$-saturation of N.

Proof. Actually we are to show that any p-primary submodule of M coincides with its $\mathcal{F}_p$-saturation. For this it suffices to verify that

$$[\tilde{\ }Ass(M) = \tilde{\ }\{p\}] \Leftrightarrow [\mathcal{F}_p M = 0].$$

If $\mathcal{F}_p M \neq 0$ then $Ass(\mathcal{F}_p M) = \emptyset$ by Corollary 1 of Proposition 9.2 and therefore $\tilde{\ }Ass(\mathcal{F}_p M) = \tilde{\ }\{p\}$. But this is impossible since $Ass(\mathcal{F}_p M) \subset \mathcal{F}_p$ and $p \notin \mathcal{F}_p$. Therefore $\mathcal{F}_p M = 0$. □

18. Associated ideals and essential embeddings. Let M be an R-module; $p \in \tilde{\ }Ass(M)$, $M(p)$ a maximal among the submodules

N' of M such that $\sim\!Ass(N')=\{\mathfrak{p}\}$. Proposition 9.4 guarantees the existence of $M(\mathfrak{p})$ and the equality $\sim\!Ass(M/M(\mathfrak{p}))=\sim\!Ass(M)\setminus\{\mathfrak{p}\}$ (see the proof).

Fix a set of submodules $\{M(\mathfrak{p})\mid \mathfrak{p}\in\sim\!Ass(M)\}$.

Proposition 1. *Let R be an $I_\ell^\succ$-noetherian ring. Then the set* $\{M(\mathfrak{p})\mid\mathfrak{p}\in\sim\!Ass(M)\}$ *possesses the following properties:*

1) $M(\mathfrak{p})\cap M(\mathfrak{p}')=\{0\}$ *if* $\mathfrak{p}\neq\mathfrak{p}'$ *in particular* $\sum_{\mathfrak{p}\in\sim Ass(M)} M(\mathfrak{p})$ *is a direct sum;*

2) $\sum_{\mathfrak{p}\in\sim Ass(M)} M(\mathfrak{p})$ *is an essential submodule of M.*

Recall that a submodule L of M is essential if $L\cap N\neq 0$ for any non-zero submodule N in M; they say that a module morphism $f: M'\longrightarrow M$ is essential if $\mathrm{im}(f)$ is an essential submodule in M.

Proof. 1) This follows from $[M'=0]\Longleftrightarrow[Ass(M')=\emptyset]$ by Corollary 1 of Proposition 9.2.

2) Let E be a non-zero submodule in M. Then since R is $I_\ell^\succ$-Noetherian there exists $\xi\in E$ such that $Ann\,\xi=$ $=p\in \widehat{Spec}_\ell R$. Let p belong to $\mathfrak{p}$. If $M(\mathfrak{p})\cap R\xi=0$ then by Corollary 1 of Proposition 9.3 $\sim\!Ass(M(\mathfrak{p})+R\xi)=$ $=\sim\!Ass(M(\mathfrak{p}))\cup\sim\!Ass(R\xi)=\{\mathfrak{p}\}$. Since $M(\mathfrak{p})\subsetneq M(\mathfrak{p})+R\xi$, we've got a contradiction with the maximality of $M(\mathfrak{p})$. Therefore $M(\mathfrak{p})\cap R\xi\neq 0$.

And aside, $R\xi\cap M(\mathfrak{p}')=0$ whenever $\mathfrak{p}\neq\mathfrak{p}'$. $\square$

Recall that $\mathcal{F}_{\mathfrak{p}}$ denotes the set $\mathcal{F}_p$ where p is a representative of a class of isomorphic ideals $\mathfrak{p}$ for any $\mathfrak{p}\in\sim\!Spec_\ell R$ (Obviously this notation holds for arbitrary classes of isomorphic ideals.)

For any $\mathfrak{p}\in\sim\!Ass(M)$ denote by $\mathcal{F}(\mathfrak{p})=\mathcal{F}_M(\mathfrak{p})$ the radical filter $\bigcap\{\mathcal{F}_{p'} \mid p' \in {}^\sim\mathrm{Ass}(M) \setminus \{p\}\}$.

Proposition 2. Let R be a I_ℓ^t- noetherian ring.

1) The following properties of $p \in {}^\sim\mathrm{Ass}(M)$ are equivalent:

a) $\mathcal{F}(p)M \neq 0$;

b) p is a maximal ideal in ${}^\sim\mathrm{Ass}(M)$;

c) ${}^\sim\mathrm{Ass}(\mathcal{F}(p)M) = \{p\}$.

2) For any $p \in {}^\sim\mathrm{Ass}(M)$ the submodule $M(p)$ coincides with its $\mathcal{F}(p)$-saturation;
in particular, $\mathcal{F}(p)M \subset M(p)$. If $\mathcal{F}(p)M \neq 0$ then it is an essential submodule in $M(p)$.

Proof. 1) (a)$\Rightarrow$(b),(c). If $\mathcal{F}(p)M \neq 0$ then $\mathrm{Ass}(\mathcal{F}(p)M) \neq \emptyset$ by Corollary 1 of Proposition 9.2. Since $\mathrm{Ass}(\mathcal{F}(p)M) \subset \mathcal{F}(p) \overset{def}{=} \bigcap\{\mathcal{F}_{p'} \mid p' \in {}^\sim\mathrm{Ass}(M) \setminus \{p\}\}$, then for any $q \in {}^\sim\mathrm{Ass}(\mathcal{F}(p)M)$ there exists not an arrow into any $p' \in {}^\sim\mathrm{Ass}(M) \setminus \{p\}$. This implies that $q = p$ and p is a maximal element in ${}^\sim\mathrm{Ass}(M)$

(b)$\Rightarrow$(a). Let p be a maximal element in ${}^\sim\mathrm{Ass}(M)$ and $\xi \in M$ such that $\mathrm{Ann}\,\xi \in p$. Then clearly $\xi \in \mathcal{F}(p)M$.

(c)$\Rightarrow$(a) is obvious.

2) Let ξ be an element of the $\mathcal{F}(p)$-saturation $M(p)_{\mathcal{F}(p)}$ of the submodule $M(p)$ in M such that $\mathrm{Ann}\,\xi = p \in \widehat{\mathrm{Spec}}_\ell R$. By definition of $\mathcal{F}(p)$-saturation there exists an ideal $m \in \mathcal{F}(p)$ such that $m\xi \subset M(p)$. There are two possibilities: either $m\xi = 0$ or $m\xi \neq 0$. In the first case $\mathrm{Ann}\,\xi \in \mathcal{F}(p)$ since $m \subset \mathrm{Ann}\,\xi$ and therefore $\xi \in \mathcal{F}(p)M$. By 1) this means that $\mathrm{Ann}\,\xi \in p$ and besides p is a maximal element in ${}^\sim\mathrm{Ass}(M)$. In the second case there

exists $x \in m$ such that $x\xi \neq 0$. Since $Ann\,\xi \in \widehat{Spec}_\ell R$ and $x \notin Ann\,\xi$, then the ideals $Ann\,\xi$ and $Ann\,x\xi = (Ann\,\xi : x)$ are isomorphic. Since $x\xi \in \mathcal{F}(p)M$ this implies that $Ann\,x\xi$ and therefore $Ann\,\xi$ is of class p.

Thus we have shown that $\sim Ass(M(p)_{\mathcal{F}(p)}) = \{p\}$. Since $M(p) \subset M(p)_{\mathcal{F}(p)}$, then the maximality of $M(p)$ implies the desired equality $M(p) = M(p)_{\mathcal{F}(p)}$.

Clearly, $\mathcal{F}(p)M \subset M(p)$ since by definition $\mathcal{F}(p)M = 0_{\mathcal{F}(p)}$ and $0_{\mathcal{F}(p)} \subset M(p)_{\mathcal{F}(p)}$.

It remains to verify the fact that $\mathcal{F}(p)M$ is essential in $M(p)$ if $\mathcal{F}(p)M \neq 0$. By 1) p is a maximal element in $\sim Ass(M)$. Therefore, if E is a non-zero submodule of $M(p)$ and ξ an element of E such that $Ann\,\xi \in \widehat{Spec}_\ell R$ then $Ann\,\xi \in \mathcal{F}(p)$ thanks to the inclusion $Ass(E) \subset p$; i.e. $\xi \in \mathcal{F}(p)M$. □

Corollary. *Let Φ be the subset of all the maximal elements of $\sim Ass(M)$ and R be $I_\ell^{\succ}$-Noetherian. Then $\sum_{p\in\Phi} \mathcal{F}(p)M + \sum_{p\in \sim Ass(M)\setminus\Phi} M(p)$ is an essential submodule of M.*

Proof. Let $\{N_i \mid i \in I\}$ be a finite family of modules over an (arbitrary) ring R; L_i an essential submodule of N_i for every $i \in I$. Then $\bigoplus_{i\in I} L_i$ is an essential submodule in $\bigoplus_{i\in I} N_i$.

In fact, it is easy to see that all the submodules $L^{(i)} = \bigoplus_{j\in I} L'_j$, where $L'_j = N_j$ for $j \in I \setminus \{i\}$ and $L'_i = L_i$, are essential. Since $\bigoplus_{i\in I} L_i = \bigcap_{i\in I} L^{(i)}$ then we deduce that $\bigoplus_{i\in I} L_i$ is essential from the following well-known fact which is subject to a straightforward verification: the intersection of a finite family of essential submodules is essential.

Since $M(\mathfrak{p})\cap M(\mathfrak{p}')=0$ for different $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathcal{F}(\mathfrak{p})M\subset$ $M(\mathfrak{p})$ by Proposition 2, then $\sum_{\mathfrak{p}\in\Phi}\mathcal{F}(\mathfrak{p})M+\sum_{\mathfrak{p}'\in \operatorname{Ass}(M)\setminus\Phi}M(\mathfrak{p}')$ is the direct sum. For every $\mathfrak{p}\in\Phi$ the submodule $\mathcal{F}(\mathfrak{p})M$ is essential in $M(\mathfrak{p})$ as is stated in Proposition 2.

Thus $\sum_{p\in\Phi}\mathcal{F}(p)M+\sum_{p'\in\tilde{\ }Ass(M)\setminus\Phi} M(p')$ is an essential submodule in $\sum_{p} M(p)$ and $\sum_{p} M(p)$ is an essential submodule in M. It remains to make use of the transitivity of the relation "X is an essential submodule in Y". □

Now turn again to the primary decompositions. Since $\sum_{p'\in\tilde{\ }Ass(M)\setminus\{p\}} M(p')$ is the direct sum for every $p\in\tilde{\ }Ass(M)$ we can (by Proposition 9.4 and Remark to it) find for every $p\in\tilde{\ }Ass(M)$ a submodule $Q(p)$ such that

(a) $\sum_{p'\in\tilde{\ }Ass(M)\setminus\{p\}} M(p')\subset Q(p)$,

(b) $Q(p)$ is maximal among the submodules Q' satisfying $\tilde{\ }Ass(Q')=\tilde{\ }Ass(M)\setminus\{p\}$.

By Proposition 9.4 $\tilde{\ }Ass(M/Q(p))=\{p\}$ and the family $\{Q(p)\mid p\in\tilde{\ }Ass(M)\}$ is the reduced primary decomposition of the zero submodule in M (Propositions 16.1 and 17.1). In particular since $\bigcap_{p\in\tilde{\ }Ass(M)} Q(p)=0$, then the canonical map $M\to\bigoplus_{p\in\tilde{\ }Ass(M)} M/Q(p)$ is injective.

Proposition 3. *Let R be an I_ℓ^{τ}-Noetherian. The monomorphism $v_M: M\to\bigoplus_{p\in\tilde{\ }Ass(M)} M/Q(p)$ is essential, i.e. its image is an essential submodule.*

Proof. (i) For every $p\in\tilde{\ }Ass(M)$ the canonical projection $M\twoheadrightarrow M/Q(p)$ induces an essential morphism $u_p: M(p)\to M/Q(p)$.

Since $Ass(M(p))\cap Ass(Q(p))=\emptyset$, then $M(p)\cap Q(p)=0$ and therefore u_p is injective.

Let $E/Q(p)$ be a submodule of $M/Q(p)$. If $p\in\tilde{\ }Ass(E)$ then repeating the arguments of the second step of the proof of Proposition 1 we see that $E\cap M(p)\neq 0$. If $p\notin\tilde{\ }Ass(E)$ then the maximality of $Q(p)$ and the inclusion $Q(p)\subset E$ imply $E=Q(p)$ and therefore $E/Q(p)=0$.

Since by hypothesis $M(p') \subset Q(p)$ for every p' different from p, then v_M induces an embedding $\bigoplus_p M(p) \xrightarrow{u} \bigoplus_p M/Q(p)$ determined by a diagonal matrix with the morphisms $u_p : M(p) \to M/Q(p)$ as diagonal entries. Since all the monomorphisms u_p are essential then so is their coproduct u (see proof of Corollary of Proposition 2).

This implies that v_M is essential since clearly [$f \circ g$ is an essential morphism] $\Longrightarrow$ [f is an essential morphism]. □

Remark 1. If p is a maximal element in $\sim\!Ass(M)$ or, equivalently, $\mathcal{F}(p)M \neq 0$ then the projection $M \twoheadrightarrow M/Q(p)$ induces an essential monomorphism $\mathcal{F}(p)M \to M/Q(p)$ since by Proposition 2 $\mathcal{F}(p)M$ is an essential submodule in $M(p)$. In particular if all the elements of $\sim\!Ass(M)$ are maximal then this and the proof of Proposition 3 implies that the restriction of v_M on an essential submodule $\bigoplus_{p \in \sim Ass(M)} \mathcal{F}(p)M$ is an essential monomorphism. □

Remark 2. If L is an essential submodule of M then for any submodule N $\subset$ M we have $Ass(N) = Ass(L \cap N)$; in particular $Ass(L) = Ass(M)$.

Therefore if Φ is a finite subset of the "reduced" left spectrum $\sim\!Spec_\ell R$ and $\{N(p) \mid p \in \Phi\}$ is a family of modules such that $\sim\!Ass(N(p)) = \{p\}$ for every $p \in \Phi$ then the existence of either of essential monomorphisms $M \rightarrowtail \bigoplus_{p \in \Phi} N(p)$ or $\bigoplus_{p \in \Phi} N(p) \to M$ implies $\sim\!Ass(M) = \Phi$.

Propositions 1 and 3 may be considered as an inversion of these simple statements. □

19. Modules over arbitrary rings and associated ideals. In the above statements on relation of Ass(M) with the structure

of M the requirement of I_t^t- noetherness of R is only imposed to guarantee the fulfilment of one of the following conditions:

(#) if M' is a nonzero submodule of M then $Ass(M') \neq \emptyset$

(##) if L is a proper submodule of M then $Ass(M/L) \neq \emptyset$.

More exactly the modifications of the obtained here results are summarized in the following table.

Propositions	Properties of M that guarantee the validity of propositions for an arbitrary R
14.1, 14.2	M is Noetherian and satisfies (#)
15.1, 17.1,2	M/N satisfies (#)
15.2	M/N satisfies (##)
16.1	M/N satisfies (#) and $^{\sim}Ass(M/N)$ is finite
18.1, 18.2, 18.3	M satisfies (#) and $^{\sim}Ass(M)$ is finite

Remark. Under Proposition 14.2 in this table we mean its part referring to $Ass(M)$ and $Supp(M)$. For the validity of the remaining part one should also require (for an arbitrary R) that $[N \not\subset \mathcal{F}_p M] \Rightarrow [Ass^{\mathcal{F}_p}(N) \neq \emptyset]$ for an arbitrary submodule $N \subset M$ and $p \in Spec_e R$. □

If every non-zero submodule of M contains a non-zero simple submodule then clearly M satisfies (#). Dually M satisfies (##)

if every non-zero quotient of M contains a non-zero simple submodule.

In particular any module of finite length satisfies (#) and (##)

20. Modules of finite length.

Proposition 1. The following property of an R-module M are equivalent:

(a) length $(M) < \infty$;

(b) M is noetherian, $\mathrm{Ass}(M/N) \neq \emptyset$ for any proper submodule N M and all the ideals of $\mathrm{Ass}(M/N)$ are isomorphic to maximal left ideals;

(c) M is noetherian, $\mathrm{Ass}(M/N) \neq \emptyset$ for any proper submodule N in M and $^{\sim}\mathrm{Supp}(M) \subset {}^{\sim}\mathrm{Max}_{\ell} R$.

Proof. (a) $\Longrightarrow$ (c). Clearly if length$(M) < \infty$ then M is noetherian. At the end of the preceding section we have already noted that $\mathrm{Ass}(M/N) \neq \emptyset$ if $N \neq M$. Let $(M_i)_{0 \leq i \leq r}$ be a simple filtration of M, i.e. $M_{i+1}/M_i \simeq R/\mathcal{M}_i$ where $\mathcal{M}_i \in \mathrm{Max}_{\ell} R$ for $0 \leq i \leq r-1$. Therefore $^{\sim}\mathrm{Supp}(M_{i+1}/M_i) = {}^{\sim}\{\mathcal{M}_i\}$. Proposition 8 implies that $^{\sim}\mathrm{Supp}(M) = \bigcup_{0 \leq i \leq r-1} \mathrm{Supp}(M_{i+1}/M_i)$. Therefore $^{\sim}\mathrm{Supp}(M) =$
$= {}^{\sim}\{\mathcal{M}_0, \dots, \mathcal{M}_{r-1}\} \subset {}^{\sim}\mathrm{Max}_{\ell} R$.

(c)$\Longrightarrow$(b) since $\mathrm{Ass}(M/N) \subset \mathrm{Supp}(M/N) \subset \mathrm{Supp}(M)$ for any submodule N of an arbitrary module M.

(b)$\Longrightarrow$(a). Making use of Proposition 14.1 for an arbitrary R (see Table 19) whose conditions constitute a part of conditions (b) select a composition series $(M_i)_{0 \leq i \leq r}$ such that $M_{i+1}/M_i \simeq R/p_i$ for $0 \leq i \leq r-1$ and $\{p_i \mid 0 \leq i \leq r-1\} \subset \widehat{\mathrm{Spec}}_{\ell} R$. Now notice that every ideal p_i is isomorphic to a maximal left ideal. In fact, $p_i \in \mathrm{Ass}(M_{i+1}/M_i) \subset \mathrm{Ass}(M/M_i)$ and by hypothesis the ideals of $Ass(M/M_i)$ are isomorphic to the maximal left ideals. Let p_i be isomorphic to an ideal $\mathcal{M}_i \in Max_\ell R$. By Proposition 7.1 this means that $p_i = (\mathcal{M}_i : x_i)$ for a finite subset $x_i = \{x_{ik} \mid 1 \leq k \leq \ell_i\}$ of $R \setminus \mathcal{M}_i$.

If $p = \bigcap\{\mathcal{M}'_j \mid 1 \leq j \leq r\}$ and all $\mathcal{M}'_j$ are different ideals from $Max_\ell R$ then the length of R/p equals r.

In fact R/p possesses a simple filtration $(R/\nu_i)_{0 \leq i \leq r}$ where $\nu_0 = R$, $\nu_i = \bigcap\{\mathcal{M}'_j \mid 1 \leq j \leq i\}$ for $1 \leq i \leq r$.

In particular if $p_i = (\mathcal{M}_i : x_i) = \bigcap\{(\mathcal{M} : x_{ik}) \mid 1 \leq k \leq \ell_i\}$ then R/p_i possesses a simple filtration $(M_j)_{0 \leq j \leq \ell_i}$ and $M_{j+1}/M_j \simeq R/\mathcal{M}_i$ for all j. □

Corollary 1. Let R be an $I_\ell^{\succ}$-Noetherian ring. The following properties of an R-module M are equivalent:

(a) length$(M) < \infty$;

(b) M is noetherian and any ideal from $\widehat{Ass^*}(M)$ is isomorphic to a maximal left ideal;

(c) M is noetherian and $^\sim Supp(M) \subset {}^\sim Max_\ell R$.

Proof. (a)$\Rightarrow$(c) follows directly from Proposition 1.

(c)$\Rightarrow$(b) since $\widehat{Ass^*}(M) \subset Supp(M)$.

(b)$\Rightarrow$(c). By Proposition 12.1 for any $p \in Supp(M)$ there exists an arrow $p' \to p$ where p' is an ideal from $\widehat{Ass^*}(M)$. By hypothesis p' is isomorphic to a maximal left ideal $\mathcal{M}$. Therefore, there exists an arrow $\mathcal{M} \to p$, and Proposition 7.1 implies $p = (\mathcal{M} : x)$ for some $x \in \mathcal{P}(R)$. In particular $p \simeq \mathcal{M}$. □

Corollary 2. $^\sim \widehat{Ass^*}(M) = {}^\sim Supp(M)$ for every M of finite length over an $I_\ell^{\succ}$-Noetherian ring.

Proof follows immediately from Corollary 1 and Proposition 12.1. □

Proposition 2. Let R be an $I_\ell^{\succ}$-Noetherian ring, M an R-module, $p \in \widehat{\mathrm{Spec}}_\ell R$. The following properties are equivalent:

(a) $\sim\mathrm{Ass}^*_{G_{\mathcal{F}_p}R}(G_{\mathcal{F}_p}M) = \sim\{G_{\mathcal{F}_p}p\}$;

(b) $\sim\mathrm{Ass}^*_{G_{\mathcal{F}_p}R}(G_{\mathcal{F}_p}M) \subset \sim\mathrm{Max}\{m \in I^{\xi}_\ell G_{\mathcal{F}_p}R \mid j_p^{-1}m \notin \mathcal{F}_p\}$;

(c) p is a minimal ideal of $\widehat{\mathrm{Ass}^*_R}(M)$ (with respect to the preordering $\longrightarrow$).

Proof. (a)$\Rightarrow$(b). If μ is an ideal from $I_\ell G_{\mathcal{F}_p}R$ such that $j_p^{-1}\mu \in \mathcal{F}_p$, then $\mu \to G_{\mathcal{F}_p}p$ by Corollary 1 of Proposition 2.9.

(b)$\Rightarrow$(c). Let $\mathfrak{p} \in \widehat{\mathrm{Ass}^*}(M)$. If $\mathfrak{p} \to p$, then $j_p^{-1}(\mu) \longrightarrow p$. Since $G_{\mathcal{F}_p}$ sends $\widehat{\mathrm{Ass}^*}(M)$ into $\mathrm{Ass}_{G_{\mathcal{F}_p}R}(G_{\mathcal{F}_p}M)$ (Proposition 11.1) and thanks to (b) $G_{\mathcal{F}_p}p$ is maximal among the left ideals ν of the ring $G_{\mathcal{F}_p}R$ such that $j_p^{-1}\nu \notin \mathcal{F}_p$, then $G_{\mathcal{F}_p}\mathfrak{p} \to G_{\mathcal{F}_p}p$ is an isomorphism. But this implies $\mathfrak{p} \simeq p$.

In fact, let $(G_{\mathcal{F}_p}p : z) \subset G_{\mathcal{F}_p}\mathfrak{p}$ for some $z \in \mathcal{P}(G_{\mathcal{F}_p}R)$, and m an ideal of $\mathcal{F}_p$ such that $m \cdot z \subset j_p(R)$. Fix $y \in m \setminus \mathfrak{p}$ and select $z_y \in \mathcal{P}(R)$ such that $j_p(z_y) = j_p(y)\cdot z$. Then

$$G_{\mathcal{F}_p}(p : z_y) = (G_{\mathcal{F}_p}p : j_p(z_y)) = (G_{\mathcal{F}_p}p : j_p(y)\cdot z) =$$
$$= ((G_{\mathcal{F}_p}p : z) : j_p(y)) = (G_{\mathcal{F}_p}\mathfrak{p} : j_p(y)) = G_{\mathcal{F}_p}(\mathfrak{p} : y).$$

Since $\mathfrak{p} \in \widehat{\mathrm{Spec}}_\ell R$ and $y \in R \setminus \mathfrak{p}$, then $(\mathfrak{p} : y) \in \widehat{\mathrm{Spec}}_\ell R$ and $\mathfrak{p} \simeq (\mathfrak{p} : y)$. Besides, $j_p^{-1}(G_{\mathcal{F}_p}(\mathfrak{p} : y)) = (\mathfrak{p} : y)_{\mathcal{F}_p} = (\mathfrak{p} : y)$ ($\nu \subset \nu_{\mathcal{F}} \subset \hat{\nu}$ for any radical filtre $\mathcal{F}$ and any $\nu \in I_\ell R \setminus \mathcal{F}$; therefore $\nu = \nu_{\mathcal{F}}$, if $\nu \in \widehat{\mathrm{Spec}}_\ell R \setminus \mathcal{F}$).

This implies $(p : z_y) \subset (\mathfrak{p} : y) \to \mathfrak{p}$.

(c)$\Rightarrow$(a). By Proposition 11.1 every ideal μ from $\mathrm{Ass}^*_{G_{\mathcal{F}_p}R}(G_{\mathcal{F}_p}M)$ is isomorphic to an ideal of the form $G_{\mathcal{F}_p}m$ where $m \in \widehat{\mathrm{Ass}^*_R}(M)$. Therefore there exists an arrow $m \longrightarrow p$ whose existence is guaranteed by the existence of $G_{\mathcal{F}_p}m \to G_{\mathcal{F}_p}p$.

The minimality of p implies $m \simeq p$, and therefore $\mu \simeq G_{\mathcal{F}_p} p$. □

Corollary. Let M be a module over an $I_\ell^\succ$-Noetherian ring R, $p \in \widehat{Spec}_\ell R$.

1) If $G_{\mathcal{F}_p} M$ is a $G_{\mathcal{F}_p} R$-module of finite length, then p is a minimal element in $Supp(M)$ or, equivalently, p is a minimal element in $\widehat{Ass}_R^*(M)$.

2) If $G_{\mathcal{F}_p}$ is an exact functor, then the contrary is true: the minimality of p in $Supp(M)$ means that $G_{\mathcal{F}_p} M$ is a $G_{\mathcal{F}_p} R$-module of finite length if and only if it is noetherian.

Proof. 1) If $G_{\mathcal{F}_p} M$ is a $G_{\mathcal{F}_p} R$-module of finite length, then $\widehat{{}^\sim Ass}^*_{G_{\mathcal{F}_p} R}(G_{\mathcal{F}_p} M) \subset {}^\sim Max_\ell G_{\mathcal{F}_p} R$ by Proposition 1, and in particular the conditions of Proposition 2 are verified.

2) By Proposition 2 the minimality of p implies

$$\widehat{{}^\sim Ass}^*_{G_{\mathcal{F}_p} R}(G_{\mathcal{F}_p} M) \subset {}^\sim Max \{ n \in I_\ell^\succ G_{\mathcal{F}_p} R \mid j_p^{-1} n \notin \mathcal{F}_p \}.$$

If $G_{\mathcal{F}_p}$ is exact, the latter set coincides with ${}^\sim Max_\ell G_{\mathcal{F}_p} R$, and it remains to make use of Proposition 1. □

Proposition 3. Let M be a module of finite length over a ring R.

1) $\{ \mathcal{F}_p M \mid p \in {}^\sim Ass(M) \}$ is the reduced primary decomposition of a zero submodule in M. This decomposition is minimal: if $\{ Q(p) \mid p \in {}^\sim Ass(M) \}$ is a reduced primary decomposition of 0 in M, then $\mathcal{F}_p M \subset Q(p)$ for every $p \in {}^\sim Ass(M)$.

2) $\mathcal{F}(p) M \neq 0$ for any $p \in {}^\sim Ass(M)$ where $\mathcal{F}(p) = \bigcap \{ \mathcal{F}_{p'} \mid p' \in {}^\sim Ass(M) \setminus \{p\} \}$; and

$${}^\sim Ass(\mathcal{F}(p) M) = \{p\}, \quad \mathcal{F}(p) M \cap \mathcal{F}(p') M = 0$$

when $p \neq p'$, and the (direct) sum $\sum_{p \in {}^\sim Ass(M)} \mathcal{F}(p) M$ is an essential submodule of M.

3) For every $p \in {}^\sim Ass(M)$ the projection $M \twoheadrightarrow M/\mathcal{F}_p M = \mathcal{F}_p' M$

indices an essential embedding

Proof. 1) For any M such that

(#) if M' is a non-zero submodule in M, then $Ass(M') \neq \emptyset$

$\bigcap\{\mathcal{F}_p M \mid p \in {}^{\sim}Ass(M)\} = 0$

In fact, if $N = \bigcap\{\mathcal{F}_p M \mid p \in {}^{\sim}Ass(M)\}$, then $Ann\,\xi \not\to p$ for any $p \in Ass(M)$ and $\xi \in N$.

For any $p \in Supp(M)$ the submodule $\mathcal{F}_p M$ is p-primary in M.

In fact, $\mathcal{F}_p M \neq M$ for every $p \in Supp(M)$ by definition of support, and therefore by Proposition 1 ${}^{\sim}Ass(\mathcal{F}_p^1 M) \neq \emptyset$ and belongs to ${}^{\sim}Max_\ell R$. For any $p' \in {}^{\sim}Ass(\mathcal{F}_p^1 M)$ there exists an arrow $p' \to p$, since $\mathcal{F}_p^1 M$ is $\mathcal{F}_p$-free, and the maximality of p' implies $p' \simeq p$.

Therefore $\{\mathcal{F}_p M \mid p \in {}^{\sim}Ass(M)\}$ is a primary decomposition of 0 in M. The fact that the decomposition is reduced and minimal follow respectively from the modifications of Propositions 17.1 and 17.2 given in 19.

2) Since all the elements of ${}^{\sim}Ass(M)$ are maximal, the statements of heading 2 of Proposition follow from Proposition 18.2 or more exactly from its modification given in Table 19.

3) Let $p \in {}^{\sim}Ass(M)$ and $N/\mathcal{F}_p M$ a simple submodule of $\mathcal{F}_p^1 M = M/\mathcal{F}_p M$. Since N is a module of finite length, we may make use of heading 2) of this proposition, according to which $\mathcal{F}(p)N \neq 0$ and $\mathcal{F}(p)N \cap \mathcal{F}_p N = 0$. Since $\mathcal{F}_p N = \mathcal{F}_p M$ and $N/\mathcal{F}_p M$ is simple, this implies that $\mathcal{F}(p)N \to N/\mathcal{F}_p M$ is isomorphism. Since $\mathcal{F}(p)N \subset \mathcal{F}_p M$ and $N/\mathcal{F}_p M$ is an arbitrary simple submodule of $\mathcal{F}_p^1 M$, then we may claim that the image of the morphism $v_p^M : \mathcal{F}(p)M \to \mathcal{F}_p^1 M$ contains the socle of $\mathcal{F}_p^1$ (recall that the socle of M' is the unique

maximal completely reducible submodule $\mathrm{Soc}\, M'$ of M' which as is easy to verify equals to the sum of all the simple submodules in M'). Obviously the socle of a module of finite length is its essential submodule. Therefore $v_{\mathfrak{p}}^{M}: \mathcal{F}(\mathfrak{p})M \to \mathcal{F}_{\mathfrak{p}}^{1}M$ is an essential monomorphism. □

Corollary. *The canonical map $u_M : M \to \bigoplus_{\mathfrak{p}\in {}^\sim\mathrm{Ass}(M)} \mathcal{F}_{\mathfrak{p}}^{1}M$ is an essential monomorphism for any M of finite length.*

Proof. Since $\mathcal{F}(\mathfrak{p})M \subset \mathcal{F}_{\mathfrak{p}'}M$ when $\mathfrak{p}, \mathfrak{p}'$ are different elements of ${}^\sim\mathrm{Ass}(M)$, u_M induces the "diagonal" map $\bigoplus_{\mathfrak{p}} \mathcal{F}(\mathfrak{p})M \longrightarrow \bigoplus_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{1}M$ whose image is an essential submodule in $\bigoplus_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{1}M$ (see the end of the proof of Proposition 18.3), since monomorphisms $v_{\mathfrak{p}}^{M}: \mathcal{F}(\mathfrak{p})M \to \mathcal{F}_{\mathfrak{p}}^{1}M$ are essential for every $\mathfrak{p} \in {}^\sim\mathrm{Ass}(M)$. □

Remark. In the commutative case the maps $\mathcal{F}(\mathfrak{p})M \to \mathcal{F}_{\mathfrak{p}}^{1}M$ are isomorphisms for all $\mathfrak{p} \in {}^\sim\mathrm{Ass}(M)$, if M is of finite length, and, besides, $\mathrm{Ass}(M) = \mathrm{Supp}(M)$. This implies that $\sum_{\mathfrak{p}} \mathcal{F}(\mathfrak{p})M \hookrightarrow M$ and $u_M : M \to \bigoplus_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^{1}M$ are isomorphisms. □

21. One more subcategory of $\mathcal{R}ings$. The requirement of $\mathrm{I}_\ell^{\succ}$-Noetherianness of R in the statements of the last eleven sections (whenever it is mentioned) can be replaced by the following weaker condition:

(†) $\{(m : x) \mid x \in R \setminus n\}$ possesses a maximal with respect to $\to$ element for any pair m, n of left ideals of R.

Denote $\mathcal{R}ings_{(\ell)}$ the full subcategory of $\mathcal{R}ings$ formed by all the rings R satisfying (†).

Proposition 1. *Let $R \in \mathrm{Ob}\,\mathcal{R}ings_{(\ell)}$. Then for any radical filter $\mathcal{F}$ of left ideals of R and arbitrary $n \in \mathrm{I}_\ell R \setminus \mathcal{F}$ there* exists $\mu \in \widehat{Spec}_\ell R \setminus \mathfrak{F}$ such that $n \to \mu$.

Proof. Consider $\{(n : x) \mid x \in R \setminus n_{\mathfrak{F}}\}$; and let $(n:\lambda)$ be its maximal element. The maximality of $(n:\lambda)$ and the identity $((n:\lambda):x) = (n : x\lambda)$ imply

$[((n:\lambda):x) \not\to (n:\lambda)] \Rightarrow [x\lambda \in n_{\mathfrak{F}}] \Rightarrow [x \in (n_{\mathfrak{F}}:\lambda) = (n:\lambda)_{\mathfrak{F}}]$.

This means by definition that $(n:\lambda) \in Spec_\ell^{\mathfrak{F}} R$ and therefore $(n:\lambda)_{\mathfrak{F}} = \widehat{(n:\lambda)} \in \widehat{Spec}_\ell R \setminus \mathfrak{F}$. Clearly, $n \to (n:\lambda)_{\mathfrak{F}}$. □

Corollary. Let $R \in Ob\, Rings_{(\ell)}$. Then $\mathfrak{F} = \bigcap \{\mathfrak{F}_\mu \mid \mu \in \widehat{Spec}_\ell R \setminus \mathfrak{F}\}$ for any radical filtre $\mathfrak{F}$ of left ideals of R.

Clearly, $Rings_{(\ell)}$ is a subcategory of $S_\ell Rings$ (see 4). In particular, $Spec\, R = \overline{Spec}\, R = \{\mu_s \mid \mu \in \widehat{Spec}_\ell R\}$ if $R \in Ob\, Rings_{(\ell)}$.

22. Associated ideals of Goldi rings. The passage from Noetherian rings to $Rings_{(\ell)}$ does not induce (at least temporary) any constructive modifications, since I do not know practically anything about both of these two classes. However, notice that is required for a ring to belong to $Rings_{(\ell)}$ mostly in order to get the following implications (see 19):

$[M$ is a non-zero module$] \Rightarrow [Ass(M) \neq \emptyset]$,

$[\mathfrak{F}$ is a radical filtre and $M \neq \mathfrak{F}M] \Rightarrow [Ass^{\mathfrak{F}}(M) \neq \emptyset]$.

Therefore if we restrict the class of modules (e.g., we'll be interested in associated or $*$-associated ideals of R), we can essential weaken the constraints.

Consider the following conditions on R:

(a) $\{(m:\lambda) \mid \lambda \in R \setminus n\}$ possesses a maximal with respect to $\to$ element for any left annulator m of a finite subset of R and an arbitrary left ideal n,

(b) the same condition for an arbitrary left annulator.

Clearly, (b)$\Rightarrow$(a). By Proposition 4.1 any prime left Goldi ring and any semiprime left Goldi ring with unit satisfy (b).

Proposition 1. Suppose one of the following conditions holds:

(1) R satisfies (a) and M is a non-zero submodule of a projective R-module;

(2) R satisfies (b) and M is a non-zero submodule of the product of a certain number of a projective R-modules.

Then $Ass(M) \neq \emptyset$ and $Ass^{\mathcal{F}}(M) \neq \emptyset$, if $\mathcal{F}$ is a radical filtre of left ideals of R such that $M \neq \mathcal{F}M$.

Proof. 1) Suppose (1) holds. Then M can be identified with a submodule of the free module $R^{(I)} = \bigoplus_{i \in I} R$. Let $\mathcal{F}$ be a radical filtre such that $M \neq \mathcal{F}M$, and $\xi \in M \setminus \mathcal{F}M$. The element ξ is the form $\bigoplus_{i \in J} \xi_i$ where J is a finite subset of I (the support of ξ) and $\{\xi_i \mid i \in J\} \subset R$. Obviously, $Ann\,\xi = (0 : \bar{\xi})$ is the annulator of $\bar{\xi} = \{\xi_i \mid i \in J\}$. By hypothesis the set $\{((0:\bar{\xi}):\lambda) = (0:\lambda\bar{\xi}) \mid \lambda \in R \setminus (0:\bar{\xi})_{\mathcal{F}}\}$ possesses a maximal with respect to $\rightarrow$ element $(0 : \lambda_0 \bar{\xi})$. It is easy to see that $(0 : \lambda_0\bar{\xi}) =$ $= Ann\,\lambda_0\xi \in Spec_{\ell}^{\mathcal{F}} R$; more generally, for any m from $I_{\ell}R \setminus \mathcal{F}$ a maximal element of $\{(m:\lambda) \mid \lambda \in R \setminus m_{\mathcal{F}}\}$ belongs to $Spec_{\ell}^{\mathcal{F}} R$, see Proposition 10.2.

(If $(m : \lambda_0)$ is maximal in $\{(m:\lambda) \mid \lambda \in R \setminus m_{\mathcal{F}}\}$ and $((m:\lambda_0):z) \not\rightarrow (m:\lambda_0)$, then $z\lambda_0 \in m_{\mathcal{F}}$; i.e. $z \in (m_{\mathcal{F}} : \lambda_0) = (m:\lambda_0)_{\mathcal{F}}$.)

Similarly, for any $\xi \in M \setminus \{0\}$ a maximal element of $\{Ann\,\lambda\xi \mid \lambda \in R \setminus Ann\,\xi\}$ belongs to $\widehat{Spec}\,R$.

2) Any submodule of the product of a family of projective modules is identified with a submodule of the product of a certain number of copies of R. This immediately implies that the ann~~u~~ihilator of any non-zero element of M is a left annulator of R. Further repeat the arguments of the first step of the proof. □

Proposition 2. *Let R be a semiprime left Goldi ring with unit, M a non-zero submodule in the product of a certain number of projective R-modules.*

1) $\mathrm{Ass}(M) \neq \emptyset$ *and* $\mathrm{Ass}(M) \subset \mathrm{Ass}(R)$.

2) *Every ideal from* $\mathrm{Ass}(M)$ *is isomorphic to a unique prime ideal and therefore* $^{\sim}\mathrm{Ass}(M)$ *is identified with a subset of* $\widehat{\mathrm{Spec}}_{\ell} R \cap \mathrm{Spec}\, R$.

3) $^{\sim}\mathrm{Ass}(M)$ *is finite and M possesses a primary decomposition.*

Proof. 1) As had been already noted, semiprime left Goldi rings with unit satisfy (b) (see above) and therefore Proposition 1 is applicable to them. Hence $\mathrm{Ass}(M) \neq \emptyset$.

Besides, $\mathrm{Ann}\,\xi$ is a proper left annulator of R for any $\xi \in M \setminus \{0\}$ (see step two of the proof of Proposition 1). Let $\mathrm{Ann}\,\xi \in \widehat{\mathrm{Spec}}_{\ell} R$ and $\mathrm{Ann}\,\xi = (0 : x)$ for a subset $x \subset R$. The ring R satisfies by Corollary 1 of Lemma 5.1 the minimality condition for left annulators with respect to inclusion; in particular, $\{(0 : Ry \mid y$ is a finite subset of $x\}$ has a minimal element $(0 : Ry_0)$. Let us show that $(0 : Ry_0) = (0 : Rx)$ or, equivalently, $(0 : Ry_0) \subset (0 : Rx)$. In fact, if $(0 : Ry_0) \not\subset (0 : Rx)$, then $(0 : Ry_0) \not\subset (0 : Ry)$ for a finite subset $y \subset x$; but then $(0 : R(y \cup y_0)) \subsetneq (0 : Ry_0)$, contradicting to the minimality of $(0 : Ry_0)$.

By Corollary 2 of Lemma 5.1 $(0 : Ry_0) \simeq (0 : y_0)$ and $(0 : Rx) \simeq$

$\simeq (0:x)$, implying that $(0:y_o) \in \widehat{Spec}_\ell R$ and that $(0:x) \subset (0:y)$ is an isomorphism in $I_\ell^{\sim} R$.

Let $y_o = \{y_{oi} \mid i \in I\}$ where $Card(I) < \infty$. Then $(0:y_{oi}) \simeq (0:y)$ for some $i \in I$.

In fact, if $(0:y_{oi}) \not\simeq (0:y_o)$ for all $i \in I$, then since $(0:y_o) \in \widehat{Spec}_\ell R$ and I is finite, the intersection $\bigcap_{i \in I}(0:y_{oi})$ also belongs to $\mathcal{F}_{(0:y_o)}$. But $\bigcap\{(0:y_{oi}) \mid i \in I\} = (0:y_o)$ and $(0:y_o) \notin \mathcal{F}_{(0:y_o)}$ by definition of $\mathcal{F}_{(0:y_o)}$.

Thus, $(0:y_{oi}) \simeq (0:y_o)$ for some $i \in I$ and therefore $(0:y_{oi}) \simeq (0:x) = Ann\,\xi$.

2) For any left annulator m of R there exists $x_m \in \mathcal{P}(R)$ such that $(m:x_m) = (m:R) = m_s$ (Corollary 2 of Lemma 5.1); therefore $m \simeq m_s$. If $m \in \widehat{Spec}_\ell R$, then $m_s = (m:x_m) \in \widehat{Spec}_\ell R \cap IR = \widehat{Spec}_\ell R \cap Spec R$. The uniqueness is obvious, since $[\{n, n'\} \subset I_\ell R,\ n \simeq n'] \Rightarrow [n_s = n'_s]$.

3) Since $Ass(M) \subset Ass(R)$ then by heading 1 of this proposition, to see that $Card(^{\sim}Ass(M)) < \infty$ it suffices to verify that $Card(^{\sim}Ass(R)) < \infty$.

For any $\mathfrak{p} \in {}^{\sim}Ass(R)$ select a maximal among the left ideals $\mathfrak{v}$ of R such that $^{\sim}Ass(\mathfrak{v}) = \{\mathfrak{p}\}$, and denote it $R(\mathfrak{p})$ (see the beginning of n.18). If $\mathfrak{p}, \mathfrak{p}'$ are different elements of $^{\sim}Ass(R)$ (which can be identified with $Ass(R) \cap Spec R$), then $Ass(R(\mathfrak{p}) \cap R(\mathfrak{p}')) = \emptyset$. By Proposition 1 this means that $R(\mathfrak{p}) \cap R(\mathfrak{p}') = 0$ (see Proposition 18.1 and its elucidation in Table 19).

Therefore $\sum_{\mathfrak{p} \in {}^{\sim}Ass(R)} R(\mathfrak{p})$ is the direct sum of non-zero left ideals of R. Since R is a Goldi ring, this sum cannot be infinite.

For every $\mathfrak{p} \in {}^{\sim}Ass(M)$ select a submodule $Q(\mathfrak{p})$ maximal among the submodules Q' of M satisfying $\tilde{}Ass(Q') = \tilde{}Ass(M) \setminus \{\mathfrak{p}\}$ (its existence is guaranteed by Proposition 9.4). Clearly, $\tilde{}Ass(\bigcap_{\mathfrak{p} \in \tilde{}Ass(M)} Q(\mathfrak{p})) = \emptyset$, and by Proposition 1 this means that $\bigcap\{Q(\mathfrak{p}) \mid \mathfrak{p} \in \tilde{}Ass(M)\} = 0$. Therefore $\{Q(\mathfrak{p}) \mid \mathfrak{p} \in \tilde{}Ass(M)\}$ is the reduced primary decomposition of M; see Proposition 17.1 and its modification from Table 19. $\square$

9. Morphisms.

1. Radicals and localizations in Abelian categories. We will find out the conditions under which a ring morphism $R \to R'$ induces a morphism of localizations $G_{\mathfrak{F}} R \to G_{\mathfrak{G}} R'$, in terms of the abstract torsion theory. Recall the main notions of the theory in a convenient form.

Let $\mathcal{A}$ be an Abelian category. A radical in $\mathcal{A}$ (or an inherited radical) is a left exact subfunctor k of the identity functor $Id_{\mathcal{A}}$ such that $k(x/k(x)) \equiv 0$. A radical k is called an idempotent one, if $k^2 = k$; i.e. if $k(k(x)) = k(x)$ for any $x \in Ob\,\mathcal{A}$.

Example. Let $\mathcal{A} = R\text{-mod}$. Any topologizing filter $\mathfrak{F}$ determines a radical $k_{\mathfrak{F}}$ assigning to every module M its $\mathfrak{F}$-torsion $\mathfrak{F}M$. On the other hand, to a radical k in R-mod the set of left ideals of R corresponds each of which is the annihilator of an element of a module of the form kM. Thus constructed maps $\mathfrak{F} \mapsto k_{\mathfrak{F}}$ and $k \mapsto \mathfrak{F}_{(k)}$ are inverse to each other, if we confine ourselves to radical filters on the one hand and the idempotent radicals on the other one (see Ch. 16 in [2]).

Fix a radical k in $\mathcal{A}$ and define a full subcategory T_k of k-periodic objects setting

$$Ob\,T_k = \{x \in Ob\,\mathcal{A} \mid k(x) = x\}$$

and the full subcategory S_k of k-semisimple objects setting

$$Ob\,S_k = \{x \in Ob\,\mathcal{A} \mid k(x) = 0\}.$$

Besides, distinguish a family $\mathfrak{M}_k$ of the morphisms

$f : x \longrightarrow y$ such that the $\operatorname{Ker} f$ and $\operatorname{Cok} f$ belong to T_k.

An object w of $\mathcal{A}$ is called k-injective, if any pair of arrows $w \xleftarrow{f} x \xrightarrow{g} y$ with $g \in \mathcal{M}_k$ extends up to a commuting diagram

$$\begin{tikzcd} x \arrow[rr, "g"] \arrow[dr] & & y \\ & w \arrow[ur] & \end{tikzcd}$$

If h is uniquely determined, then w is called strictly k-injective.

It is not difficult to verify that a k-injective object is strictly k-injective if and only if it is k-semisimple. The full subcategory of $\mathcal{A}$ formed by strict k-injective objects will be denoted by $\mathcal{A}(k)$

A k-localization of an object x of $\mathcal{A}$ will be called an object $Q_k x$ of $\mathcal{A}(k)$ such that there exists a universal arrow $\underline{\mu}_x : x \longrightarrow Q_k x$ i.e. any morphism $x \longrightarrow y$, where $y \in \mathcal{O}b\,\mathcal{A}(k)$, uniquely factors through $\underline{\mu}_x$. As always in a similar situation, the map $x \mapsto Q_k x$ uniquely extends up to a partial (i.e. determined on a subcategory of $\mathcal{A}$) functor with values in $\mathcal{A}(k)$. This functor is everywhere defined and left adjoint to the embedding $\mathcal{A}(k) \hookrightarrow \mathcal{A}$ when $\mathcal{A}$ is the Grothendieck category.

Recall that the Grothendieck category is an Abelian category with coproducts and generators such that $(\sup_i a_i) \cap b = \sup_i (a_i \cap b)$ for any directed family of subobjects $\{a_i \mid i \in I\}$ of an object x and any subobject $b \hookrightarrow x$.

Example. If $\mathcal{A} = R\text{-}mod$ and $k = k_{\mathcal{F}}$ for a radical filter of left ideals $\mathcal{F}$, then it is easy to show that $\mathcal{A}(k) = R\text{-}mod_{\mathcal{F}}$. Therefore the $k_{\mathcal{F}}$-localization coincides with the $\mathcal{F}$-localization of modules. $\square$

It is natural to crown this short list of notions with the following fundamental fact which connects them all:

Theorem (Popesku-Gabriel). Let $\mathcal{A}$ be a Grothendieck category; $X \in Ob\,\mathcal{A}$; R the ring dual to the ring of endomorphisms of X; h^o_X the functor $\mathcal{A} \to R\text{-}mod$ determined on objects Y and morphisms f from $\mathcal{A}$ by the formulas

$$h^o_X(Y) = \mathcal{A}(X,Y), \quad h^o_X(f) = \mathcal{A}(1_X, f)$$

with an obvious R-module structure on $\mathcal{A}(X,Y)$. The following conditions are equivalent:

1) X is a generator of $\mathcal{A}$;

2) h^o_X is full and faithful;

3) h^o_X induces an equivalence of categories $\mathcal{A} \simeq R\text{-}mod(k)$, where k is the maximal of the radicals k' in R-mod such that $h^o_X(Y)$, $Y \in Ob\,\mathcal{A}$, are strictly k'-injective.

Proof see in [2] or [7].

2. Morphisms of localizations. Turn a family of pairs $(\mathcal{A}, k)$, where $\mathcal{A}$ is an abelian category ($Hom\,\mathcal{A}$ belongs to a fixed univers), k a radical in $\mathcal{A}$, into a "localization" category $\mathcal{L}Ab$ as follows: the morphisms from $(\mathcal{A}, k)$ into $(\mathcal{A}', k')$ are the functors $F: \mathcal{A} \to \mathcal{A}'$, such that $Fx \in \mathcal{A}'(k')$ for every $x \in \mathcal{A}(k)$, with the natural composition. We are interested here in the subcategory $\mathcal{G}Ab$ of $\mathcal{L}Ab$ formed by the objects $(\mathcal{A}, k)$ such

that $\mathcal{A}$ is a Grothendieck category and k an idempotent radical, and morphisms $(\mathcal{A},k)\xrightarrow{F}(\mathcal{A}',k')$ such that F possesses a left adjoint.

Note the main for us property of morphisms in $\mathcal{L}Ab$:

Let $(\mathcal{A},k)$ and $(\mathcal{A}',k')$ be categories with radicals such that the embeddings $\mathcal{A}(k)\hookrightarrow\mathcal{A}$ and $\mathcal{A}'(k')\hookrightarrow\mathcal{A}'$ possess left adjoint functors Q_k and $Q_{k'}$ respectively, and $(\mathcal{A},k)\xrightarrow{F}(\mathcal{A}',k')$ a morphism in $\mathcal{L}Ab$. Then for any $x\in Ob\,\mathcal{A}'$, $y\in Ob\,\mathcal{A}$ and any arrow $f: x\to Fy$ there exists a unique morphism $f_{kk'}: Q_{k'}x\to FQ_k y$ for which the following diagram commutes

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\mu_{k'}(x)"'] & Fy \arrow[d, "F\mu_k(y)"] \\
Q_{k'}x \arrow[r, "f_{kk'}"] & FQ_k y
\end{tikzcd}$$

Proposition. Let k and k' be radicals in abelian categories $\mathcal{A}$ and $\mathcal{A}'$ respectively, $F:\mathcal{A}\to\mathcal{A}'$ a functor with left adjoint F^L.

1) For the following properties we have (a)$\Leftrightarrow$(b)$\Rightarrow$(c):

(a) $F^L(T_{k'})\subset T_k$ and $F^L f\in\mathcal{M}_k$ for any monomorphism $f\in\mathcal{M}_{k'}$;

(b) $F^L(\mathcal{M}_{k'})\subset\mathcal{M}_k$;

(c) F is a morphism $(\mathcal{A},k)\longrightarrow(\mathcal{A}',k')$.

2) If the embedding $\mathcal{A}(k)\hookrightarrow\mathcal{A}$ possesses a left adjoint, then (c)$\Rightarrow$(b).

Proof. (a)$\Rightarrow$(b). Let $f: x\to y$ be an arbitrary arrow from $\mathcal{M}_{k'}$. The functor F^L is right exact (thanks to the fact that it possesses a right adjoint) and in particular, sends an exact sequence $x \xrightarrow{f} y \to \mathrm{Cok}\, f \to 0$ into the exact sequence $F^L x \xrightarrow{F^L f} F^L y \to F^L \mathrm{Cok}\, f \to 0.$ By hypothesis $\mathrm{Cok}\, f \in T_k$, and therefore $\mathrm{Cok}\, F^L f \in T_k$.

To the factoring of f into the composition $x \twoheadrightarrow x/\ker f \xrightarrow{\bar f} y$ the commuting diagram with exact rows corresponds:

$$\begin{tikzcd}
F^L \ker f \arrow[r] & F^L x \arrow[r] & F^L(x/\ker f) \arrow[r] & 0 \\
F^L \ker f \arrow[r] \arrow[u, "id"] & \mathrm{Ker}\, F^L f \arrow[r] \arrow[u, tail] & \mathrm{Ker}\, F^L \bar f \arrow[r] \arrow[u, tail] & 0
\end{tikzcd} \qquad (2)$$

By hypothesis $\{F^L \ker f, \mathrm{Ker}\, F^L \bar f\} \subset \mathrm{Ob}\, T_k$. Recall the main properties of the radicals:

Lemma. *For any exact sequence* $0 \to x \to y \to z \to 0$ *in* $\mathcal{A}$ *the following implications hold:*

(i) $[\{x, y\} \subset \mathrm{Ob}\, T_k] \Rightarrow [y \in \mathrm{Ob}\, T_k]$;

(ii) $[y \in \mathrm{Ob}\, T_k] \Rightarrow [\{x, z\} \subset \mathrm{Ob}\, T_k]$.

In particular, if in an exact sequence $x \to y \to z \to 0$ *the objects x and z are k-periodic, then so is y.*

To see this it suffices to look at the commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & kx \arrow[r] \arrow[d, tail] & ky \arrow[r] \arrow[d, tail] & kz \arrow[d, tail] & \\
0 \arrow[r] & x \arrow[r] & y \arrow[r] & z \arrow[r] & 0
\end{tikzcd} \qquad (3)$$

Therefore the exactness of the lowest row (2) and the k-periodicity of $F^L \mathrm{Ker}\, f$ and $\mathrm{Ker}\, F^L \bar f$ implies the k-periodicity of $\mathrm{Ker}\, F^L f$.

(b) $\Rightarrow$ (a). Clearly, an object x of $\mathcal{A}$ is k-periodic if and only if $0_x \in \mathcal{M}_k$. Therefore (b) implies
$[x \in \mathrm{Ob}\, T_k] \Leftrightarrow [0_x \in \mathcal{M}_k] \Rightarrow [0_{F^L x} = F^L(0_x) \in \mathcal{M}_k] \Leftrightarrow [F^L x \in \mathrm{Ob}\, T_k]$.

(b) $\Longrightarrow$ (c). Let $x \xrightarrow{\varphi} y \in \mathcal{M}_{k'}$, $f: x \longrightarrow Fw$ an arbitrary morphism; $\psi_{x,w}$ the conjugation isomorphism $\mathcal{A}'(x, Fw) \xrightarrow{\simeq} \mathcal{A}(F^L x, w)$. If $w \in Ob\,\mathcal{A}(k)$, then there exists a unique arrow $g: F^L y \to w$ such that

$$\psi_{x,w}(f) = g \circ F^L \varphi$$

This means that $g' \circ \varphi = f$ for the unique arrow g', $g' = \psi^{-1}_{y,w}(g)$. Therefore, $Fw \in Ob\,\mathcal{A}(k')$.

(c) $\Longrightarrow$ (b) (when the embedding $\mathcal{A}(k) \hookrightarrow \mathcal{A}$ possesses a left adjoint functor Q_k). Let $\varphi: x \longrightarrow y$ be an arrow from $\mathcal{M}_{k'}$. The strict k'-injectiveness of $FQ_k F^L x$ implies the existence of a unique arrow $y \xrightarrow{g} FQ_k F^L x$ such that the diagram

$$\begin{tikzcd} x \arrow[rr, "\varphi"] \arrow[dr, "\tilde{\mu} = \psi^{-1}_{x, Q_k F^L_x}(\mu(x))"'] & & y \arrow[dl, "g"] \\ & FQ_k F^L x & \end{tikzcd}$$

commutes. To the "adjoint" morphism $\tilde{g} = \psi^{-1}_{y, Q_k F^L_x}(g): F^L y \longrightarrow Q_k F^L x$ a unique arrow $g': Q_k F^L y \longrightarrow Q_k F^L x$ corresponds for which the following diagram commutes

$$\begin{tikzcd} F^L x \arrow[r, "F^L \varphi"] \arrow[d] & F^L y \arrow[d] \arrow[dl, "\tilde{g}"'] \\ Q_k F^L x & Q_k F^L y \arrow[l, "g'"] \end{tikzcd}$$

The standard uniqueness considerations imply that g' is inverse to $Q_k F^L \varphi$. Now notice that an arrow f from $\mathcal{A}$ belongs to $\mathcal{M}_k$ if and only if $Q_k \varphi$ is invertible. $\square$

Corollary 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be radical filters of left ideals of R and R′ respectively. The following properties of a ring morphism $\varphi: R \to R'$ are equivalent:*

1) The change of base functor φ_* is a morphism $(R'\text{-mod}, k_{\mathcal{G}}) \longrightarrow (R\text{-mod}, k_{\mathcal{F}})$.

2) The following conditions hold:

(γ) $[m \in \mathcal{F}] \Longrightarrow [(R', \varphi(m)) \in \mathcal{G}]$;

(δ) if M is a submodule of an R-module N and $\mathcal{F}(N/M) = N/M$, then the kernel of a natural morphism $R' \otimes_R M \longrightarrow R' \otimes_R N$ belongs to $\mathcal{T}_{k_{\mathcal{G}}}$.

Proof. The functor $\varphi_* : R'\text{-mod} \to R\text{-mod}$ possesses the left adjoint $\varphi_*^L = R'^{(1)} \otimes_R -$ where $R'^{(1)}$ is the ring obtained from R' by accrueing the unit. It is not difficult to see that (γ) is equivalent to $k_{\mathcal{G}}$-periodicity of the R'-module $R' \otimes_R M'$ for any $k_{\mathcal{F}}$-periodic R-module M'. Therefore (γ) and (δ) is a specialization of condition (a) of Proposition 2.

Remark. Consider the following properties:

$(\gamma 1)$ $[n \in I_\ell R', \varphi^{-1} n \in \mathcal{F}] \Rightarrow [n \in \mathcal{G}]$;

$(\gamma 2)$ $[M \in Ob\, R'\text{-mod}, \varphi_* M \in \mathcal{T}_{k_{\mathcal{F}}}] \Rightarrow [M \in \mathcal{T}_{k_{\mathcal{G}}}]$;

$(\gamma 3)$ $[\xi \in Hom_{R'}(M, N), \varphi_* \xi \in \mathcal{M}_{k_{\mathcal{F}}}] \Rightarrow [\xi \in \mathcal{M}_{k_{\mathcal{G}}}]$;

$(\delta 1)$ For any $m \in \mathcal{F}$ the kernel of the canonical morphism $R' \otimes_R m \to R'$ belongs to $\mathcal{T}_{k_{\mathcal{F}}}$;

$(\delta 2)$ For any $m \in \mathcal{F}$ the R'-module $Tor_1(R', R/m)$ is $k_{\mathcal{G}}$-periodic.

$(\delta 3)$ If an R-module M is $k_{\mathcal{F}}$-periodic, then the R'-module $Tor_1(R', M)$ is $k_{\mathcal{G}}$-periodic.

It is not difficult to verify the equivalence of conditions (γi), (δi) for $i = 1, 2, 3$ to conditions (γ) and (δ) of Corollary 1 respectively. □

Corollary 2. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be radical filters of left ideals of R and R′ respectively;* $\varphi: R \longrightarrow R'$ *a morphism satisfying the equivalent conditions of Corollary 1, and* $M \in \mathcal{O}b\, R\text{-mod}$, $N \in \mathcal{O}b\, R'\text{-mod}$.

Then any R-module morphism $f: M \longrightarrow \varphi_* N$ *determines the unique R-module morphism* $f_{\mathcal{F},\mathcal{G}}: G_{\mathcal{F}} M \longrightarrow \varphi_* G_{\mathcal{F}} N$ *such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & \varphi_* N \\ {\scriptstyle j_{\mathcal{F},M}}\downarrow & & \downarrow{\scriptstyle \varphi_* j_{\mathcal{G},N}} \\ G_{\mathcal{F}} M & \xrightarrow{f_{\mathcal{F},\mathcal{G}}} & \varphi_* G_{\mathcal{G}} N \end{array}$$

commutes. In particular, there exists a unique ring morphism $\varphi_{\mathcal{F},\mathcal{G}}: G_{\mathcal{F}} R \longrightarrow G_{\mathcal{G}} R'$ *for which the following diagram commutes*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R' \\ \downarrow & & \downarrow \\ G_{\mathcal{F}} R & \xrightarrow{\varphi_{\mathcal{F},\mathcal{G}}} & G_{\mathcal{G}} R' \end{array}$$

Before we leave the radicals in Abelian categories and confine ourselves again to radical filters of left ideals, note that the results of the first headings of (Theorem 1 and its "geometric" corollaries) may be reformulated in terms of radicals in Abelian categories and related notions, constructing therefore a "geometrization" of Abelian categories. This point of view is an abstract expression of the fact that the main object of the study in non-commutative (algebraic) geometry are not ring spaces but spaces with category of quasicoherent sheaves of modules over them.

3. Morphisms of $\amalg$-semischemes. Denote $\overline{\text{Rings}}_\ell$ the category whose objects are the pairs $(R, \mathcal{F})$, where R is a ring and $\mathcal{F}$ a radical filter of left ideals of R; the morphisms $(R,\mathcal{F}) \to (R',\mathcal{G})$ are the ring morphisms $\varphi : R \to R'$, satisfying the conditions of Corollary 1 in 2, with the natural composition.

Let $(R, \mathcal{T})$ and $(R', \mathcal{T}')$ be $\amalg$-semischemes. A morphism $(R,\mathcal{T}) \longrightarrow (R',\mathcal{T}')$ is a pair (ψ,φ), where ψ is a precositi morphism $\underline{\mathcal{T}} \longrightarrow \underline{\mathcal{T}}'$ (i.e. a monotonous function $\psi : \mathcal{T}' \longrightarrow \mathcal{T}$ such that $\psi(\mathcal{F}\amalg\mathcal{G}) = \psi\mathcal{F}\amalg\psi\mathcal{G}$, $\psi\mathcal{F} = \bigcap\{\psi\mathcal{F}_i \mid i\in I\}$ for any $\{\mathcal{F},\mathcal{G}\}\subset\mathcal{T}'$ and any cocovering $\{\mathcal{F}\hookrightarrow\mathcal{F}_i \mid i \in I\} \in \widetilde{\mathcal{T}}'$, and $\varphi : (R',\mathcal{F}) \to (R, \psi\mathcal{F})$ a morphism from $\overline{\text{Rings}}_\ell$. It is easy to verify that for any pair of morphisms $(R,\mathcal{T}) \xrightarrow{(\psi,\varphi)} (R',\mathcal{T}') \xrightarrow{(\psi',\varphi')} (R'',\mathcal{T}'')$ the pair $(\psi'\circ\psi, \varphi\circ\varphi')$ is a morphism.

Therefore a composition is defined which turns the collection of $\amalg$-semischemes into a category which we will denote by $\amalg\text{-}\mathcal{S}h_\ell$.

Proposition. *Let $(\psi,\varphi) : (R,\mathcal{T}) \longrightarrow (R',\mathcal{T}')$ be a semischeme morphism.*

1) (ψ,φ) *uniquely determines a ringed precositi morphism $(\psi, \varphi^a) : (\underline{\mathcal{T}}, R^a_{\mathcal{T}}) \longrightarrow (\underline{\mathcal{T}}', R'^a_{\mathcal{T}'})$. The correspondence $(\psi,\varphi) \longmapsto (\psi,\varphi^a)$ is functorial.*

2) *Suppose that for any $p \in \mathcal{S}pec_\ell(R,\mathcal{T})$ and $\mathcal{F}\in\mathcal{T}'$ the ideal $\varphi^{-1}p$ belongs to $\mathcal{F}$, if $p\in\psi\mathcal{F}$. Then $m \longmapsto \varphi^{-1}m$ induces a continuous map ${}^s\varphi : \mathcal{S}pec_\ell(R,\mathcal{T}) \to \mathcal{S}pec_\ell(R',\mathcal{T}')$ which uniquely extends up to a morphism of ringed spaces (see 4.10)*

$$({}^s\varphi, \varphi^h) : (\mathcal{S}pec_\ell(R,\mathcal{T}), \widetilde{R}^a_{\mathcal{T}}) \to (\mathcal{S}pec_\ell(R',\mathcal{T}'), \widetilde{R}'^a_{\mathcal{T}'}).$$

Proof. 1) The definition of arrows in $\amalg\text{-}Sh_\ell$ shows that (ψ,φ) determines a morphism $(\psi,\{\varphi_{\mathcal{F}}\}):(\underline{\mathcal{T}},R_{\mathcal{T}})\to$ $\longrightarrow(\underline{\mathcal{T}}',R'_{\mathcal{T}'})$ to which in turn, the morphism $(\psi,\varphi^a):(\underline{\mathcal{T}},R^a_{\mathcal{T}})\to(\underline{\mathcal{T}}',R'^a_{\mathcal{T}'})$ corresponds. The functorial property of the correspondence $(\psi,\varphi)\mapsto(\psi,\varphi^a)$ is obvious.

2) Let $p\in Spec_\ell(R,\mathcal{T})$; $\mathcal{F}$ and $\mathcal{G}$ be filters from $\mathcal{T}'$ such that $\varphi^{-1}p\in\mathcal{F}\amalg\mathcal{G}$. Definition of a morphism implies that $(R,\varphi(\varphi^{-1}p))$ and therefore p belong to $\psi(\mathcal{F}\amalg\mathcal{G})=\psi\mathcal{F}\amalg\psi\mathcal{G}$ (see Remark after Corollary 2 in 1, condition ($\gamma 1$)). Since $p\in Spec_\ell(R,\mathcal{T})$, then $p\in\psi\mathcal{F}\cup\psi\mathcal{G}$. By hypothesis this implies $\varphi^{-1}p\in\mathcal{F}\cup\mathcal{G}$. Therefore we have established that the map $m\mapsto\varphi^{-1}m$ induces the map of sets ${}^s\varphi: Spec_\ell(R,\mathcal{T})\to Spec_\ell(R',\mathcal{T}')$. This map is continuous. The preimage of any closed set $V(\mathcal{F})=$ $=Spec_\ell(R',\mathcal{T}')\cap\mathcal{F}$ coincides with $V(\psi\mathcal{F})$. Finally, ${}^s\varphi$ extends up to a morphism $({}^s\varphi,\varphi^\natural)$ of ringed spaces uniquely determined by the commutativity of the diagram

$$\begin{tikzcd}
(\underline{\mathcal{T}},R^a_{\mathcal{T}}) \arrow[r, "(\psi,\varphi^a)"] & (\underline{\mathcal{T}}',R'^a_{\mathcal{T}'}) \\
(\underline{Spec_\ell(R,\mathcal{T})},\widetilde{R}^a_{\mathcal{T}}) \arrow[u] \arrow[r, "({}^s\underline{\varphi},\varphi^\natural)"] & (Spec_\ell(R',\mathcal{T}'),\widetilde{R}'^a_{\mathcal{T}'}) \arrow[u]
\end{tikzcd}$$

The details of the proof of this fact (passage to the limit) are left to the reader. □

4. Morphisms of rings and left spectrum. In the commutative case any morphisms of unitary rings induce morphisms of the corresponding affine schemes. Passing to

non-commutative rings and left affine (quasi)schemes complicates the picture. The responsibility for this complicatedness takes the same category $\mathrm{I}_\ell^{\succ} R$.

Denote $\widetilde{\text{Rings}}_\ell$ the subcategory of Rings formed by all the morphisms $\varphi: R \to R'$ such that the map $m \mapsto \varphi^{-1}m$ satisfies

$$(1_\ell) \quad [p' \in \mathrm{Spec}_\ell R', m \in \mathrm{I}_\ell R', m \to p'] \Rightarrow [\varphi^{-1}m \to \varphi^{-1}p'].$$

Proposition. Let $\varphi: R \to R'$ be a morphism from $\widetilde{\text{Rings}}_\ell$. Then the map $m \mapsto \varphi^{-1}m$ induces the maps $\varphi_\ell: \mathcal{U}_\ell(\varphi(R)) \to \mathrm{Spec}_\ell R$ and $\hat{\varphi}_\ell: \hat{\mathcal{U}}_\ell(\varphi(R)) \to \mathrm{S\hat{p}ec}_\ell R$ continuous with respect to $\mathfrak{T}_0$ and $\mathfrak{T}$.

Proof. 1) Let $p' \in \mathrm{Spec}_\ell R'$ and $n \in \mathrm{I}_\ell R$. Since $\varphi^{-1}(p' : \varphi(y)) = (\varphi^{-1}p' : y)$ for any $y \subset R$, then

$[(\varphi^{-1}p' : x) \not\to \varphi^{-1}p'$ for any $x \in \mathcal{P}(n)] \Rightarrow [(p' : \varphi(x)) \not\to p'$ for any $x \in \mathcal{P}(n)] \Rightarrow [(R', \varphi(n)) \longrightarrow p'] \Rightarrow$
$\Rightarrow [\varphi^{-1}(R', \varphi(n)) \to \varphi^{-1}p'] \Rightarrow [n \to \varphi^{-1}p']$

If $p' \in \mathrm{S\hat{p}ec}_\ell R$, then for any $x \in \mathcal{P}(R)$
$[(\varphi^{-1}p' : x) \not\to p'] \Rightarrow [(p' : \varphi(x)) \not\to p'] \Rightarrow [\varphi(x) \subset p'] \Leftrightarrow$
$\Leftrightarrow [x \subset \varphi^{-1}p']$; i.e. $\varphi^{-1}p' \in \mathrm{S\hat{p}ec}_\ell R \cup \{R\}$.

2) The map $\varphi_\ell: \mathcal{U}_\ell(\varphi(R)) \to \mathrm{Spec}_\ell R$ is continuous with respect to $\mathfrak{T}_0$.

In fact, let $W \subset \mathrm{Spec}_\ell R$ be closed in the topology $\mathfrak{T}_0$; i.e. $[p' \in \mathrm{Spec}_\ell R$ and $p \to p'$ for some $p \in W] \Rightarrow [p' \in W]$. Then $\overline{\varphi_\ell^{-1} W}$ coincides with the set itself:

$[\{p', p\} \subset \mathcal{U}_\ell(\varphi(R)),\ p \to p',\ \varphi^{-1}p \in W] \Rightarrow [\varphi^{-1}p \to \varphi^{-1}p',$
$\varphi^{-1}p \in W] \Rightarrow [\varphi^{-1}p' \in W].$

3) φ_ℓ is continuous with respect to $\mathfrak{T}$. More exactly, $\varphi_\ell^{-1}(V_\ell(\alpha)) = V_\ell(\alpha_\varphi) \cap U_\ell(\varphi(R))$ for any $\alpha \in IR$, where $\alpha_\varphi = (R', \varphi(\alpha), R')$ is the two-sided ideal generated by $\varphi(\alpha)$.

In fact,

$$\varphi_\ell^{-1}(V_\ell(\alpha)) = \{p' \in U_\ell(\varphi(R)) \mid \alpha \subset \varphi^{-1}p'\} = \{p' \mid (R', \varphi(\alpha)) \subset p'\}.$$

Since φ is a morphism from $\widetilde{\mathrm{Rings}}_\ell$, then

$$[\alpha \subset \varphi^{-1}p,\ p \to p',\ p' \in \mathcal{S}pec_\ell R'] \Rightarrow [\alpha \to \varphi^{-1}p'] \Leftrightarrow [\alpha \subset \varphi^{-1}p'] \Leftrightarrow$$
$$\Leftrightarrow [(R', \varphi(\alpha)) \subset p'].$$

In particular, if $p \in \widehat{\mathcal{S}pec}_\ell R'$, then $[\alpha \subset \varphi^{-1}p] \Rightarrow$ $\Rightarrow [(R', \varphi(\alpha)) \subset (p:z)$ for all $z \in \mathfrak{I}(R')] \Rightarrow$ $\Rightarrow [(R', \varphi(\alpha), R') \subset p]$. In other words, $\varphi_\ell^{-1}(\hat{V}_\ell(\alpha)) =$ $= \hat{V}_\ell(\alpha_\varphi) \cap U_\ell(\varphi(R))$. This and Proposition 1.6 imply the required identity $\varphi_\ell^{-1}(V_\ell(\alpha)) = V_\ell(\alpha_\varphi) \cap U_\ell(\varphi(R))$.

5. Some subcategories of $\widetilde{\mathrm{Rings}}_\ell$.

5.1. Denote by SRings the family of ring morphisms $\varphi : R \to R'$ such that

There exists a finite chain $R_0 \subset R_1 \subset \ldots \subset R_{k+1}$ of subrings R' such that R_i is a two-sided ideal in R_{i+1}, $0 \le i \le k$, $R_0 = \varphi(R)$ and $R_{k+1} = R'$.

It is easy to verify that the composition of morphisms from Srings belongs to SRings and therefore SRings is a subcategory of Rings.

Proposition. SRings is a subcategory of $\widetilde{\mathrm{Rings}}_\ell$.

Proof. Clearly, it suffices to verify that ring epimorphisms and embeddings of two-sided ideals belong to Rings. We have already verified the latter in heading b) of the proof of Proposition 5.9.

Let $\varphi: R \longrightarrow R'$ be an epimorphism $\{m, n\} \subset I_\ell R'$ and $m \rightarrow n$. If $(m:\bar{x}) \subset n$ for some $\bar{x} \in \mathcal{P}(R')$ and x is an element of $\mathcal{P}(R)$ such that $\bar{x} = \varphi(x)$, whose existence is guaranteed by epimorphicy of φ, then $(\varphi^{-1}m : x) = \varphi^{-1}(m:\bar{x}) \subset \varphi^{-1}n$. Therefore the map $\mu \mapsto \varphi^{-1}\mu$ is a functor $I_\ell^{\succ} R' \rightarrow I_\ell^{\succ} R$ and therefore $\varphi \in \mathrm{Hom}\, \widetilde{\mathrm{Rings}}_\ell$.

5.2. Category Rings_ℓ and left normal morphisms. Denote Rings_ℓ the subcategory of **R**ings formed by all the ring morphisms $\varphi: R \longrightarrow R'$ such that $m \longmapsto \varphi^{-1}m$ is a functor $I_\ell^{\succ} R' \longrightarrow I_\ell^{\succ} R$.

As we have just verified, all the ring epimorphisms belong to Rings_ℓ. Let us give much more subtle "estimate from below" of this category.

For an arbitrary ring morphism $\varphi: R \rightarrow R'$ set $\mathcal{N}_\ell(\varphi) = \{z \in R' \mid \varphi(x) z \in (R', \varphi(x)) \text{ for any } x \in R\}$.

A morphism $\varphi: R \rightarrow R'$ will be called left normal if $\varphi(R)$ and $\mathcal{N}_\ell(\varphi)$ generate R'.

It is not difficult to verify that left normal morphisms form a subcategory of **R**ings which we will denote by N_ℓ Rings.

Proposition. $\mathcal{N}_\ell$ Rings is a subcategory of Rings_ℓ.

Proof. Let $\varphi: R \rightarrow R'$ be an arrow from $\mathcal{N}_\ell\,\mathrm{Rings}$; $\{n, m\} \subset I_\ell R'$ and $(n:y) \subset m$ for some $y \in \mathcal{P}(R)$. Let us show that there exists a finite subset $w \subset R$ such that either $\varphi^{-1}n \subset \varphi^{-1}m$ or $(\varphi^{-1}n : w) \subset \varphi^{-1}m$.

1) First notice that $\varphi^{-1}\nu \subset \varphi^{-1}(\nu : z)$ for any $z \in \mathcal{N}_\ell(\varphi)$ and any left ideal ν of R'.

2) Suppose that a Z-module y is generated (over Z) by an element $u \in R'$ and consider different possibilities.

a) If $u \in \mathcal{N}_\ell(\varphi)$, then 1) implies $\varphi^{-1} n \subset \varphi^{-1} m$.

b) If $u = \varphi(x) z$ where $z \in \mathcal{N}_\ell(\varphi)$, then $(n : u) = ((n : z) : \varphi(x))$ and therefore

$$\varphi^{-1}(n : u) = \varphi^{-1}((n : z) : \varphi(x)) = (\varphi^{-1}(n : z) : x) \supset (\varphi^{-1} n : x).$$

c) If $u = z \varphi(x)$ where $z \in \mathcal{N}_\ell(\varphi)$, then

$$\varphi^{-1}(n : u) = \varphi^{-1}((n : \varphi(x)) : z) \supset \varphi^{-1}(n : \varphi(x)) = (\varphi^{-1} n : x).$$

d) Possibilities (b) and (c) and the standard induction yield that if u is the product of several elements of the form $\varphi(x_1), \ldots, \varphi(x_k)$ by elements $z_1, \ldots, z_r$ from $\mathcal{N}_\ell(\varphi)$ (the factors are arranged in an arbitrary order), then $\varphi^{-1}(n : u) \supset (\varphi^{-1} n : x_{i_1} \ldots x_{i_k})$ where $i_1, \ldots, i_k$ are numbers of factors in the order of appearing of $\varphi(x_j)$ in the expression of u (from left to right). Extend φ up to a morphism $\varphi_1 : R^{(1)} \longrightarrow R'^{(1)}$.

(e) Since φ_1 is a morphism from $\mathcal{N}_\ell$ Rings, then any element $u \in R'$ is of the form $u_1 + \ldots + u_s$ where each summand is the product of elements from $\varphi(R^{(1)})$ by elements from $\mathcal{N}_\ell(\varphi_1)$. Therefore for every u_i there exists according to (d) an element $x_i \in R^{(1)}$ such that $(\varphi_1^{-1} n : x_i) \subset \varphi_1^{-1}(n : u_i)$. Therefore

$$\varphi_1^{-1}(n : u) \supset \bigcap_{1 \le i \le s} \varphi_1^{-1}(n : u_i) \supset \bigcap_{1 \le i \le s} (\varphi_1^{-1} n : x_i) = (\varphi_1^{-1} n : \{x_i \mid 1 \le i \le s\}).$$

3) This clearly implies that for any finite family u of generators of the Z-module y there exists a finite subset x of elements from R such that $\varphi^{-1} n \cap (\varphi^{-1} n : x) \subset \varphi^{-1}(n : u)$. $\square$

5.3. A ring morphism $\varphi: R \longrightarrow R'$ is called a central extension if its centralizer $Z(\varphi) \overset{def}{=} \{z \in R' \mid \varphi(x)z = z\varphi(x)$ for every $x \in R\}$ and $\varphi(R)$ generate R'.

Clearly the central extensions form a subcategory of $\mathcal{N}_\ell \mathcal{R}ings$.

Besides, the map $p \mapsto \varphi^{-1}p$ induces a continuous map ${}^a\varphi: \mathcal{S}pec\,R' \to \mathcal{S}pec\,R$, if φ is a central extension.

In fact, if $p' \in \mathcal{S}pec\,R'$ and $\{\alpha, \beta\} \subset IR$, then

$[\alpha\beta \subset \varphi^{-1}p'] \Leftrightarrow [\varphi(\alpha)\varphi(\beta) \subset p'] \Rightarrow$

$\Rightarrow [\varphi(\alpha)\varphi(\beta) + \varphi(\alpha)\varphi(R)Z(\varphi)\varphi(\beta) = (\varphi(\alpha), R')\varphi(\beta) \subset p'] \Rightarrow$

$\Rightarrow$ [either $\varphi(\alpha) \subset p'$ or $\varphi(\beta) \subset p'$].

The verification of the identities ${}^a\varphi^{-1}(V(\alpha)) = V(\alpha_\varphi)$, $\alpha_\varphi = (R', \varphi(\alpha), R')$, for any $\alpha \in IR$ is left to the reader. □

6. Morphisms preserving $\overline{\mathcal{S}pec}\,R$. Denote by $\overline{\mathcal{R}ings}_\ell$ the subcategory of rings formed by all the morphisms $\varphi: R \to R'$ such that $rad_\ell(\varphi^{-1}n) \subset \varphi^{-1}rad_\ell(n)$ for any $n \in I_\ell R'$.

Proposition. 1) $\widetilde{\mathcal{R}ings}_\ell \subset \overline{\mathcal{R}ings}_\ell$.

2) For any morphism $\varphi: R \longrightarrow R'$ from $\overline{\text{Rings}}_\ell$ the map $m \mapsto \varphi^{-1}m$ induces a continuous map

$$\bar{\varphi} : \overline{U}(\varphi(R)) \longrightarrow \overline{\mathcal{S}pec}\,R .$$

Proof. 1) By Proposition 4 $m \longmapsto \varphi^{-1}m$ sends $\mathcal{S}pec_\ell R'$ into $\mathcal{S}pec_\ell R \cup \{R\}$ and, in particular, $\varphi^{-1}V_\ell(n) \subset V_\ell(\varphi^{-1}n) \cup \{R\}$ for any $n \in I_\ell R'$ when $\varphi \in \mathrm{Hom}\,\widetilde{\mathcal{R}ings}_\ell$. Therefore $\varphi^{-1}rad_\ell(n) = \bigcap\{\varphi^{-1}p' \mid p' \in V_\ell(n)\} \supset \bigcap\{p \mid p \in V_\ell(\varphi^{-1}n)\} \overset{def}{=} rad_\ell(\varphi^{-1}n)$.

2) Now let $\varphi : R \to R'$ be an arrow from $\overline{Rings}_\ell$.

(a) $rad_\ell(\varphi^{-1}p) = (\varphi^{-1}p)_s = \varphi^{-1} rad_\ell(p)$ for any $p \in Spec_\ell R'$. In fact, $\varphi^{-1}(p_s) = \varphi^{-1}p \cap \varphi^{-1}(p : R') \subset$
$\subset \varphi^{-1}p \cap \varphi^{-1}(p : \varphi(R)) = \varphi^{-1}p \cap (\varphi^{-1}p : R) \overset{def}{=} (\varphi^{-1}p)_s$.

On the other hand, $(\varphi^{-1}p)_s \subset \varphi^{-1}(p_s)$, since $p_s = rad_\ell(p)$, $(\varphi^{-1}p)_s \subset rad_\ell(\varphi^{-1}p)$ and by hypothesis

$$rad_\ell(\varphi^{-1}p) \subset \varphi^{-1} rad_\ell(p).$$

b) It follows from (a) that $\varphi^{-1}p = rad_\ell(\varphi^{-1}p)$ for any $p \in \overline{Spec}\ R$.

It remains to show that the ideal $\varphi^{-1}p$ of R is prime for any $p \in \overline{Spec}\ R$; i.e. $\alpha \subset (\varphi^{-1}p : x)$ implies $\alpha \subset \varphi^{-1}p$ for any $\alpha \in IR$ and $x \in R \setminus \varphi^{-1}p$.

Since α is a two-sided ideal, then
$[\alpha \subset (\varphi^{-1}p : x) = \varphi^{-1}(p : \varphi(x))] \Rightarrow [\alpha \subset rad_\ell(\varphi^{-1}(p : \varphi(x))) =$
$= \varphi^{-1} rad_\ell((p : \varphi(x))) = \varphi^{-1}(p : (R', \varphi(x)))]$.

This means that $\alpha_\varphi \overset{def}{=} (R', \varphi(\alpha), R') \subset (p : \varphi(x))$. Since by hypothesis $\varphi(x) \notin p$ and p is prime then $\alpha_\varphi \subset p$ and therefore $\alpha \subset \varphi^{-1}p$ as required.

3) Clearly $\bar\varphi : \overline{U}(\varphi(R)) \longrightarrow \overline{Spec}\ R$ is continuous, since

$$\bar\varphi^{-1}(\overline{V}(\alpha)) = \{p \in \overline{U}(\varphi(R)) \mid \alpha \subset \varphi^{-1}p\} = \overline{V}(\alpha_\varphi) \cap \overline{U}(\varphi(R))$$

for any $\alpha \in IR$. □

7. <u>The categories $\widetilde{Rings}^1_\ell$ and $\overline{Rings}^1_\ell$</u>. For any ring morphism $\varphi : R \to R'$ and left ideal n of R denote by $K_{\varphi,n}$ the kernel of the natural morphism $R' \otimes_R n \longrightarrow R'$.

<u>Proposition</u>. <u>Consider the following properties</u> of $\varphi : R \to R'$

($\tilde{a}$) $\varphi \in \mathrm{Hom}\,\widetilde{\mathrm{Rings}}_\ell$ and $K_{\varphi,n} = \mathfrak{F}_p K_{\varphi,n}$ for any $p \in \mathrm{Spec}_\ell R'$ and $n \in \mathfrak{F}_{\varphi^{-1}p}$;

($\tilde{b}$) for any $p \in \mathrm{Spec}_\ell R'$ the functor φ_* of "restricting of scalars" is a morphism

$$(R'\text{-mod}, k_{\mathfrak{F}_p}) \longrightarrow (R\text{-mod}, k_{\mathfrak{F}_{\varphi^{-1}p}}) ;$$

($\tilde{c}$) for any closed subset $W \subset (\mathrm{Spec}_\ell R, \mathfrak{T}_o)$ the functor φ_* defines a morphism

$$(R'\text{-mod}, k_{\mathfrak{F}_W}) \longrightarrow (R\text{-mod}, k_{\mathfrak{F}_{\varphi^{-1}W}}) ;$$

($\bar{a}$) $\varphi \in \mathrm{Hom}\,\overline{\mathrm{Rings}}_\ell$ and $\mathrm{rad}_\ell(n) \subset \varphi^{-1}\mathrm{rad}_\ell(\mathrm{Ann}\,\xi)$ for any $n \in I_\ell R$ and $\xi \in K_{\varphi,n}$;

($\bar{b}$) for any $p \in \mathrm{Spec}_\ell R$ the functor φ_* determines a morphism $(R'\text{-mod}, k_{\mathfrak{F}(p_s)}) \to (R\text{-mod}, k_{\mathfrak{F}(\varphi^{-1}p_s)})$ where $\mathfrak{F}(p') = \{ n \in I_\ell R' \mid \mu_s \notin p' \text{ for any } \mu \in V_\ell(n)\}$ (5.5.e).

($\bar{c}$) φ_* determines a morphism $(R'\text{-mod}, k_{\mathfrak{F}_{V_\ell(\alpha_\varphi)}}) \to$ $\to (R\text{-mod}, k_{\mathfrak{F}_{V_\ell(\alpha)}})$ for any $\alpha \in IR$.

The following implications hold:

$$(\tilde{c}) \Leftrightarrow (\tilde{b}) \Leftrightarrow (\tilde{a}) \Rightarrow (\bar{a}) \Leftrightarrow (\bar{b}) \Leftrightarrow (\bar{c}).$$

Proof. ($\tilde{a}$)$\Longleftrightarrow$($\tilde{b}$). Clearly, "$\varphi \in \mathrm{Hom}\,\widetilde{\mathrm{Rings}}_\ell$" may be expressed in the form

(ч) $[n \in I_\ell R', \varphi^{-1}n \in \mathfrak{F}_{\varphi^{-1}p}] \Rightarrow [n \in \mathfrak{F}_p]$ for any $p \in \mathrm{Spec}_\ell R'$.

This condition is a specialization of the condition ($\gamma 1$) from Remark to Corollary 1 of Proposition 2 for the filters $\mathfrak{F}_p$, $\mathfrak{F}_{\varphi^{-1}p}$ and all $p \in \mathrm{Spec}_\ell R'$. The identities $K_{\varphi,n} = \mathfrak{F}_p K_{\varphi,n}$ for all $p \in \mathrm{Spec}_\ell R'$ and $n \in \mathfrak{F}_{\varphi^{-1}p}$ are the corresponding family of specializations of condition ($\delta 1$) from the same remark. Therefore, Corollary 1 of Proposition 2 implies that ($\tilde{a}$)$\Leftrightarrow$($\tilde{b}$).

$(\tilde{a}) \Longleftrightarrow (\tilde{c})$. Clearly, (4) implies the following statement

$$[\varphi^{-1}n \in \mathcal{F}_{\varphi^{-1}W} \overset{def}{=} \cap\{\mathcal{F}_{\varphi^{-1}p} \mid p \in W^{\perp}\}] \Rightarrow [n \in \mathcal{F}_W]$$

for any $W \subset \mathrm{Spec}_\ell R'$ and $n \in I_\ell R'$.

If $n \in \mathcal{F}_{\varphi^{-1}p}$, then $\mathcal{F}_p K_{\varphi,n} = K_{\varphi,n}$ for all $p \in W$ by $(\tilde{a})$, and therefore $\mathcal{F}_W K_{\varphi,n} = K_{\varphi,n}$.

Now obviously $(\tilde{a}) \Longleftrightarrow (\tilde{c})$ follows from Corollary 1 of Proposition 2 in the same way as in the above step of the proof.

$(\bar{a}) \Longleftrightarrow (\bar{c})$.

Clearly, the condition $rad_\ell(\varphi^{-1}n) \subset \varphi^{-1}rad_\ell(n)$ for all $n \in I_\ell R'$ (the fact that $\varphi \in \overline{Rings}_\ell$) is equivalent to the condition that, if $\alpha \in IR$, $n \in I_\ell R'$ and $\varphi^{-1}n \in \mathcal{F}_{V_\ell(\alpha)}$, then $n \in \mathcal{F}_{V_\ell(\alpha)}$.

Similarly, the second half of $(\bar{a})$ allows the following reformulation:

$$\mathcal{F}_{V_\ell(\alpha_\varphi)} K_{\varphi,n} = K_{\varphi,n} \qquad \text{for any } \alpha \in IR$$

and $n \in \mathcal{F}_{V_\ell(\alpha)}$.

Therefore $(\bar{a})$ is equivalent to

$(\bar{c})$ φ_* determines a morphism $(R'\text{-}mod, k_{\mathcal{F}_{V_\ell(\alpha_\varphi)}}) \to$ $\to (R\text{-}mod, k_{\mathcal{F}_{V_\ell(\alpha)}})$ for any $\alpha \in IR$.

$(\tilde{c}) \Longrightarrow (c)$. Obvious.

$(\bar{c}) \Longleftrightarrow (\bar{b})$. As we showed in 5.5.C, $\mathcal{F}_{(p)} = \cup\{\mathcal{F}_{V_\ell(\alpha)} \mid \alpha \not\subset p\}$. On the other hand, $\mathcal{F}_{V_\ell(\alpha)} = \cap\{\mathcal{F}_{(p_s)} \mid p \in U_\ell(\alpha)\}$ for any $\alpha \in IR$. Therefore $(\bar{c})$, $(\bar{b})$ imply similar conditions relating $\mathcal{F}_{(p_s)}$, $\mathcal{F}_{\varphi^{-1}p_s}$ and φ. $\square$

Morphisms from Rings satisfying the equivalence conditions $(\tilde{a})$ - $(\tilde{c})$ form clearly a subcategory that we will denote by $\widetilde{\text{Rings}}_\ell^1$. Denote by $\overline{\text{Rings}}_\ell^1$ the subcategory of Rings formed by all the morphisms φ with one of the equivalent properties $(\bar{a})$ - $(\bar{c})$.

Corollary. (1) The map $R \mapsto (\mathrm{Spec}_\ell R, {}^o\mathcal{O}_R)$ extends ~~up~~ to a functor from $\widetilde{\text{Rings}}_\ell^{1\,op}$ into the category of preringed spaces.

(2) The map $R \mapsto (\overline{\mathrm{Spec}} R, \bar{\mathcal{O}}_R^a)$ extends to a functor from $\overline{\text{Rings}}_\ell^{1\,op}$ into the category of ringed spaces.

Proof. The property $(\tilde{c})$ characterizing the arrows from Rings means that

$$(\varphi, \{\mathcal{F}_W \mapsto \mathcal{F}_{\varphi_\ell^{-1} W} \mid W \in \mathfrak{T}_o^R\})$$

is a semischeme morphism $(R, \{\mathcal{F}_W \mid W \in \mathfrak{T}_o^R\}) \to (R', \{\mathcal{F}_{W'} \mid W' \in \mathfrak{T}_o^{R'}\})$.

This obviously implies (1).

Similarly, the property $(\bar{c})$ of the arrows from $\overline{\text{Rings}}_\ell^1$ means that every morphism $\varphi : R \to R'$ from $\overline{\text{Rings}}_\ell^1$ induces of $\amalg$ -semischeme morphism

$$(R, \{\mathcal{F}_V \mid V \in \mathfrak{T}^R\}) \to (R', \{\mathcal{F}_{V'} \mid V' \in \mathfrak{T}^{R'}\})$$

which in turn defines a preringed space morphism $(\bar{\varphi}, \bar{\varphi}^\natural) : (\overline{\mathrm{Spec}}\, R', \bar{\mathcal{O}}_{R'}) \to (\overline{\mathrm{Spec}}\, R, \bar{\mathcal{O}}_R)$. Clearly, the correspondence $\varphi \mapsto (\bar{\varphi}, \bar{\varphi}^\natural)$ is a functor from $\overline{\text{Rings}}_\ell^{1\,op}$ to the category of preringed spaces. In the second heading of Corollary we are speaking about the composition of this functor with the sheafication functor. $\square$

Appendix

§1. Left radical, l-systems and Levitzky's radical.

1. l-systems. Fix an associative ring with unit R. A subset $\mathcal{S}\subset\mathcal{P}(R)$ will be called an l-system if for any $t\in\mathcal{S}$ there exists $a_t\in\mathcal{P}(R)$ such that $\mathcal{S}a_t t\subset\mathcal{S}$, i.e. $t'a_t t\in\mathcal{S}$ for any $t'\in\mathcal{S}$.

Obviously, any multiplicative subset $\mathcal{S}$ of $\mathcal{P}(R)$ (i.e. such that $st\in\mathcal{S}$ for any $(s,t)\in\mathcal{S}\times\mathcal{S}$) is an l-system. Another series of examples of l-systems is provided with the following

Lemma. *A left ideal p of R belongs to* $\mathrm{Spec}_1 R$ *if and only if* $\mathcal{S}_p \overset{\text{def}}{=} \mathcal{P}(R)\setminus\mathcal{P}(p)$ *is an l-system.*

Proof. By definition p belongs to $\widehat{\mathrm{Spec}}_\ell R$ if and only if $(p:s)\to p$ for any $s\in\mathcal{S}_p$. This means exactly that $(p:a_s s)=((p:s):a_s)\subset p$ for some $a_s\in\mathcal{P}(R)$. Clearly,

$$[(p:a_s s)\subset p]\Longleftrightarrow[\mathcal{S}_p a_s s\subset\mathcal{S}_p].\ \square$$

Proposition. *If S is an l-system of R, then* $F_{\mathcal{S}}=\{n\in I_\ell R\mid\mathcal{P}((n:x))\cap\mathcal{S}\neq\emptyset$ *for any* $x\in\mathcal{P}(R)\}$ *is a radical filter.*

Proof. Let $m\in F_{\mathcal{S}}$, $n\in F_{\mathcal{S}}\circ\{m\}$; i.e. $\mathcal{P}((n:y))\cap\mathcal{S}\neq\emptyset$ for any $y\in\mathcal{P}(m)$ and $\mathcal{P}((m:x))\cap\mathcal{S}\neq\emptyset$ for any $x\in\mathcal{P}(R)$. We should demonstrate that $\mathcal{P}((n:x))\cap\mathcal{S}\neq\emptyset$ for any $x\in\mathcal{P}(R)$.

Let s be an element of S such that $sx\in\mathcal{P}(m)$, element of $\mathcal{P}(R)$ such that $\mathcal{S}a_s s\subset\mathcal{S}$. Since $a_s sx\in\mathcal{P}(m)$, there exists $t\in\mathcal{S}$ such that $ta_s sx\subset n$; i.e. $ta_s s\in\mathcal{P}((n:x))$. But by conjecture $ta_s s\in\mathcal{S}$. $\square$

2. Levitzky's radical. A ring R' is locally nilpotent if every finite subset of its elements X generates a nilpotent subring. This means that there exists $N = N(X)>0$

such that $x_1 \cdot \ldots \cdot x_N = 0$ for any $(x_1, \ldots, x_N) \in X \times \ldots \times X$. Ideals are called locally nilpotent if so they are as rings.

The following facts take place ([5], Ch. 8, § 3):

1) A two-sided ideal generated by a left or right locally nilpotent ideal is locally nilpotent;

2) the sum $\mathcal{L}(R)$ of all the locally nilpotent ideals of R is an (obviously, two-sided) nilpotent ideal.

$\mathcal{L}(R)$ is called the Levitzky radical of R.

Proposition. The following properties of a left ideal m of R are equivalent:

(a) [S is an l-system and $S \cap \mathcal{P}(m) \neq \emptyset$] $\Rightarrow$ [$\{0\} \in S$];

(b) S is a multiplicative subset in (R) and $S \cap \mathcal{P}(m) \neq \emptyset$] $\Rightarrow$ [$\{0\} \in S$]

(c) m is locally nilpotent.

Proof. (a) $\Longrightarrow$ (b), Since any multiplicative subset of $\mathcal{P}(R)$ is an l-system.

(b) $\Longrightarrow$ (c). It is not difficult to see that m is locally nilpotent if and only if for any $t \in \mathcal{P}(R)$ there exists N=N(t) such that $t^N = \{0\}$; i.e. $\{0\} \in (t) \overset{def}{=} \{t^k \mid k \geqslant 1\}$.

(c) $\Longrightarrow$ (a). Let S be an l-system and $t \in \mathcal{P}(m) \cap S$. By definition of an l-system there exists $a_t \in \mathcal{P}(R)$ such that $S a_t t \subset S$. In particular, $t a_t t \in S$, $(t a_t t) a_t t \in S, \ldots, t(a_t t)^k \in S$ for all $k \geqslant 1$. Since $a_t t \in \mathcal{P}(m)$ together with t, then, by hypothesis, there exists $k_0 \geqslant 1$ such that $(a_t t)^{k_0} = \{0\}$, Therefore $\{0\} = t(a_t t)^{k_0} \in S$. □

Corollary. A left radical of an arbitrary associative ring contains Levitzky's radical.

Proof. Let m be a left ideal of R such that $m \not\subset \widehat{rad}_\ell(R)$. This means exactly that $m \not\subset p$ or, equivalently, $\mathcal{P}(m) \cap S_p \neq \emptyset$ for some $p \in \widehat{Spec}_\ell R$. If m were locally nilpotent, this would imply (by Proposition 2 and Lemma 1) $\{0\} \in S_p$ which is impossible by definition of S_p. Therefore for an arbitrary left ideal m of R we have $[m \not\subset \widehat{rad}_\ell(R)] \Rightarrow [m \not\subset \mathcal{L}(R)]$; so that $\mathcal{L}(R) \subset \widehat{rad}_\ell(R)$. □

Thus we have improved the estimate from the low for the left radical: have passed from $\mathfrak{b}(R) \subset \widehat{rad}_\ell(R) \subset J(R)$ to $\mathcal{L}(R) \subset \widehat{rad}_\ell(R) \subset J(R)$.

Our nearest aim is to improve the estimate from above.

3. The upper nil-radical. A ring R' is called a nil-ring if every its element is nilpotent. The ideals are called nil-ideals if they are nil-rings. The following fact holds ([5], see ch. 8, § 1):

the sum $K(R)$ of all the two-sided nil-ideals of R is a nil-ideal, the greatest two-sided nil-ideal of R.

K(R) is called the upper nil-radical or the Kethe radical, as by the way all the other radicals involved here, is a torsion (see 5.15); i.e.

a) There is no non-zero two-sided nil-ideals in R/K(R) or, equivalently, K(R/K(R))=0;

b) $f(K(R)) \subset K(f(R))$ for any ring morphism

c) $K(\alpha) = \alpha \cap K(R)$ for any two-sided ideal α of R.

Proposition. Left radical of an arbitrary associative ring is contained in its upper nil-radical.

Proof. Obviously, it suffices to show that $rad_\ell^R(K(R)) = K(R)$ for any R. Let α be a two-sided ideal of R. There exists a natural isomorphism $\widehat{rad}_\ell(R/\alpha) \xrightarrow{\sim} rad_\ell^R(\alpha)/\alpha$ which follows from the bijectivity of the map $V_\ell(\alpha) \longrightarrow \mathrm{Spec}_\ell R/\alpha$, $\mu \mapsto \mu/\alpha$, (Proposition 5.9). Therefore $[rad_\ell^R(\alpha) = \alpha] \Leftrightarrow [\widehat{rad}_\ell(R/\alpha)=0]$ and, in particular, $rad_\ell^R(K(R)) = K(R)$ if and only if $R/K(R)$ is $\widehat{rad}_\ell$-semisimple. Since $\widehat{rad}_\ell$ and K are hereditary with respect to two-sided ideals and, in particular, $\widehat{rad}_\ell(R) = R \cap \widehat{rad}_\ell(R^{(1)})$, $K(R) = R \cap K(R^{(1)})$, we can (and will) assume that R is with unit.

Thus, we should show that the ring with unit $\widetilde{R}=R/K(R)$ is $\widehat{rad}_\ell$-semisimple. The following fact takes place (16, Theorem 6.1.1.):

Theorem. *If* R' *has no non-zero two-sided nil-ideals, then* R'[t] *is semisimple.*

Since $\widehat{rad}_\ell \hookrightarrow J$, this theorem implies that $\widehat{rad}_\ell(\widetilde{R}[t])=0$, where, as above, $\widetilde{R} = R/K(R)$.

Now notice that the natural embedding $R \hookrightarrow R[t]$ is a central extension (9 , Example 3) and, therefore, the map $\mu \longmapsto \mu \cap R$ is a morphism from $I_\ell^{\succ} R[t]$ into $I_\ell^{\succ} R$; and, in particular, the correspondence $\mu \mapsto \mu \cap R$ sends $\widehat{\mathrm{Spec}}_\ell R[t]$ into $\widehat{\mathrm{Spec}}_\ell R$. Therefore $\widehat{rad}_\ell(\widetilde{R}) \subset \widetilde{R} \cap \widehat{rad}_\ell(\widetilde{R}[t]) = 0$. □

4. Left radical and Levitzky's radical. Let us perform the last step. Pass from the estimate $\mathcal{L}(R) \subset \widehat{rad}_\ell(R) \subset$ $\subset K(R)$ to the equality $\widehat{rad}_\ell(R) = \mathcal{L}(R)$. The arguments of the preceding section hint how to perform this.

Consider the polynomial ring $R_\infty = R[t_1, t_2, t\ ,\dots]$ in infinitely many non-commuting variables.

Lemma. If R has no non-zero locally nilpotent ideals, then R_∞ has no non-zero nil-ideals, i.e. $K(R_\infty) = 0$.

Proof. Denote by $\mathcal{N}_f$ the set of finite ordered sets of positive integers; for every $J = \{i_1, \dots, i_k\} \in \mathcal{N}_f$ denote $t_{i_1} \cdot \ldots \cdot t_{i_k}$ by t^J. Suppose $K(R_\infty) \neq 0$ and let $\sum\limits_{J \in E} a_J t^J$ be a non-zero element of $K(R_\infty)$ (here E is a finite subset of $\mathcal{N}_f$). First of all let us show that the subring of R generated by $\{a_J \mid J \in E\}$ is nilpotent.

In fact, by hypothesis, $x(\sum\limits_{J \in E} a_J t^J)$ is a nilpotent element of R_∞ for every $x \in R_\infty$. For x take t_{k_0} where k_0 is the index not encountered among the elements of J, $J \in E$; i.e. consider $\sum\limits_{J \in E} a_J t_{k_0} t^J$ instead of $\sum\limits_{J \in E} a_J t^J$. Then the ν-th power of $\sum\limits_{J \in E} a_J t_{k_0} t^J$ vanishes for some $\nu \geqslant 1$. Since $(\sum\limits_{J \in E} a_J t_{k_0} t^J)^\nu = \sum\limits_{(J_1, \dots, J_\nu) \in E \times \dots \times E} a_{J_1} \cdot \ldots \cdot a_{J_\nu} t_{k_0} t^{J_1} t_{k_0} \dots t_{k_0} t^{J_\nu}$, then $(\sum\limits_J a_J t_{k_0} t^J)^\nu = 0$ if and only if $a_{J_1} \cdot \ldots \cdot a_{J_\nu} = 0$ for all $(J_1, \dots, J_\nu) \in E \times \dots \times E$.

Now let us show that the left ideal, generated by $\{a_J \mid J \in E\}$, is locally nilpotent. In other words, we should show that for any set $\{b_J \mid J \in E\}$ of elements of R the subring generated by $\{a_J, b_J a_J\}_{J \in E}$ is nilpotent.

Select positive integers $k_0, k_J, J \in E$, so as all k_J

were different among themselves and different from k_o, and neither k_o nor any of k_J is encountered among the indices of the sets J' for $J' \in E$. Consider the linear form $t_{k_o} + \sum_{J \in E} b_J t_{k_J}$. By hypothesis $f = (t_{k_o} + \sum_{J \in E} b_J t_{k_J}) \cdot (\sum_{J \in E} a_J t^J)$ is an element of $K(R_\infty)$. As we have just shown, this implies that the set of coefficients of the polynomial

$$f = \sum_{J \in E} a_J t_{k_o} t^J + \sum_{(J, J_1) \in E \times E} b_{J_1} a_J t_{k_{J_1}} t^J$$

generates a nilpotent subring in R. Obviously, if a set of elements of R generates a nilpotent subring, then so does any its subset, in particular, $\{a_J, b_J a_J \mid J \in E\}$. $\square$

Proposition. $\widehat{rad}_\ell(R) = \mathcal{L}(R)$ for any associative ring R.

Proof. Since we have already established that $\mathcal{L}(R) \subset \widehat{rad}_\ell(R)$, it only suffices to verify inverse inclusion. Taking the quotient of R modulo $\mathcal{L}(R)$ with help of Proposition 5.9, we reduce the desired statement to the following one:

If R has no non-zero locally nilpotent ideals, then $\widehat{rad}_\ell(R) = 0$.

Proof of this fact follows the scenario of the proof of Proposition 3 with R[t] being replaced by $R_\infty = R[t_1, t_2, \dots]$.

The natural embedding $R \hookrightarrow R_\infty$ is a central extension and, therefore, the map $\mu \mapsto \mu \cap R$ sends the ideals from $\widehat{Spec}_\ell R_\infty$ into the ideals from $\widehat{Spec}_\ell R$. Therefore $\widehat{rad}_\ell(R) \subset R \cap \widehat{rad}_\ell(R_\infty)$. But $\widehat{rad}_\ell(R_\infty) \subset K(R_\infty)$ by Proposition 3 and, as the above lemma claims, $K(R_\infty) = 0$ if $\mathcal{L}(R) = 0$; and therefore $\widehat{rad}_\ell(R) = 0$. $\square$

Corollary. For any associative ring R the set $\overline{Spec}\,R$ (of the points of the base space of the affine (quasi) scheme of R) consists of all the primary ideals $\mathfrak{p}$ such that $R/\mathfrak{p}$ has no non-zero local^{1y} nilpotent ideals.

Proof. By Proposition 5.9 $\widehat{rad}_\ell(R/\mathfrak{p}) \simeq rad_\ell(\mathfrak{p})/\mathfrak{p}$. If $R/\mathfrak{p}$ has no non-zero local nilpotent ideals, then $\widehat{rad}_\ell(R/\mathfrak{p}) = \mathcal{L}(R/\mathfrak{p}) = 0$ and, therefore, $rad_\ell(\mathfrak{p}) = \mathfrak{p}$. The latter equality is the definition of membership $\mathfrak{p} \in \overline{Spec}\,R$. Conversely, if $\mathfrak{p} \in \overline{Spec}\,R$, then $\mathcal{L}(R/\mathfrak{p}) = \widehat{rad}_\ell(R/\mathfrak{p}) = rad_\ell(\mathfrak{p})/\mathfrak{p} = 0$. $\square$

Now it is the high time to compare the left geometry with the right one. First of all, $\widehat{rad}_\ell(R) = \mathcal{L}(R) = \widehat{rad}_r(R)$ for any associative ring R, where $\widehat{rad}_r(R) = \bigcap\{\mathfrak{p} \mid \mathfrak{p} \in \overline{Spec}_r R\}$ and $\overline{Spec}_r R$ is the obviously defined right spectrum of R; in particular, $rad_\ell^R(\underline{\alpha}) = rad_r^R(\underline{\alpha})$ for any two-sided ideal $\underline{\alpha}$ in R.

This (or Corollaries of Proposition 4) make it clear that $\overline{Spec}_\ell R = \overline{Spec}_r R$. Therefore the difference between left and right (quasi)schemes manifests itself in the structural sheaves, the base spaces are the same.

The corollary of Proposition 4 suggests to call the space $\overline{Spec}_\ell R = \overline{Spec}\,R$ nameless so far the Levitzky spectrum of R.

Finally, we notice with satisfaction that $\widehat{rad}_\ell$-reduced left schemes -- the first pretenders for the role of non-commutative (left) algebraic varieties--are the left schemes $(X, \mathcal{O})$ such that for every $x \in X$ the stalk $\mathcal{O}_x$ of $\mathcal{O}$ at x has no non-zero local^{1y} nilpotent ideals.

§ 2. Coherent sheaves and locally trivial bundles.

In this Section R is an associative ring with unit, and all the modules are unitary.

1. Categories $\{\mathcal{F}\}_{\otimes}$. Localizations of projective modules. Fix a radical filter $\mathcal{F}$ and denote by $\{\mathcal{F}\}_{\otimes}$ a full subcategory of R-mod formed by all the R-modules M such that the canonical morphism $\chi^{\mathcal{F}}_{M}: G_{\mathcal{F}}R \otimes_{R} M \longrightarrow G_{\mathcal{F}}M$ is an isomorphism.

Proposition. 1) Consider the following conditions:

a) there exists a morphism φ of the projective module

P onto M such that $G_{\mathcal{F}}\varphi$ is an epimorphism;

β) for any R-module epimorphism $\psi : \mathcal{N} \longrightarrow M$ the map $G_{\mathcal{F}}\psi$ is epimorphism;

c) the canonical arrow $\chi^{\mathcal{F}}_{M} : G_{\mathcal{F}}R \otimes_R M \longrightarrow G_{\mathcal{F}}M$ is epimorphism;

d) M belongs to the category $\{\mathcal{F}\}_{\otimes}$

These conditions are related as follows:

$$a) \Leftrightarrow b) \Leftarrow c) \Leftarrow d).$$

If M is finitely generated then a), b) and c) are equivalent.

If M is finitely presentable then all the four conditions are equivalent.

If $\mathcal{F}$ is a filter of finite type then a)-d) are equivalent for an arbitrary R-module M.

2) $\{\mathcal{F}\}_{\otimes}$ is closed with respect to finite (and if $\mathcal{F}$ is a filter of finite type then with respect to arbitrary) coproducts and with any module contains all its retracts

(a map $V \xrightarrow{\alpha} M$ is a retraction if there exists $M \xrightarrow{\beta} V$ such that $\beta \circ \alpha = id_M$).

3) $\{\mathcal{F}\}_{\otimes}$ contains all the projective modules of finite type. If $\mathcal{F}$ is of finite type then $\{\mathcal{F}\}_{\otimes}$ contains all the projective modules.

Proof. 1) Clearly, b) $\Rightarrow$ a) and d) $\Rightarrow$ c).

a) $\Longrightarrow$ b). Let $\varphi : P \to M$ be a morphism spoken about in a), $\psi : \mathcal{N} \twoheadrightarrow M$ and arbitrary R-module epimorphism. Since P is projective, there exists a commutative triangle

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr, "\varphi"] & \\ \mathcal{N} \arrow[rr, two heads, "\psi"] & & M \end{tikzcd}$$

Since $G_{\mathcal{F}}\varphi = G_{\mathcal{F}}\psi \circ G_{\mathcal{F}}\gamma$ the epimorphicy of $G_{\mathcal{F}}\varphi$ implies that of $G_{\mathcal{F}}\psi$.

c) $\Longrightarrow$ b). Let $\psi : \mathcal{N} \twoheadrightarrow \mathcal{M}$ be an epimorphism. Then in the commuting diagram

$$\begin{tikzcd} G_{\mathcal{F}}\mathcal{N} \arrow[r, "G_{\mathcal{F}}\psi"] & G_{\mathcal{F}}\mathcal{M} \\ G_{\mathcal{F}}R \otimes_R \mathcal{N} \arrow[u] \arrow[r] \arrow[ur] & G_{\mathcal{F}}R \otimes_R \mathcal{M} \arrow[u] \end{tikzcd}$$

the diagonal is epimorphism thanks to the epimorphicy of $G_{\mathcal{F}}R \otimes_R \mathcal{M} \to G_{\mathcal{F}}\mathcal{M}$. Therefore $G_{\mathcal{F}}\psi$ is also epimorphism.

b) $\Longrightarrow$ c) for a finitely generated M. In fact, in this case there exists an epimorphism of a free module $R^{(I)}$ of finite rank (i.e. $R^{(I)}$ is the direct sum of I copies of R where $\mathrm{Card}(I) < \infty$) onto M. Since $G_{\mathcal{F}}$ commutes with finite coproducts, $G_{\mathcal{F}}R \otimes_R R^{(I)} \simeq (G_{\mathcal{F}}R)^{(I)}$.

Therefore in the commuting diagram

$$\begin{tikzcd} G_{\mathcal{F}}R^{(I)} \arrow[r, two heads] & G_{\mathcal{F}}\mathcal{M} \\ G_{\mathcal{F}}R \otimes_R R^{(I)} \arrow[u, "\wr"] \arrow[r, two heads] \arrow[ur] & G_{\mathcal{F}}R \otimes_R \mathcal{M} \arrow[u] \end{tikzcd}$$

the diagonal is also epimorphism (this time thanks to the epimorphicy of the upper and left arrows) and therefore the right arrow is epimorphic.

c) $\Longrightarrow$ d) for a finitely presentable M. By definition there exists an exact sequence

$$R^{(J)} \to R^{(I)} \to \mathcal{M} \to 0$$

where $R^{(J)}$ and $R^{(I)}$ are free modules of finite rank. Thus we have a commuting diagram

$$\begin{tikzcd}
G_{\mathcal{F}} R^{(\mathcal{J})} \arrow[r] & G_{\mathcal{F}} R^{(I)} \arrow[r] & G_{\mathcal{F}} M \\
G_{\mathcal{F}} R \otimes_R R^{(\mathcal{J})} \arrow[r] \arrow[u, "\wr"] & G_{\mathcal{F}} R \otimes_R R^{(I)} \arrow[r] \arrow[u, "\wr"] & G_{\mathcal{F}} R \otimes_R M \arrow[r] \arrow[u, "\chi_M^{\mathcal{F}}"'] & 0
\end{tikzcd} \qquad (1)$$

with exact horizontal rows and two isomorphic vertical arrows. This clearly implies the monomorphicy of $\chi_M^{\mathcal{F}}$.

If $\mathcal{F}$ is a filter of finite type then $G_{\mathcal{F}}$ commutes with arbitrary coproducts. Therefore for an arbitrary R-module M we can draw diagram (1) with infinite, in general, sets I and $\mathcal{J}$ in which the two left vertical arrows are isomorphisms as earlier. This implies the monomorphicy $\chi_M^{\mathcal{F}}: G_{\mathcal{F}} R \otimes_R M \to G_{\mathcal{F}} M$. Now the equivalence of the conditions a)-d) is obvious.

2) The closedness of $\{\mathcal{F}\}_{\otimes}$ with respect to finite (or arbitrary if $\mathcal{F}$ is a filter of finite type) coproducts follows from the fact that $G_{\mathcal{F}}$ commutes with finite (resp. arbitrary) coproducts. Clearly if $\varkappa: M' \to M$ is a retraction then in the commuting diagram $(\tau \circ \varkappa = id_{M'})$

$$\begin{tikzcd}
G_{\mathcal{F}} M' \arrow[r, tail, "G_{\mathcal{F}} \varkappa"] & G_{\mathcal{F}} M \arrow[r, two heads, "G_{\mathcal{F}} \tau"] & G_{\mathcal{F}} M' \\
G_{\mathcal{F}} R \otimes_R M' \arrow[r, tail] \arrow[u, "\chi_{M'}^{\mathcal{F}}"] \arrow[ur, "e_1"] & G_{\mathcal{F}} R \otimes_R M \arrow[r, two heads] \arrow[u, "\chi_M^{\mathcal{F}}"'] \arrow[ur, "e_2"] & G_{\mathcal{F}} R \otimes_R M' \arrow[u, "\chi_{M'}^{\mathcal{F}}"']
\end{tikzcd}$$

e_1 is monomorphism and e_2 is epimorphism. The monomorphicy of e_1 implies the monomorphicy of $\chi_{M'}^{\mathcal{F}}$, and the epimorphicy of e_2 implies the epimorphicy of $\chi_{M'}^{\mathcal{F}}$.

3) follows from 2) and the fact that R belongs to $\{\mathcal{F}\}_{\otimes}$ which we have already used during the proof. $\square$

During these simple arguments we have established two facts deserving to be mentioned specially (see implication c) $\Longrightarrow$ d)):

1) if M is finitely presentable, then for any radical filter $\mathcal{F}$ the canonical morphism $\chi_M^{\mathcal{F}}: G_{\mathcal{F}}R\otimes_R M\to$ $\to G_{\mathcal{F}}M$ is monomorphism;

2) if a radical filter $\mathcal{F}$ is of finite type, then $\chi_M^{\mathcal{F}}$ is monomorphism for any M.

These facts imply

Corollary. *Let* $\mathcal{F}$ *be a radical filter and* M *an R-module.*

1) *If* $\varphi: M'\to M$ *is an* R-*module monomorphism and* M' *is finitely presentable then* $1_{G_{\mathcal{F}}R}\otimes_R\varphi$ *is also monomorphism.*

2) *If* $\mathcal{F}$ *is of finite type then* $G_{\mathcal{F}}R$ *is a flat right* R-*module.*

Proof. In fact, let $\varphi: M'\to M$ be an R-module monomorphism. Consider the commuting diagram

$$\begin{array}{ccc} G_{\mathcal{F}}M' & \xrightarrow{G_{\mathcal{F}}\varphi} & G_{\mathcal{F}}M \\ \chi_{M'}^{\mathcal{F}}\uparrow & 1_{G_{\mathcal{F}}R}\otimes_R\varphi & \uparrow\chi_M^{\mathcal{F}} \\ G_{\mathcal{F}}R\otimes_R M' & \longrightarrow & G_{\mathcal{F}}R\otimes_R M \end{array} \qquad (2)$$

Since $G_{\mathcal{F}}$ is left exact, $G_{\mathcal{F}}\varphi$ is monomorphism. If M' is finitely presentable or $\mathcal{F}$ is of finite type, then $\chi_{M'}^{\mathcal{F}}$ is monomorphism. The monomorphicy of $\chi_{M'}^{\mathcal{F}}$ and $G_{\mathcal{F}}\varphi$ and the commutativity of (2) implies the monomorphicy of $1_{G_{\mathcal{F}}R}\otimes_R\varphi$. □

2. Local and global properties of finiteness. For any family $\mathcal{T}$ of radical filters denote by $\mathcal{T}_{\otimes}$ the intersection of the categories $\{\mathcal{F}\}_{\otimes}, \mathcal{F}\in\mathcal{T}$, and by $\mathcal{T}_{\otimes}^{epi}$ the full subcategory of the category of left R-modules formed by the modules M such that $\chi_M^{\mathcal{F}}: G_{\mathcal{F}}R\otimes_R M\to$

$\longrightarrow G_{\mathcal{F}}M$ is epimorphism for $\mathcal{F} \in \mathcal{T}$.

Proposition. Let $\mathcal{T}$ be a finite family of radical filters of left ideals of R such that $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$ and M a left R-module.

1) Suppose every $G_{\mathcal{F}}M$, $\mathcal{F} \in \mathcal{T}$, is a $G_{\mathcal{F}}R$-module of finite type and one of the following conditions holds:

(i) All the filters of $\mathcal{T}$ are of finite type;

(ii) M belongs to $\mathcal{T}_{\otimes}^{epi}$.

Then M is an R-module of finite type.

2). If $M \in Ob\, \mathcal{T}_{\otimes}$ then M is a finitely presentable module if and only if $G_{\mathcal{F}}R$ -module $G_{\mathcal{F}}M$ is finitely presentable for every $\mathcal{F} \in \mathcal{T}$.

Proof. 1) Let $\{N_\alpha \mid \alpha \in \mathfrak{A}\}$ be an ordered family of submodules M such that $\bigcup\{N_\alpha \mid \alpha \in \mathfrak{A}\} = M$.

(i) If all the filters of $\mathcal{T}$ are of finite type then by 2. $G_{\mathcal{F}}M = \bigcup\{G_{\mathcal{F}}N_\alpha \mid \alpha \in \mathfrak{A}\}$ for every $\mathcal{F}$ of $\mathcal{T}$. Since $G_{\mathcal{F}}M$ is a $G_{\mathcal{F}}R$ - module of finite type then $G_{\mathcal{F}}M = G_{\mathcal{F}}N_{\alpha_{\mathcal{F}}}$ for some $\alpha_{\mathcal{F}} \in \mathfrak{A}$. Making use of the finiteness of $\mathcal{T}$ we can choose $\alpha_0 \in \mathfrak{A}$ that majorizes all $\alpha_{\mathcal{F}}, \mathcal{F} \in \mathcal{T}$. Then $G_{\mathcal{F}}M = G_{\mathcal{F}}N_{\alpha_0}$ for all $\mathcal{F} \in \mathcal{T}$. The identity $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$ and Theorem 4.1 immediately imply that $N_{\alpha_0} = M$ (which is also easy to prove directly).

(ii) Now suppose that (ii) holds and φ is an epimorphism of a free R-module $R^{(I)}$ onto M. Present $R^{(I)}$ as the union of an directed family $\{\mathcal{L}_\alpha \mid \alpha \in \mathfrak{A}\}$ of free submodules of finite ring. Let $\mathcal{N}_\alpha = \varphi(\mathcal{L}_\alpha)$ be the image of $\mathcal{L}_\alpha$ in M for any $\alpha \in \mathfrak{A}$. Fix $\mathcal{F} \in \mathcal{T}$.

Since $(G_{\mathcal{F}} R)^{(T)} = \bigcup \{ G_{\mathcal{F}} \mathcal{L}_\alpha \mid \alpha \in \mathfrak{A} \}$ and $\chi_{\mathcal{M}}^{\mathcal{F}}$ is an epimorphism by hypothesis $G_{\mathcal{F}}\mathcal{M} = \bigcup \{ G_{\mathcal{F}} \varphi (G_{\mathcal{F}} \mathcal{L}_\alpha) \mid \alpha \in \mathfrak{A} \}$. Since $G_{\mathcal{F}} M$ is a $G_{\mathcal{F}} R$ -module of finite type there exists $\alpha_{\mathcal{F}} \in \mathfrak{A}$ such that $G_{\mathcal{F}} \varphi | G_{\mathcal{F}} \mathcal{L}_{\alpha_{\mathcal{F}}}$ is an epimorphism. And so it is for any $\mathcal{F} \in \mathcal{T}$. Choose some α_0 majorizing all $\alpha_{\mathcal{F}}$ and denote by φ_{α_0} the restriction of the epimorphism φ onto $\mathcal{L}_{\alpha_0}$.
The epimorphicy of $G_{\mathcal{F}} \varphi_{\alpha_0}$ in the commuting diagram

$$\begin{tikzcd} G_{\mathcal{F}} \mathcal{L}_{\alpha_0} \arrow[r] \arrow[dr, "G_{\mathcal{F}} \varphi_{\alpha_0}"'] & G_{\mathcal{F}} \mathcal{N}_{\alpha_0} \arrow[d] \\ & G_{\mathcal{F}} M \end{tikzcd}$$

implies the epimorphicy and hence isomorphicy of the embedding $G_{\mathcal{F}} \mathcal{N}_{\alpha_0} \to G_{\mathcal{F}} M$. Therefore $G_{\mathcal{F}} \mathcal{N}_{\alpha_0} = G_{\mathcal{F}} M$ for any $\mathcal{F} \in \mathcal{T}$ and therefore (see the last sentence of the heading (i) of the proof) $\mathcal{M} = \mathcal{N}_{\alpha_0}$ i.e. φ_{α_0} is epimorphism.

2.1) Clearly, the finite presentability of an R-module M and the fact that M belongs to $\{\mathcal{F}\}_\otimes$ imply the finite presentability of the $G_{\mathcal{F}} R$ -module $G_{\mathcal{F}} \mathcal{M}$.

2.2) Now let M be a module of $\mathcal{T}_\otimes$ such that every $G_{\mathcal{F}} R$ -module $G_{\mathcal{F}} M$ is finitely presentable. This in particular implies as we have just proved that M is a module of finite type, i.e. there exists an exact sequence

$$0 \to K \to \mathcal{L} \xrightarrow{\varphi} M \to 0$$

where $\mathcal{L}$ is a free R-module of finite rank.

Recall the following fact (see [3], Ch.I, 2, No.8, Lemma 9).

If A is a ring and E a finitely presentable A-module then for any exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0$$

where G is a module of finite type, the module F is also of finite type.

In particular, the exactness of the sequences

$$0 \longrightarrow G_{\mathcal{F}}\mathcal{K} \longrightarrow G_{\mathcal{F}}\mathcal{L} \xrightarrow{G_{\mathcal{F}}\varphi} G_{\mathcal{F}}\mathcal{M} \longrightarrow 0$$

yields that the $G_{\mathcal{F}}R$-module $G_{\mathcal{F}}\mathcal{K}$ is of finite type for any $\mathcal{F} \in \mathcal{T}$. The exactness of the horizontal line in the commuting diagram

$$\begin{tikzcd}
G_{\mathcal{F}}R \otimes_R \mathcal{K} \arrow[r] \arrow[dr, "\chi^{\mathcal{F}}_{\mathcal{K}}"'] & G_{\mathcal{F}}R \otimes_R \mathcal{L} \arrow[r] \arrow[d, "\wr"] & G_{\mathcal{F}}R \otimes_R \mathcal{M} \arrow[r] \arrow[d] & 0 \\
G_{\mathcal{F}}\mathcal{K} \arrow[ur] & G_{\mathcal{F}}\mathcal{L} \arrow[r] & G_{\mathcal{F}}\mathcal{M} &
\end{tikzcd}$$

yields the epimorphicy of $\chi^{\mathcal{F}}_{\mathcal{K}} : G_{\mathcal{F}}R \otimes_R \mathcal{K} \to G_{\mathcal{F}}\mathcal{K}$. Therefore 1), (ii) of the proved statement implies that the R-module K is of finite type as required. $\square$

3. Discussion. Therefore, the notions of the finiteness of type and of finite representability of modules are invariant with respect to Gabriel functors $G_{\mathcal{F}}$ (when modules belong to $\{\mathcal{F}\}_{\otimes}$) and "local". By of language these notions are "geometric", since without any obstructions could be extended onto (semi)schemes. Is it possible to say the same on projective modules of finite type? More exactly, our question runs as follows:

Let $\mathcal{T}$ be a finite set of radical filters such that $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$, M an R-module of finite type from $\mathcal{T}_{\otimes}$ and $G_{\mathcal{F}} M$ a projective $G_{\mathcal{F}}R$-module for every $\mathcal{F} \in \mathcal{T}$. Is M a projective R-module?

Conjecture. In general, the answer is negative even if $\mathcal{T}$ consists of radical filters of the form

$$\mathcal{F}_{V_\ell(\alpha)} = \bigcap\{\mathcal{F}_p \mid p \in U_\ell(\alpha)\}\ ,\quad \alpha \in IR.$$

As we will presently see, the answer is positive if we confine ourselves to the modules that we will call normal ones.

4. Normal modules. Let $\bar{\xi} = \{\xi_i \mid i \in I\}$ be a set of elements of an R-module M, $\bar{a} = \{a_i \mid i \in I\}$ a set of elements of R. Denote by $R_{\bar{a},\bar{\xi}}$ the set $\{r \in R \mid \sum a_i r e_i = 0\}$. It is easy to verify that $R_{\bar{a},\bar{\xi}}$ is a subring R.

Definitions. 1) A subring A of R will be called left normal if $\sup\{A \cap m_i \mid i \in I\} = A$ for any family $\{m_i \mid i \in I\}$ of left ideals of R such that $\sup\{m_i \mid i \in I\} = R$ (see Example 3.2.4).

2) A set of generators $\bar{\xi} = \{\xi_i \mid i \in I\}$ of a module M will be called *normal* if for any set $\bar{a} = \{a_i \mid i \in I\}$ with finite support such that $\sum_{i \in I} a_i \cdot \xi_i = 0$ the subring $R_{\bar{a},\bar{\xi}}$ is left normal.

3) A module M is called *normal* if it possesses a normal set of generators. □

Example. For every element $a \in R$ denote by ${}_aR$ the set $\{\lambda \in R \mid a\lambda \in Ra\}$. Clearly, ${}_aR$ is a subring of R. If $\bigcap\{{}_{a_i}R \mid i \in I\}$ is a left normal subring for any finite subset $\{a_i \mid i \in I\}$, then any family of generators of an arbitrary module is normal, as is easy to verify. □

This example shows that over the rings that possess some properties of "generalized commutativity" all the modules are normal. The following example is much more essential for us and, therefore, deserves special subsection.

5. Symmetric modules and Artin submodules. Let N be an (R,R)-bimodule. The centre $Z_R(N)$ of N is by definition the set $\{x \in N \mid x.\lambda = \lambda \cdot x$ for every $\lambda \in R\}$. A bimodule M is called an Artin bimodule (in honour of M. Artin) if it is generated as a one-sided module by its centre. Clearly, R is an Artin bimodule. The category $\mathcal{B}i_R$ of Artin R-bimodules is closed with respect to coproducts. In particular, it contains free bimodules -- direct sums of several copies of R.

A set of generators $\bar{\xi} = \{\xi_i \mid i \in I\}$ of an R-module M is *symmetric*, if $\sum_{i \in I} a_i \lambda \xi_i = 0$ for every set $\bar{a} = \{a_i \mid i \in I\} \subset R$ with finite support such that $\sum_{i \in I} a_i \xi_i = 0$, and for all $\lambda \in R$; i.e. $R_{\bar{a},\bar{\xi}} = R$. A module M is symmetric if it possesses a symmetric family of generators.

Therefore all the symmetric modules are normal. Denote by res_2 the functor from the category of (R,R)-bimodules into R - mod forgetting the right R-action.

Proposition. The following conditions on a left R-module M are equivalent.

1) $M = res_2 \widetilde{M}$ for an Artin bimodule $\widetilde{M}$;

2) M is symmetric;

3) there exists an epimorphism of a free module $R^{(I)}$ onto M whose kernel is a subbimodule of the bimodule $R^{(I)}$

Proof. 1) $\Longrightarrow$ 2). Let $\widetilde{M}$ be an Artin R-bimodule, $\bar{\xi}$ a family of generators of the $\mathfrak{Z}(R)$-module $Z_R(\widetilde{M})$. Clearly, $\bar{\xi}$ is a symmetric family of generators of the left module $res_2 \widetilde{M}$.

2) $\Longrightarrow$ 3). Let $\bar{\xi} = \{\xi_i \mid i \in I\}$ be a symmetric family of generators of R-module M; $\varphi_{\bar{\xi}} : R^{(I)} \to M$ an epimorphism sending an element e_i of the canonical basis of the free module $R^{(I)}$ into ξ_i, and $K_{\bar{\xi}}$ the kernel of $\varphi_{\bar{\xi}}$. Due to symmetricity of $\bar{\xi} = \{\xi_i \mid i \in I\}$ we have $[\sum_{i\in I} a_i e_i \in K_{\bar{\xi}}] \Rightarrow [\sum_i a_i r e_i \in K_{\bar{\xi}}$ for any $r \in R]$. This exactly means that $K_{\bar{\xi}}$ is a subbimodule of $R^{(I)}$ (we denote by the same symbol a free bimodule and its image with respect to the forgetting functor res_2).

3) $\Longrightarrow$ 1). Follows immediately from the following fact: if $M \longrightarrow M'$ is a bimodule epimorphism and M

an Artin bimodule, then so is M'. $\square$

6. Flat and locally flat normal modules. Let $\xi = \{\xi_i \mid i \in I\}$ be a family of elements R-module M, φ_ξ the morphism from $R^{(I)}$ into M sending the element e_i of the canonical basis into ξ_i, and K_ξ the kernel of φ_ξ. For any $u = \sum_{i \in I} a_i e_i \in R^{(I)}$ denote by $K_{\xi,u}$ the set of elements of the form $\sum_i a_i r e_i$, $r \in R$, belonging to K_ξ.

Proposition. 1) *If M is a flat R-module, $\mathcal{F}$ a radical filter and $M \in Ob\{\mathcal{F}\}_\otimes$, then $G_\mathcal{F} M$ is a flat $G_\mathcal{F} R$-module.*

2) *Let $\mathcal{T}$ be a family of radical filters such that $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$; M a module from $\mathcal{T}_\otimes^{epi}$ such that*

a) $G_\mathcal{F} M$ is a flat $G_\mathcal{F} R$-module for any $\mathcal{F} \in \mathcal{T}$;

b) there is a normal family of generators $\xi = \{\xi_i \mid i \in I\}$ in M such that for every element $u = \sum_i a_i e_i$ from K_ξ the set $K_{\xi,u}$ is contained in a finitely generated submodule of K_ξ.

Then M is a flat R-module.

Proof. 1) This is a particular case of the well known (see e.g. [3], Ch.1, § 2, No.7, Corollary 2) and very easy to verify fact:

If M is a flat R-module and $R \longrightarrow A$ a morphism of rings with unit, then $A \otimes_R M$ is a flat A-module.

2) As the main tool in the proof of this statement we will make use of the following

Proposition. (Villamayor). *Let* A *be a ring and*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{N} \longrightarrow 0$$

an exact sequence of left A-*modules, where* $\mathcal{L}$ *is free. Then the following statements are equivalent:*

a) $\mathcal{N}$ *is a flat module.*

b) *For any* $u \in \mathcal{K}$ *there exists an* A-*module morphism* $\theta: \mathcal{L} \longrightarrow \mathcal{K}$ *such that* $\theta(u) = u$.

c) *For any* $u_1, \dots, u_n$ *there exists a morphism* $\theta: \mathcal{L} \longrightarrow \mathcal{N}$ *such that* $\theta(u_i) = u_i$ *for* $i = 1, \dots, n$.

Proof see in [2], v.I, Ch.11, No.27. □

Let an R-module N satisfying the conditions of heading 2). We will show that for an arbitrary $u = \sum_i a_i e_i$ there exists a morphism $\theta: R^{(I)} \to K_\xi$ such that $\theta(u) = u$.

$G_{\mathcal{F}} R^{(I)}$ is a submodule of $G_{\mathcal{F}}(R^{(I)})$ for any radical filter $\mathcal{F}$; i.e. $\chi^{\mathcal{F}}_{R^{(I)}}$ is a monomorphism. If $\chi^{\mathcal{F}}_{M}$ is an epimorphism, then, as is clear from the commuting diagram with exact horizontal lines

$$\begin{tikzcd}
 & G_{\mathcal{F}} R \otimes_R K \arrow[r] \arrow[d] & G_{\mathcal{F}} R^{(I)} \arrow[r] \arrow[d, tail] & G_{\mathcal{F}} R \otimes_R M \arrow[r] \arrow[d, "\chi^{\mathcal{F}}_M"] & 0 \\
0 \arrow[r] & G_{\mathcal{F}} K \arrow[r] & G_{\mathcal{F}}(R^{(I)}) \arrow[r, "G_{\mathcal{F}}\varphi_\xi"] & G_{\mathcal{F}} M &
\end{tikzcd},$$

the restriction $G_{\mathcal{F}}\varphi_\xi$ onto $G_{\mathcal{F}} R^{(I)}$ is an epimorphism and, therefore, the sequence

$$0 \longrightarrow G_{\mathcal{F}} K \cap G_{\mathcal{F}} R^{(I)} \longrightarrow G_{\mathcal{F}} R^{(I)} \longrightarrow G_{\mathcal{F}} M \longrightarrow 0$$

is exact. (Notice that $G_{\mathcal{F}} K \cap G_{\mathcal{F}} R^{(I)}$ is the image of $\chi^{\mathcal{F}}_{K}$) . Let $\widetilde{K}_u$ be a finitely generated submodule of $K_{\bar{\xi}}$ containing $K_{\bar{\xi},u}$. By Villamayor criterion ($a) \Longleftrightarrow c)$) there exists an $G_{\mathcal{F}}R$ -module morphism $\theta_{\mathcal{F}} : G_{\mathcal{F}}R^{(I)} \to G_{\mathcal{F}}K \cap G_{\mathcal{F}}R^{(I)}$ for every $\mathcal{F} \in \mathcal{T}$ such that $j_{\mathcal{F}}(u') = \theta_{\mathcal{F}}(j_{\mathcal{F}}(u'))$ for all $u' \in \widetilde{K}_u$ (and, in particular, for all $u' \in K_{\bar{\xi},u}$). Here $j_{\mathcal{F}} = j_{\mathcal{F},R^{(I)}}$ is the canonical morphism $R^{(I)} \longrightarrow G_{\mathcal{F}}(R^{(I)})$. Set $J_u = \{i \in I \mid a_i \neq 0\}$. The morphism $\theta_{\mathcal{F}}$ (as any other morphism from $G_{\mathcal{F}}R^{(I)}$) is uniquely determined by its values on the natural basis $\{\tilde{e}_i = j_{\mathcal{F}}(e_i)\}_{i \in I}$ and we may assume that $v_i^{\mathcal{F}} = \theta_{\mathcal{F}}(\tilde{e}_i)$, when $i \notin J_u$. Since J_u is finite, there exists an ideal $m_{\mathcal{F}} \in \mathcal{F}$ such that every $v_i^{\mathcal{F}}$ is the image of a (uniquely determined) $\tilde{v}_i^{\mathcal{F}} \in Hom_R(m_{\mathcal{F}}, \mathcal{F}^1 K)$ with respect to the coprojection $Hom_R(m_{\mathcal{F}}, \mathcal{F}^1 K) \to G_{\mathcal{F}}K$. The equality $\theta_{\mathcal{F}}(j_{\mathcal{F}}(u')) = j_{\mathcal{F}}(u')$ means exactly that the morphism $\theta_{\mathcal{F}}(j_{\mathcal{F}}(u')) = \sum_i b_i v_i^{\mathcal{F}}$ from $(m_{\mathcal{F}} : \{b_i \mid i \in J_u\}) = \bigcap\{(m_{\mathcal{F}} : b_i) \mid i \in J_u\}$ in $\mathcal{F}^1 K$ sends every x into $x \cdot \varphi^{\mathcal{F}}(u') = \sum_i x b_i \varphi^{\mathcal{F}}(e_i)$, where $\varphi^{\mathcal{F}} = \varphi^{\mathcal{F}}_{R^{(I)}}$ is the canonical morphism $R^{(I)} \longrightarrow \mathcal{F}^1 R^{(I)}$

(ii). For better understanding we first complete the proof in a simplified situation. Namely, suppose that K is an $\mathcal{F}$-torsion free module for all $\mathcal{F} \in \mathcal{T}$ (e.g. R is a left $\mathcal{F}$-torsion free module for all $\mathcal{F} \in \mathcal{T}$). Since $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$, then $\sup\{m_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{T}\} = R$; and since R is a ring with unit, then $\sup\{m_{\mathcal{F}} \mid \mathcal{F} \in \mathcal{T}'\} = R$ for a finite subset $\mathcal{T}' \subset \mathcal{T}$. Thanks to the normality of the set of generators $\bar{\xi} = \{\xi_i \mid i \in I\}$ the subring $R_{\bar{a},\bar{\xi}} = \{\lambda \mid \sum_i a_i \lambda \xi_i = 0\} = \{\lambda \mid \sum_i a_i \lambda e_i \in K\}$

is left normal. In particular, $\sup\{m_{\mathcal{F}} \cap R_{\bar{a},\xi} \mid \mathcal{F} \in \mathcal{T}'\} = R_{\bar{a},\xi}$, or, equivalently, for every $\mathcal{F} \in \mathcal{T}'$ there exists $\lambda_{\mathcal{F}} \in$ $\in m_{\mathcal{F}} \cap R_{\bar{a},\xi}$ such that $\sum_{\mathcal{F}\in\mathcal{T}'} \lambda_{\mathcal{F}} = 1$. Now set $v_i =$ $= \sum_{\mathcal{F}\in\mathcal{T}'} \tilde{v}_i^{\mathcal{F}}(\lambda_{\mathcal{F}})$, and let θ be the map $R^{(I)} \longrightarrow K$ sending $\sum_{i\in I} b_i e_i$ into $\sum_{j\in J_u} b_j v_j$.

The morphism θ preserves $u = \sum a_i e_i$, since $\{\sum_i a_i \lambda_{\mathcal{F}} e_i \mid \mathcal{F} \in \mathcal{T}'\} \subset K_{\xi,u}$ and $(m_{\mathcal{F}} : \{a_i \lambda_{\mathcal{F}} \mid i \in J_u\}) =$ $= R$, and, therefore,

$$\theta(\sum_i a_i e_i) = \sum_i a_i v_i = \sum_{\mathcal{F}\in\mathcal{T}'} \sum_i a_i \tilde{v}_i^{\mathcal{F}}(\lambda_{\mathcal{F}}) = \sum_{\mathcal{F}\in\mathcal{T}'} \sum_i (a_i \lambda_{\mathcal{F}} \tilde{v}_i^{\mathcal{F}})(1) =$$

$$= \sum_{\mathcal{F}\in\mathcal{T}'} \sum_i a_i \lambda_{\mathcal{F}} e_i = \sum_i a_i (\sum_{\mathcal{F}\in\mathcal{T}'} \lambda_{\mathcal{F}}) e_i = \sum_i a_i e_i$$

(iii) General case. First of all, embed $\mathcal{F}^1 K$ into $H_{\mathcal{F}} K = \varinjlim_{m\in\mathcal{F}} \mathrm{Hom}_R(m, K)$. For every $x \in m_{\mathcal{F}}$ there exists an ideal $\nu_x^{\mathcal{F}} \in \mathcal{F}$ such that $\tilde{v}_i^{\mathcal{F}}(x)$ is the image of an R-module morphism $\hat{v}_i^{\mathcal{F}}(x,-) : \nu_x^{\mathcal{F}} \to K$, $z \mapsto \hat{v}_i^{\mathcal{F}}(x,z)$. Clearly, $\nu^{\mathcal{F}} = \sum_{x\in m_{\mathcal{F}}} \nu_x^{\mathcal{F}} x$ is an ideal from $\mathcal{F} \circ \{m_{\mathcal{F}}\}$ and therefore $\nu^{\mathcal{F}} \in \mathcal{F}$. The following copies of implications of step (ii) take place:

$$[\cap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}] \Rightarrow [\sup\{\nu^{\mathcal{F}} \mid \mathcal{F} \in \mathcal{T}'\} = R$$

for a finite subset $\mathcal{T}' \subset \mathcal{T}$] $\Longrightarrow$ [there exists $\lambda_{\mathcal{F}} \in \nu^{\mathcal{F}} \cap$ $\cap R_{\bar{a},\xi}$, $\mathcal{F} \in \mathcal{T}'$, such that $\sum_{\mathcal{F}\in\mathcal{T}'} \lambda_{\mathcal{F}} = 1$]. Every $\lambda_{\mathcal{F}}$ represents (in general non-uniquely) in the form $\sum_{x\in m_{\mathcal{F}}} y_x^{\mathcal{F}} \cdot x$ where $y_x^{\mathcal{F}} \in \nu_x^{\mathcal{F}}$ and $\mathrm{Card}\{x \mid y_x^{\mathcal{F}} \neq 0\} < \infty$. Set $\bar{v}_i = \sum_{\mathcal{F}\in\mathcal{T}'} \sum_{x\in m_{\mathcal{F}}} \hat{v}_i^{\mathcal{F}}(x, y_x^{\mathcal{F}})$ and let θ be the map $R^{(I)} \longrightarrow K$ assigning to every $\sum b_i e_i$ the element $\sum_{j\in J_u} b_j \bar{v}_j$ of K. The verification of the equality

$\Theta(u)=u$ is performed in approximately the same way as in the particular case considered in (ii). The details are left to the reader. $\square$

Corollary 1. (of the proof). *Let* $\mathcal{T}$ *be a family of radical filters such that* $\bigcap\{\mathcal{F} \mid \mathcal{F}\in\mathcal{T}\}=\{R\}$ *and* M *a normal module from* $\mathcal{T}^{epi}_{\otimes}$. *If the* $G_{\mathcal{F}}R$ *-module* $G_{\mathcal{F}}M$ *is projective for every* $\mathcal{F}\in\mathcal{T}$, *then* M *is flat.*

Proof. Select a normal system of generators $\xi=\{\xi_i \mid i\in I\}$ of M and the corresponding to this system exact sequences similar to the considerations of step (i) of the proof of Proposition:

$$0\longrightarrow K_{\xi}\xrightarrow{\ \gamma\ } R^{(I)}\xrightarrow{\ \varphi_{\xi}\ } M\longrightarrow 0$$

$$0\longrightarrow G_{\mathcal{F}}K_{\xi}\cap G_{\mathcal{F}}R^{(I)}\xrightarrow{\ \gamma_{\mathcal{F}}\ } G_{\mathcal{F}}R^{(I)}\longrightarrow G_{\mathcal{F}}M\longrightarrow 0$$

Since $G_{\mathcal{F}}R$ -modules $G_{\mathcal{F}}M$ are projective for $\mathcal{F}\in\mathcal{T}$, there are morphisms $\theta_{\mathcal{F}}: G_{\mathcal{F}}R^{(I)}\to G_{\mathcal{F}}K_{\xi}\cap G_{\mathcal{F}}R^{(I)}$ such that $\theta_{\mathcal{F}}\circ\gamma_{\mathcal{F}}=id$. Now we can repeat the arguments of the main part of the proof word for word. $\square$

A module M will be called finitely connected if there exists an exact sequence

$$0\longrightarrow \mathcal{K}\longrightarrow \mathcal{L}\longrightarrow M\longrightarrow 0$$

where $\mathcal{L}$ is free and $\mathcal{K}$ is finitely generated.

Remark. Unfortunately, this term nice per se is one more homonym. For instance, in [2] by finitely connected and finitely representable modules the same thing is denoted. $\square$

Corollary 2. *Let* $\mathcal{T}$ *be a family of radical filters* such that $\bigcap\{\mathcal{F} \mid \mathcal{F}\in\mathcal{T}\}=\{R\}$ and M a finitely connected normal R-module. Consider the following properties:

1) M is projective;

2) M is projective and belongs to $\mathcal{T}_{\otimes}^{epi}$;

3) M belongs to $\mathcal{T}_{\otimes}^{epi}$ and $G_{\mathcal{F}}M$ is a flat $G_{\mathcal{F}}R$-module for every $\mathcal{F}\in\mathcal{T}$.

The following implications hold: 1) $\Longleftarrow$ 2) $\Longleftrightarrow$ 3)

If M is a module of finite type and all the filters from $\mathcal{T}$ are of finite type, then 1) $\Longleftrightarrow$ 2).

Proof. 2) $\Longrightarrow$ 1) is trivial.

2) $\Longrightarrow$ 3). Every projective module is flat, as is known. Therefore it suffices to show that the projectivity of M and its membership to $\{\mathcal{F}\}_{\otimes}^{epi}$ imply the flatness of the $G_{\mathcal{F}}R$-module $G_{\mathcal{F}}M$.

Let φ be the epimorphism of a free R-module $R^{(I)}$ onto M, and $\theta: M\to R^{(I)}$ its right inverse: $\varphi\circ\theta=id_M$. Consider the commuting (thanks to the functorial property of $\chi_{-}^{\mathcal{F}}$) diagram

$$\begin{tikzcd}
G_{\mathcal{F}}(R^{(I)}) & & G_{\mathcal{F}}M \arrow[ll, "G_{\mathcal{F}}\theta"'] \\
G_{\mathcal{F}}R^{(I)}\simeq G_{\mathcal{F}}R\otimes_R R^{(I)} \arrow[u, "\chi^{\mathcal{F}}_{R^{(I)}}"] & & G_{\mathcal{F}}R\otimes_R M \arrow[ll, "1_{G_{\mathcal{F}}R}\otimes_R\theta"'] \arrow[u, "\chi^{\mathcal{F}}_M"']
\end{tikzcd}$$

We see that since $\chi_M^{\mathcal{F}}$ is an epimorphism, the image of $G_{\mathcal{F}}\theta$ is contained in the free submodule $G_{\mathcal{F}}R^{(I)}$ of $G_{\mathcal{F}}(R^{(I)})$ (we identify $G_{\mathcal{F}}R^{(I)}$ with its image in $G_{\mathcal{F}}(R^{(I)})$). Therefore $G_{\mathcal{F}}M$ is the retract of a free $G_{\mathcal{F}}R$-module.

3) $\Longrightarrow$ 2). It is known ([2], v.1, Ch.11, No.30)

that any finitely connected flat module is projective (this is a direct corollary of the Villiamayor criterion). Since $M \in Ob\, \mathcal{T}_{\otimes}^{epi}$, then the finite connectivity of the R-module M implies that of the $G_{\mathcal{F}}R$ - modules $G_{\mathcal{F}}M$ for all $\mathcal{F} \in \mathcal{T}$. Since by hypothesis $G_{\mathcal{F}}M$ are flat, they are projective and it remains to make use of Corollary 1.

The last statement follows from Proposition 1. □

7. Characterizations of normal projective modules of finite type. Actually we continue the list of corollaries started in the preceding section.

Proposition. *Let M be a normal R-module. The following properties are equivalent:*

1) M *is a projective module of finite type.*

2) M *is a finitely representable module and there exists a family of radical filters* Ω *such that* $\bigcap\{\mathcal{G} \mid \mathcal{G} \in \Omega\} = \{R\}$ *and the* $G_{\mathcal{G}}R$ - *module* $G_{\mathcal{G}}M$ *is flat for all* $\mathcal{G} \in \Omega$.

3) *There exist families of radical filters* Ω *and* $\mathcal{T}$ *such that* $\bigcap\{\mathcal{G} \mid \mathcal{G} \in \Omega\} = \{R\} = \bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\}$, *the family* $\mathcal{T}$ *is finite*, $M \in Ob(\mathcal{T} \cup \Omega)_{\otimes}^{epi}$ *the* $G_{\mathcal{F}}R$ -*module* $G_{\mathcal{F}}M$ *is finitely representable for every* $\mathcal{F} \in \mathcal{T}$ *and the* $G_{\mathcal{F}}R$ -*module* $G_{\mathcal{G}}M$ *is flat for every* $\mathcal{G} \in \Omega$.

4) *There exists a finite family* $\mathcal{T}$ *of radical filters such that* $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$, $M \in Ob\, \mathcal{T}_{\otimes}^{epi}$ *and the* $G_{\mathcal{F}}R$ - *module* $G_{\mathcal{F}}M$ *is projective for every* $\mathcal{F} \in \mathcal{T}$.

Proof. 1) $\Longrightarrow$ 2). It is known ([3] , Ch.1, § 2, No.8, Lemma 8) and easy to verify that any projective module of finite type is finitely representable. By Proposition 1, if M is a projective module of finite type, then $M \in Ob\{\mathfrak{F}\}_{\otimes}$ and $G_{\mathfrak{F}}M$ is projective (and, therefore, flat) $G_{\mathfrak{F}}R$ -module for any radical filter.

2) $\Longrightarrow$ 3). Take $\mathcal{T} = \{\{R\}\}$.

1) $\Longrightarrow$ 4). Similarly.

4) $\Longrightarrow$ 3). Obviously.

3) $\Longrightarrow$ 1). By Proposition 2 applied to M and $\mathcal{T}$ we deduce that M is a finitely representable R-module. It follows from Proposition 6 that M is a flat R-module. As had been noted above, this implies the projectivity of M. □

Corollary 1. The following properties of the normal module M *are equivalent:*

1) M *is a projective module of finite type;*

2) M *belongs to the subcategory* $\{\mathfrak{F}_\mu \mid \mu \in Max_\ell R\}^{epi}_{\otimes}$ *and the* $G_{\mathfrak{F}_\mu}R$ *-module* $G_{\mathfrak{F}_\mu}M$ *is flat for every*

There exists a finite set $\mathcal{T}$ *of radical filters such that* $M \in Ob\,\mathcal{T}^{epi}_{\otimes}$ *and the* $G_{\mathfrak{F}}R$ *-module* $G_{\mathfrak{F}}M$ *is finitely representable for every* $\mathfrak{F} \in \mathcal{T}$.

Proof. The statements of corollary differ from the corresponding statements of the Proposition (1) and 3)) only by specification of Ω . Therefore we are only to verify the triviality of the intersection $\bigcap\{\mathfrak{F}_\mu \mid \mu \in Max_\ell R\}$.

By definition $[\nu \in \bigcap\{\mathcal{F}_\mu \mid \mu \in Max_\ell R\}] \Leftrightarrow [\nu \not\to \mu$ for any $\mu \in Max_\ell R]$. This implication can be extended: $\Rightarrow [\nu \not\subset \mu$ for every $\mu \in Max_\ell R]$. Since R is a ring with unit, there is only one left ideal which does not belong to any maximal left ideal: R itself. $\square$

Corollary 2. Let M be a normal R-module. Suppose there exists a subset $X \subset Spec_\ell^{fl} R$ such that $\bigcap\{\mathcal{F}_p \mid p \in X\} = \{R\}$; then the following conditions are equivalent:

1) M is a projective module of finite type;

2) $G_{\mathcal{F}_p} M$ is a flat $G_{\mathcal{F}_p} R$ -module and for any $p \in X$, there exists a finite family $\mathcal{T}$ of radical filters such that $\bigcap\{\mathcal{G} \mid \mathcal{G} \in \mathcal{T}\} = \{R\}$ and $G_{\mathcal{G}} M$ is a finitely representable $G_{\mathcal{G}} R$ -module for every $\mathcal{G} \in \mathcal{T}$.

Proof. By definition $Spec_\ell^{fl} R$ consists of the points p of the left spectrum such that $\{\mathcal{F}_p\}_\otimes = R\text{-}mod$. Therefore we are under the conditions of Proposition with $\Omega = \{\mathcal{F}_p \mid p \in X\}$. $\square$

Corollary 3. Let R be a left hereditary ring. Then the following properties of a normal R-module M are equivalent:

1) M is a projective module of finite type.

2) $G_{\mathcal{F}_\mu} M$ is a flat $G_{\mathcal{F}_\mu}$ -module for any There exists a final family $\mathcal{T}$ of radical filters such that $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$ and the $G_{\mathcal{F}} R$ -module $G_{\mathcal{F}} M$ is finitely representable for every $\mathcal{F} \in \mathcal{T}$.

3) There exists a finite family $\mathcal{T}$ of radical filters such that $\bigcap\{\mathcal{F} \mid \mathcal{F} \in \mathcal{T}\} = \{R\}$ and $G_{\mathcal{F}} M$ is a

projective module of finite type over $G_{\mathcal{F}} R$ for every $\mathcal{F} \in \mathcal{T}$.

Proof. If R is left hereditary, then $\{\mathcal{F}\}_{\otimes} = R\text{-mod}$ for every radical filter $\mathcal{F}$ (see 2.2, property 6)). The implications 1) $\Longleftrightarrow$ 2) require less; it suffices that $\{\mathcal{F}_{\mu}\}_{\otimes} = R\text{-mod}$ for every $\mu \in Max_{\ell} R$. The implications 1) $\Longleftrightarrow$ 3) follow from the heading 4) of Proposition. $\square$

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