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A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra

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Abstract

Given an n-dimensional Lie algebra $\mathfrak{g}$ over a field $\boldsymbol{k} \supset \mathbb{Q}$, together with its vector space basis $X_1^0, \dots, X_n^0$, we give a formula, depending only on the structure constants, representing the infinitesimal generators, $X_i = X_i^0 t$ in $\mathfrak{g} \otimes_{\boldsymbol{k}} \boldsymbol{k}[[t]]$, where t is a formal variable, as a formal power series in t with coefficients in the Weyl algebra A_n. Actually, the theorem is proved for Lie algebras over arbitrary rings $\boldsymbol{k} \supset \mathbb{Q}$.

We provide three different proofs, each of which is expected to be useful for generalizations. The first proof is obtained by direct calculations with tensors. This involves a number of interesting combinatorial formulas in structure constants. The final step in calculation is a new formula involving Bernoulli numbers and arbitrary derivatives of $\coth(x/2)$. The dimensions of certain spaces of tensors are also calculated. The second method of proof is geometric and reduces to a calculation of formal right-invariant vector fields in specific coordinates, in a (new) variant of formal group scheme theory. The third proof uses coderivations and Hopf algebras.

Keywords: Deformations of algebras; Lie algebras; Weyl algebra; Bernoulli numbers; Representations; Formal schemes

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1. Introduction and the statement of the main theorem

We consider here a remarkable special case of the following problem: given a commutative ring $\boldsymbol{k}$, when may a given associative $\boldsymbol{k}$-algebra U with n generators, say $X_1^0, \dots, X_n^0$, be represented as a formal 1-parameter deformation of a commutative polynomial subalgebra $\boldsymbol{k}[x_1, \dots, x_n]$ of the Weyl algebra $A_{n,\boldsymbol{k}} := \boldsymbol{k}[x_1, \dots, x_n, \partial^1, \dots, \partial^n]/\langle \partial^j x_i - x_i \partial^j - \delta_i^j \rangle$, where the whole deformation is within the Weyl algebra itself. More explicitly, we look for the deformations of the form

$$X_i^0 t = x_i + \sum_{N=1}^{\infty} P_{N,i} t^N, \quad P_{N,i} \in A_{n,\boldsymbol{k}},$$

where t is a deformation parameter. Because the deformation can introduce only infinitesimal noncommutativity, we rescaled the generators X_i^0 by factor t in the very formulation of the problem, i.e. the actual algebra realized as a deformation is U_t, what is the positive degree $U_t = \bigoplus_{i>0} U t^i$ part of $U \otimes_{\boldsymbol{k}} \boldsymbol{k}[[t]]$. Such a deformation, if it exists, does not need to be unique. In this paper we find a universal formula which provides such a deformation when U is the enveloping algebra of any Lie algebra $\mathfrak{g}$ over any unital ring $\boldsymbol{k}$ containing the field $\mathbb{Q}$ of rational numbers. Still, the underlying $\boldsymbol{k}$-module of $\mathfrak{g}$ will be assumed free (only projective when formula given in invariant form) and finitely generated.

Given a basis $X_1^0, \dots, X_n^0$ of a free finite-rank $\boldsymbol{k}$-module underlying a Lie algebra $\mathfrak{g}$, the structure constants $(C^0)_{ij}^k$ of $\mathfrak{g}$ are defined by $[X_i^0, X_j^0] = \sum_{k=1}^n (C^0)_{ij}^k X_k$ and are clearly antisymmetric in the *lower* two indices. As usual, the choice of the basis will be considered as an isomorphism $X^0 : \boldsymbol{k}^n \to \mathfrak{g}$ given by $X^0(e_i) = X_i^0$, where $e_1, \dots, e_n$ is the standard basis of $\boldsymbol{k}^n$.

Let $\mathfrak{g} \otimes_{\boldsymbol{k}} \boldsymbol{k}[[t]] = \bigoplus_{i=0}^{\infty} \mathfrak{g} t^i$ be the Lie algebra $\mathfrak{g}$ but with scalars extended to include formal power series in one variable. Its positive degree part $\mathfrak{g}_t := \bigoplus_{i=1}^{\infty} \mathfrak{g} t^i$ is a Lie subalgebra of $\mathfrak{g} \otimes_{\boldsymbol{k}} \boldsymbol{k}[[t]]$ over $\boldsymbol{k}[[t]]$ with basis $X_1, \dots, X_n$ where $X_i = X_i^0 t$. Then $[X_i, X_j] = C_{ij}^k X_k$ where the new structure constants $C_{ij}^k := (C^0)_{ij}^k t$ are also of degree 1 in t and may be interpreted as infinitesimal. Denote by $\mathcal{U}(\mathfrak{g}_t)$ the universal enveloping $\boldsymbol{k}[[t]]$-algebra of $\mathfrak{g}_t$. It naturally embeds into $(\mathcal{U}(\mathfrak{g}))_t := \bigoplus_{i>0} \mathcal{U}(\mathfrak{g}) t^i$.

Define a matrix $\mathbf{C}$ over $A_{n,\boldsymbol{k}}$ by $\mathbf{C}_j^i = \sum_{k=1}^n C_{jk}^i \partial^k$.

Main theorem. *In above notation, if the structure constants are totally antisymmetric, then for any number $\lambda \in \boldsymbol{k}$, the formula*

$$X_i \mapsto \sum_{\alpha} x_\alpha \varphi_i^\alpha,$$

where

$$\varphi_\beta^\alpha := \sum_{N=0}^{\infty} \frac{(-1)^N B_N}{N!} \left(\mathbf{C}^N\right)_\beta^\alpha \in \boldsymbol{k}\left[\partial^1, \dots, \partial^n\right][[t]] \hookrightarrow A_{n,\boldsymbol{k}}[[t]], \tag{1}$$

and B_n are Bernoulli numbers, extends to an embedding of associative $\boldsymbol{k}[[t]]$-algebras Φ : $\mathcal{U}(\mathfrak{g}_t) \hookrightarrow A_{n,\boldsymbol{k}}[[t]]$. If $\boldsymbol{k} = \mathbb{C}$ (or $\mathbb{Q}[\sqrt{-1}]$), and if the basis is chosen such that $C_{jk}^i \in \mathbb{R}\sqrt{-1}$, and $\lambda \in \mathbb{R}$, then the same holds for the more general formula

$$X_i \mapsto \sum_\alpha \lambda x_\alpha \varphi_i^\alpha + (1-\lambda)\varphi_i^\alpha x_\alpha.$$

Note that, for $\lambda = \frac{1}{2}$, the expressions for X_i are hermitean (invariant with respect to the standard antilinear involution on $A_{n,\mathbb{C}}$).

In particular, for $\lambda = 1$

$$X_i \mapsto x_i + \frac{1}{2}C_{ij}^k x_k \partial^j + \frac{1}{12}C_{ij}^{k'}C_{k'j'}^k x_k \partial^j \partial^{j'} - \frac{1}{720}C_{ij}^{k'}C_{k'j'}^{k''}C_{k''j''}^k x_k \partial^j \partial^{j'} \partial^{j''} + \cdots$$

Clearly, the image $\Phi(\mathcal{U}(\mathfrak{g}_t))/(t\Phi(\mathcal{U}(\mathfrak{g}_t)))$ modulo the subspace of all elements of degree 2 and higher in t is the polynomial algebra in $\dim_{\boldsymbol{k}} \mathfrak{g}$ commuting variables $x_i = \Phi(X_i)$. Thus this embedding may be considered as a realization of $\mathcal{U}(\mathfrak{g}_t)$ as a deformation *of the commutative algebra $\boldsymbol{k}[x_1, \ldots, x_n]$, where the whole deformation is taking place within the Weyl algebra $A_{n,\boldsymbol{k}}$.*

In Section 2 we show that the generalization to general λ (when $\boldsymbol{k} = \mathbb{C}$) is easy. In Sections 3–6 three of us (S.M., A.S., Z.Š.) motivate and prove the theorem by direct computation with tensors. In Sections 7–9 the first author (N.D.) gives an alternative proof and interpretation using formal geometry. In this second part, a completed Weyl algebra is used instead of working with a deformation parameter t to make sense of the power series expressions in our formulas. Over an arbitrary ring $\boldsymbol{k} \supset \mathbb{Q}$, the completed Weyl algebra is identified with the algebra of formal vector fields on a formal neighborhood of the origin in our Lie algebra $\mathfrak{g}$, considered here as a formal variety. Similarly to the classical Lie theory over $\boldsymbol{k} = \mathbb{C}$ or $\mathbb{R}$, where the elements of a Lie algebra can be interpreted as (say, right) invariant vector fields on a Lie group, we identify the elements of a Lie algebra $\mathfrak{g}$ over any ring $\boldsymbol{k} \supset \mathbb{Q}$ to the right-invariant formal vector fields on a suitable formal group, and compute them in terms of the coordinate chart given by an appropriate version of the exponential map. As a consequence, in this part (Sections 7–9), we actually construct a deformation of the abelian subalgebra generated by the ∂^i within the Weyl algebra, rather than the subalgebra of coordinates, but the difference is inessential: the automorphism of the Weyl algebra mapping $x_i \mapsto -\partial^i$, $\partial^i \mapsto x_i$ interchanges the formulas between the first and the second parts of the work. However, we kept the different conventions as the deformation of "space" coordinates x_i is our initial motivation, while the representation via vector fields is also a valuable geometric point of view. In Section 10, N.D. adds a third proof using coalgebra structure and coderivations. In some sense this proof is obtained by "dualizing" the previous geometric proof; this makes the proof shorter but more difficult to understand.

As we learned from D. Svrtan after completing our first proof, one can find a superimposable formula in E. Petracci's work [Petracci] on representations by coderivations. More precisely, Theorem 5.3 and formula (20) of her work, once her formulas (13) and (15) are taken into account, correspond to our 10.10, i.e. the invariant form of the main formula of present work, expressed in the language of coderivations, prior to any Weyl algebra identifications (Weyl algebras are in fact never mentioned in [Petracci]) and explicit coordinate computations. Moreover, our formulas of 10.6 essentially appear in [Petracci, Remark 3.4].

Notation. Throughout the paper, for $\rho = 1, \ldots, n$, we use the $\boldsymbol{k}$-linear derivations $\delta_\rho := \frac{d}{d(\partial^\rho)}$ of $A_{n,\boldsymbol{k}}$. If r is a real number, then $\lfloor r \rfloor$ denotes the largest integer smaller or equal to r (integer part or floor of r).

All considered modules over unital rings will be unital. For a $\boldsymbol{k}$-module M, $S(M)$, $T(M)$ will denote its symmetric and tensor $\boldsymbol{k}$-algebra, respectively.

2. Reduction to $\lambda = 1$

2.1. Let ψ and χ be matrices of expressions depending on ∂s. Assume Einstein convention (summation over each pair of repeated indices). Then

$$\begin{aligned}
\left[x_\alpha\psi^\alpha_\mu, x_\beta\chi^\beta_\nu\right] &= x_\gamma\left(\left(\delta_\rho\psi^\gamma_\mu\right)\chi^\rho_\nu - \left(\delta_\rho\chi^\gamma_\nu\right)\psi^\rho_\mu\right),\\
\left[x_\alpha\psi^\alpha_\mu, \chi^\beta_\nu x_\beta\right] &= x_\gamma\left(\left(\delta_\rho\psi^\gamma_\mu\right)\chi^\rho_\nu - \left(\delta_\rho\chi^\gamma_\nu\right)\psi^\rho_\mu\right) - \psi^\alpha_\mu\left(\delta_\alpha\delta_\beta\chi^\beta_\nu\right),\\
\left[\psi^\alpha_\mu x_\alpha, \chi^\beta_\nu x_\beta\right] &= x_\gamma\left(\left(\delta_\rho\chi^\rho_\nu\right)\psi^\gamma_\mu - \left(\delta_\rho\psi^\rho_\nu\right)\chi^\gamma_\nu\right),\\
\left[\psi^\alpha_\mu x_\alpha, x_\beta\chi^\beta_\nu\right] &= x_\gamma\left(\left(\delta_\rho\chi^\rho_\nu\right)\psi^\gamma_\mu - \left(\delta_\rho\psi^\rho_\nu\right)\chi^\gamma_\nu\right) + \psi^\alpha_\mu\left(\delta_\alpha\delta_\beta\chi^\beta_\nu\right).
\end{aligned}$$

2.2. There is a $\boldsymbol{k}$-linear antiautomorphism $\dagger$ of the Weyl algebra $A_{n,\boldsymbol{k}}$ given on generators by $x \mapsto x$ and $\partial \mapsto -\partial$. In particular, $(x_\alpha\varphi^\alpha_\mu)^\dagger = (\varphi^\alpha_\mu)^\dagger x_\alpha$.

In the case when $\boldsymbol{k} = \mathbb{C}$, we also have the conjugation—the antilinear involution which will be also denoted by $\dagger$. In that case, it is easy to check that for φ given by formula (1) we have $\varphi = \varphi^\dagger$.

2.3. From the formulas in 2.1, or, even easier, directly, we obtain

$$\left[x_\alpha\varphi^\alpha_\mu, x_\beta\varphi^\beta_\nu\right] + \left[\varphi^\alpha_\mu x_\alpha, \varphi^\beta_\nu x_\beta\right] = \left[x_\alpha\varphi^\alpha_\mu, \varphi^\beta_\nu x_\beta\right] + \left[\varphi^\alpha_\mu x_\alpha, x_\beta\varphi^\beta_\nu\right]. \tag{2}$$

Therefore

$$\begin{aligned}
&\left[\lambda x_\alpha\varphi^\alpha_\mu + (1-\lambda)\left(x_\alpha\varphi^\alpha_\mu\right)^\dagger, \lambda x_\beta\varphi^\beta_\nu + (1-\lambda)\left(x_\beta\varphi^\beta_\nu\right)^\dagger\right]\\
&\quad= \left[\lambda x_\alpha\varphi^\alpha_\mu + (1-\lambda)\varphi^\alpha_\mu x_\alpha, \lambda x_\beta\varphi^\beta_\nu + (1-\lambda)\varphi^\beta_\nu x_\beta\right]\\
&\quad\overset{(2)}{=} \lambda(\lambda+1-\lambda)\left[x_\alpha\varphi^\alpha_\mu, x_\beta\varphi^\beta_\nu\right] + (1-\lambda)(1-\lambda+\lambda)\left[\varphi^\alpha_\mu x_\alpha, \varphi^\beta_\nu x_\beta\right]\\
&\quad= \lambda\left[x_\alpha\varphi^\alpha_\mu, x_\beta\varphi^\beta_\nu\right] + (1-\lambda)\left(\left[x_\beta\varphi^\beta_\nu, x_\alpha\varphi^\alpha_\mu\right]\right)^\dagger.
\end{aligned}$$

Thus it is sufficient to prove the $\lambda = 1$ identity

$$\left[x_\alpha\varphi^\alpha_\mu, x_\beta\varphi^\beta_\nu\right] = C^\rho_{\mu\nu}x_\gamma\varphi^\gamma_\rho \tag{3}$$

and the general identity is $\lambda\,\text{Eq. (3)} + (1-\lambda)\,\text{Eq. (3)}^\dagger$.

3. Covariance and the universal case

3.1. In calculational approach, we first try a more general Ansatz, and then gradually inspect various identities a formula should satisfy in each order in the deformation parameter in order to provide a Lie algebra representation. After long calculations we conclude that specializing the coefficients to Bernoulli numbers ensures that the identities hold.

Our more general Ansatz, with nice covariance properties under $GL(n,\boldsymbol{k})$-action, is a special case of a more general requirement of functoriality. We want that our formula be universal under change of rings, and universal for all Lie algebras over a fixed ring. Thus we build the Ansatz from tensors in structure constants of certain kind, and we want the shape to be controlled (covariant in some sense) under the morphisms of Lie algebras, where we allow the underlying ring, the Lie algebra, and its basis to change. To this end we will define certain universal ring, which is not a field, and later a universal Lie algebra over it where our calculations in fact take place. By specialization, the formulas then imply the formulas for "concrete" Lie algebras.

3.2. Definition. Consider the affine $\boldsymbol{k}$-space $\boldsymbol{k}^{n^3}$. Define the affine function algebra $\boldsymbol{k}[\mathcal{C}_n]$ of the affine variety $\mathcal{C}_n$ ("the variety of generic structure constants of generic rank n Lie $\boldsymbol{k}$-algebra") to be the polynomial algebra in n^3-variables $\bar{C}^i_{jk}$, $i,j,k=1,\dots,n$, modulo the homogeneous relations

(i) $\bar{C}^i_{jk}=\bar{C}^i_{kj}$ (antisymmetry in lower indices),
(ii) $\sum_\alpha \bar{C}^\alpha_{ij}\bar{C}^l_{\alpha k}+\bar{C}^\alpha_{jk}\bar{C}^l_{\alpha i}+\bar{C}^\alpha_{ki}\bar{C}^l_{\alpha j}=0\ \forall i,j,k,l$ (Jacobi identity).

Let $\mathcal{L}_n$ be the Lie $\boldsymbol{k}[\mathcal{C}_n]$-algebra over $\boldsymbol{k}[\mathcal{C}_n]$, free as a $\boldsymbol{k}[\mathcal{C}_n]$-module with basis $X:(\boldsymbol{k}[\mathcal{C}_n])^n \xrightarrow{\cong} \mathcal{L}_n$ and bracket $[X_k,X_l]=\bar{C}^i_{kl}X_i$. It will be sometimes also called universal.

3.3. The correspondence which to each $\boldsymbol{k}$-algebra associates the set of all Lie brackets on the free $\boldsymbol{k}$-module $\boldsymbol{k}^n$ of rank n extends to a covariant functor from the category of (unital associative) $\boldsymbol{k}$-algebras to the category of sets. It is clearly represented by $\boldsymbol{k}[\mathcal{C}_n]$. If we take an arbitrary rank n free Lie algebra $\mathfrak{g}$ over $\boldsymbol{k}$ as a Lie algebra, and fix a basis $X^0=(X^0_1,\dots,X^0_n):\boldsymbol{k}^n\to\mathfrak{g}$, it can be therefore considered as a point of affine $\boldsymbol{k}$-variety $\mathcal{C}_n$. The map of $\boldsymbol{k}$-algebras $\mathrm{ev}_\mathfrak{g}:=\mathrm{ev}_{\mathfrak{g},X^0}:\boldsymbol{k}[\mathcal{C}_n]\to\boldsymbol{k}$ determined by $\bar{C}^i_{jk}\mapsto C^i_{jk}$ is called the *evaluation map*.

3.4. Let $A_{n,\boldsymbol{k}}[[t]]$ be the $\boldsymbol{k}[[t]]$-algebra of formal power series in one indeterminate t with coefficients in $A_{n,\boldsymbol{k}}$. Any choice of a basis in $\mathfrak{g}$ provides an isomorphism of $\boldsymbol{k}$-modules from $S(\mathfrak{g})\otimes S(\mathfrak{g}^*)$ to $A_{n,\boldsymbol{k}}$, where $S(\mathfrak{g})$ is the symmetric (polynomial) algebra in $X^0_1,\dots,X^0_n$. Algebra $S(\mathfrak{g})\otimes S(\mathfrak{g}^*)$ acts on the left and right on $S(\mathfrak{g})$, namely the elements of $S(\mathfrak{g})$ act by multiplication, and the elements of $\mathfrak{g}^*$ act by derivations.

3.5. $GL_{\boldsymbol{k}}(\mathfrak{g})\cong GL(n,\boldsymbol{k})$ naturally acts on $\mathfrak{g}$, $\mathfrak{g}^*$, $T(\mathfrak{g})\otimes T(\mathfrak{g}^*)$ and $S(\mathfrak{g})\otimes S(\mathfrak{g}^*)$. We will take our Lie algebra $\mathfrak{g}$ to be free as $\boldsymbol{k}$-module to be able to work with tensor components. As our main interest is in the formulas for generators, given $\mathcal{O}\in GL_{\boldsymbol{k}}(\mathfrak{g})$, we find more convenient to consider the matrix elements $\mathcal{O}^\alpha_i$ for the expansion of a new basis $\mathcal{O}X$ in terms of old X, rather than the more customary matrix elements for the expansion of the contragradient vector components: $(\mathcal{O}X)_i=:\sum_\alpha\mathcal{O}^\alpha_i X_\alpha$. The structure constants $C^{(\mathcal{O})i}_{jk}$ in the new frame $(\mathcal{O}X)_1,\dots,(\mathcal{O}X)_n$ can be easily described

$$\big[(\mathcal{O}X)_i,(\mathcal{O}X)_j\big]=C^{(\mathcal{O})\sigma}_{ij}(\mathcal{O}X)_\sigma, \tag{4}$$

where $C^{(\mathcal{O})\sigma}_{ij}:=\mathcal{O}^\alpha_i\mathcal{O}^\beta_j C^\gamma_{\alpha\beta}(\mathcal{O}^{-1})^\sigma_\gamma$ (clearly the structure constants make a tensor which may be considered as living in $\mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}$, but we here present everything in coordinates).

3.6. The natural $GL(\mathfrak{g}, \boldsymbol{k})$-action on $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$ transports to the Weyl algebra $A_{n,\boldsymbol{k}}$ via the identification of their underlying $\boldsymbol{k}$-modules. Both actions may be considered as factored from $T(\mathfrak{g}) \otimes T(\mathfrak{g}^*)$. It is crucial that the induced action is compatible both with the product in $A_{n,\boldsymbol{k}}$ and with the product in $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$, in the sense that $\mathcal{O}(x \cdot y) = \mathcal{O}(x)\mathcal{O}(y)$ for any $x, y \in A_{n,\boldsymbol{k}}$, $\mathcal{O} \in GL(\mathfrak{g}, \boldsymbol{k}) \cong GL_n(\boldsymbol{k})$. This is because the $GL(\mathfrak{g}, \boldsymbol{k})$-action is factored from the action on $T(\mathfrak{g}) \otimes T(\mathfrak{g}^*)$. Namely, the defining ideal, both for $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$ and for $A_{n,\boldsymbol{k}}$, is $GL_n(\boldsymbol{k})$-invariant (in the case of $A_{n,\boldsymbol{k}}$, $\partial^j x_i - x_i \partial^j - \delta_i^j$ are components of a tensor for which all components are included in the ideal). For this $GL_n(\boldsymbol{k})$-action, $x = (x_1, \ldots, x_n)$ are $GL_n(\boldsymbol{k})$-cogradient with respect to X, and $\partial = (\partial^1, \ldots, \partial^n)$ are contragradient, i.e. $(\mathcal{O}x)_i = \mathcal{O}_i^\alpha x_\alpha$ and $(\mathcal{O}\partial)^i = (\mathcal{O}^{-1})^i_\alpha \partial^\alpha$.

Contractions of tensors with respect to the product in $A_{n,\boldsymbol{k}}$ have the same covariance, as if they would be contractions with respect to the product in $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$. Given a contraction, $c : (\mathfrak{g}^*)^m \otimes \mathfrak{g}^n \to (\mathfrak{g}^*)^{m-1} \otimes \mathfrak{g}^{n-1}$ (e.g. the pairing $\mathfrak{g}^* \otimes \mathfrak{g} \to \boldsymbol{k}$), and tensors $A \in \mathfrak{g}^*$, $B \in \mathfrak{g}$ one usually considers the behavior or $c(A, B)$ under the action $c(A, B) \mapsto c(\mathcal{O}A, \mathcal{O}B)$ and likewise for multiple contractions. Writing down $\mathbf{C}$ in terms of C^i_{jk} and ∂^l and using induction it is direct to show that

$$\left(\mathcal{O}\mathbf{C}^N\right)^\sigma_\rho = \sum_{i,j} \mathcal{O}^\sigma_i \left(\mathbf{C}^N\right)^i_j \left(\mathcal{O}^{-1}\right)^j_\rho,$$

$$\mathcal{O}\left(\sum_\alpha A_\alpha x_\alpha \left(\mathbf{C}^N\right)^\alpha_\beta\right) = \sum_\sigma \mathcal{O}^\sigma_\beta \sum_\alpha A_\alpha x_\alpha \left(\mathbf{C}^N\right)^\alpha_\sigma.$$

Thus looking for the solution in terms of a series $\sum_\alpha A_\alpha x_\alpha (\mathbf{C}^N)^\alpha_\beta$ *is seeking for a solution which behaves covariantly with respect to the change of coordinates under* $GL_n(\boldsymbol{k})$.

3.7. In the case of the universal rank n Lie algebra $\mathcal{L}_n$ over $\boldsymbol{k}[\mathcal{C}_n]$, the components of the structure tensor are identical to the generators of the ground ring. Hence $\mathcal{O}$ induce an automorphism of $\boldsymbol{k}[\mathcal{C}_n]$ as a $\boldsymbol{k}$-module. It is a remarkable fact, however, that $C^{(\mathcal{O})i}_{jk}$ satisfy the same relations as C^i_{jk}, i.e. $\mathcal{O}$ induces a $\boldsymbol{k}$-*algebra* automorphism of $\boldsymbol{k}[\mathcal{C}_n]$. For example, if we want to show the Jacobi identity, we consider

$$\mathcal{O}\left(C^\alpha_{ij} C^l_{\alpha k}\right) = \mathcal{O}^r_i \mathcal{O}^s_j C^\sigma_{rs} \left(\mathcal{O}^{-1}\right)^\alpha_\sigma \mathcal{O}^t_\alpha \mathcal{O}^u_k C^\tau_{tu} \left(\mathcal{O}^{-1}\right)^l_\tau = \mathcal{O}^r_i \mathcal{O}^s_j \mathcal{O}^u_k C^t_{rs} C^\tau_{tu} \left(\mathcal{O}^{-1}\right)^l_\tau$$

and two other summands, cyclically in (i, j, k). Now rename (r, s, t) apropriately in the two other summands to force the same factor $\mathcal{O}^r_i \mathcal{O}^s_j \mathcal{O}^u_k$ in all three summands. Then the CC-part falls in a form where the Jacobi identity can be readily applied. The covariance follows from the functoriality under the canonical isomorphism of $\boldsymbol{k}[\mathcal{C}_n]$-Lie algebras from the pullback $\mathcal{O}^*\mathcal{L}_n$ to $\mathcal{L}_n$ for all $\mathcal{O}$ in $GL_n(\boldsymbol{k})$.

3.8. A more general covariant Ansatz $X_\beta = \sum_{IJK} A_{IJK} x_\alpha (\operatorname{Tr} \mathbf{C}^{2I})^J (\mathbf{C}^K)^\alpha_\beta$ is likely useful for finding new representations for specific Lie algebras. In the universal case ($\mathfrak{g} = \mathcal{L}_n$), however, the traces are contraction-disconnected from the rest of expression, and different trace factors cannot be mixed. Namely, the defining ideal of $\boldsymbol{k}[\mathcal{C}_n]$ does not have "mixed" (different J and K) elements, and commuting x_α with such tensors also does not produce them either.

4. Differential equation and recursive relations

Introduce the "star" notation for higher order connected tensors. Namely, in order to spot better just the relevant indices, if we have the contraction of an upper index in one of the C-factors with one lower index of the next C-factor, we may just write the $*$ symbol on the two places. In that notation, the derivative $\delta_\rho = \frac{\partial}{\partial(\partial^\rho)}$ applied to $\mathbf{C}^I$ equals

$$\delta_\rho(\mathbf{C}^I)^\gamma_\mu = C^*_{\mu\rho}\mathbf{C}^*_*\mathbf{C}^*_*\cdots\mathbf{C}^\gamma_* + \mathbf{C}^*_\mu C^*_{*\rho}\mathbf{C}^*_*\cdots\mathbf{C}^\gamma_* + \cdots + \mathbf{C}^*_\mu\mathbf{C}^*_*\mathbf{C}^*_*\cdots C^\gamma_{*\rho}.$$

From $[x_\alpha\varphi^\alpha_\mu, x_\beta\varphi^\beta_\nu] = C^\sigma_{\mu\nu}x_\gamma\varphi^\gamma_\rho$ (C1) (see Section 2), equating the polynomials in ∂s in front of x_γ, we obtain the system of differential equations manifestly antisymmetric with respect to the interchange $\mu \leftrightarrow \nu$:

$$\boxed{(\delta_\rho\varphi^\gamma_\mu)\varphi^\rho_\nu - (\delta_\rho\varphi^\gamma_\nu)\varphi^\rho_\mu = C^\sigma_{\mu\nu}\varphi^\gamma_\sigma.}$$

We use Ansatz $\varphi^i_j = \sum_{N=0}^\infty A_I(\mathbf{C}^I)^i_j$ where $A_0 = 1$ and $(\mathbf{C}^0)^i_j = \delta^i_j$. In any order $N \geqslant 1$ in expansion in t (hence in C^i_{jk}s and also in ∂^is) we thus obtain

$$\sum_{I=1}^N A_I A_{N-I}\big\{\big[\delta_\rho(\mathbf{C}^I)^\gamma_\mu\big](\mathbf{C}^{N-I})^\rho_\nu - \big[\delta_\rho(\mathbf{C}^I)^\gamma_\nu\big](\mathbf{C}^{N-I})^\rho_\mu\big\} = A_{N-1}C^\sigma_{\mu\nu}(\mathbf{C}^{N-1})^\gamma_\sigma. \tag{5}$$

Notice that for $N = I$,

$$\begin{aligned}\delta_\rho(\mathbf{C}^I)^\gamma_\mu(\mathbf{C}^{N-I})^\rho_\nu &= \delta_\nu(\mathbf{C}^N)^\gamma_\mu \\ &= C^*_{\mu\nu}\mathbf{C}^*_*\cdots\mathbf{C}^\gamma_* + \mathbf{C}^*_\mu C^*_{*\nu}\mathbf{C}^*_*\cdots\mathbf{C}^\gamma_* + \cdots + \mathbf{C}^*_\mu\mathbf{C}^*_*\cdots C^\gamma_{*\nu} \\ &=: M'_0 + M'_1 + \cdots + M'_{N-1},\end{aligned}$$

where we use the star notation, as explained above. Of course, the tensors $M'_I = (M'_I)^\gamma_{\mu\nu}$ have three *suppressed* indices μ, ν, γ. In Eq. (5) these tensors come in the combination $(M_I)^\gamma_{\mu\nu} := (M'_I)^\gamma_{\mu\nu} - (M'_I)^\gamma_{\nu\mu}$ antisymmetrized in the lower two indices. In detail,

$$\begin{aligned}M_0 = (M_0)^\gamma_{\mu\nu} &:= 2C^*_{\mu\nu}\mathbf{C}^*_*\mathbf{C}^*_*\cdots\mathbf{C}^\gamma_*, \\ M_I = (M_I)^\gamma_{\mu\nu} &:= (\mathbf{C}^{I-1})^*_\mu C^*_{*\nu}(\mathbf{C}^{N-I-1})^\gamma_* - (\mu\leftrightarrow\nu), \quad 1 \leqslant I \leqslant N-1, \\ M_{N-1} = (M_{N-1})^\gamma_{\mu\nu} &:= \mathbf{C}^*_\mu(\mathbf{C}^{N-2})^*_*\mathbf{C}^\gamma_\nu - \mathbf{C}^*_\nu(\mathbf{C}^{N-2})^*_*\mathbf{C}^\gamma_\mu.\end{aligned}$$

One can easily show that if the Lie algebra in question is $\mathfrak{su}(2)$ then $M_0 = M_1 = \cdots = M_{N-1}$ and the rest of the proof is much simpler. We checked using *Mathematica*™ that already for $\mathfrak{su}(3)$ the M_Is are mutually different.

In particular, the $N = 1$ equation is obvious: $A_1(C^\gamma_{\mu\nu} - C^\gamma_{\nu\mu}) = C^\gamma_{\mu\nu}$ what by antisymmetry in $(\mu \leftrightarrow \nu)$ forces $A_1 = \frac{1}{2}$.

The Jacobi identity $[[X_i, X_j], X_k] + cyclic = 0$ will be used in the form

$$\sum_{\alpha} C^{\alpha}_{ij} C^{\beta}_{\alpha k} + C^{\alpha}_{jk} C^{\beta}_{\alpha i} + C^{\alpha}_{ki} C^{\beta}_{\alpha j} = 0.$$

Together with the antisymmetry in the lower indices it implies

$$\mathbf{C}^{*}_{i} C^{\gamma}_{*k} = \mathbf{C}^{*}_{k} C^{\gamma}_{*i} + C^{*}_{ik} \mathbf{C}^{\gamma}_{*}. \tag{6}$$

In the 2nd order ($N = 2$),

$$A_1^2 C^{\gamma}_{*\mu} \mathbf{C}^{*}_{\nu} + A_2 \big(C^{*}_{\nu\mu} \mathbf{C}^{\gamma}_{*} + \mathbf{C}^{*}_{\mu} C^{\gamma}_{\nu *} \big) - (\mu \leftrightarrow \nu) = A_1 C^{*}_{\mu\nu} \mathbf{C}^{\gamma}_{*}.$$

After applying (6) with $\mu = i$, $\nu = k$, we obtain $A_1^2 + 3A_2 = A_1$ (up to a common factor, which is in general nonzero), hence $A_2 = 1/12$.

In higher order, we will recursively show that the odd coefficients are zero (except A_1). That means that, among all products $A_I A_{N-I}$, only the terms $I = 1$ and $I = N - 1$, where the product equals $A_1 A_{N-1} = 1/2 A_{N-1}$ survive. Now, A_{N-1} is at the both sides of the equation, and as we suppose those to be nonzero (what is justified afterwards), we divide the equation by A_{N-1} to obtain

$$C^{\gamma}_{\mu\rho} \big(\mathbf{C}^{2K} \big)^{\rho}_{\nu} + \big[\mathbf{C}^{\rho}_{\nu} \delta_{\rho} \big(\mathbf{C}^{2K} \big)^{\gamma}_{\mu} \big] - (\mu \leftrightarrow \nu) = 2 C^{\sigma}_{\mu\nu} \big(\mathbf{C}^{2K} \big)^{\gamma}_{\sigma},$$

where $2K + 1 = N$.

In even order $N = 2k \geqslant 4$, the RHS is zero. Because of the different shape of the tensors involved, we split the LHS into the part $I = N$, and the rest, which we then move to the RHS to obtain

$$-A_N (M_0 + M_1 + \cdots + M_{N-1}) = \sum_{I=1}^{N-1} A_I A_{N-I} \big\{ \big[\delta_{\rho} \big(\mathbf{C}^{I} \big)^{\gamma}_{\mu} \big] \big(\mathbf{C}^{N-I} \big)^{\rho}_{\nu} - (\mu \leftrightarrow \nu) \big\} \tag{7}$$

for N even. The expression in the curly brackets will be denoted by $(K_{I,N-I})^{\gamma}_{\mu\nu}$.

As γ, μ, ν will be fixed, and we prove the identities for all triples (γ, μ, ν) we will just write $K_{I,N-I}$ without indices. In this notation (7) reads,

$$-A_N (M_0 + M_1 + \cdots + M_{N-1}) = \sum_{I=1}^{N-1} A_I A_{N-I} K_{I,N-I}. \tag{8}$$

As an extension of this notation, we may also denote

$$K_{0,N} = M_0 + M_1 + \cdots + M_{N-1}.$$

Notice that

$$\big[x_{\alpha} \big(\mathbf{C}^{I} \big)^{\alpha}_{\mu}, x_{\beta} \big(\mathbf{C}^{I} \big)^{\beta}_{\nu} \big] = x_{\gamma} (K_{I,N-I})^{\gamma}_{\mu\nu}, \quad I = 0, 1, \ldots, N.$$

Lemma. *For $L = 0, 1, 2, \dots$ and $1 \leqslant \mu, \nu, \gamma \leqslant n$*

$$\mathbf{C}_\nu^\rho C_{\mu\rho}^* \big(\mathbf{C}^L\big)_*^\gamma - \mathbf{C}_\mu^\rho C_{\nu\rho}^* \big(\mathbf{C}^L\big)_*^\gamma = C_{\mu\nu}^\sigma \big(\mathbf{C}^{L+1}\big)_\sigma^\gamma, \tag{9}$$

$$C_{\mu\rho}^\gamma \big(\mathbf{C}^L\big)_\nu^\rho + \big[\mathbf{C}_\nu^\rho \delta_\rho \big(\mathbf{C}^L\big)_\mu^\gamma\big] - (\mu \leftrightarrow \nu) = 2C_{\mu\nu}^\sigma \big(\mathbf{C}^L\big)_\sigma^\gamma. \tag{10}$$

Proof. Equation (9) follows easily by applying the Jacobi identity in form (6) to the expression $\mathbf{C}_\nu^\rho C_{\mu\rho}^* - \mathbf{C}_\mu^\rho C_{\nu\rho}^*$ and contracting with $\mathbf{C}^L$.

Equation (10) will be proved by induction. For $L = 0$ it boils down to (6). We need to verify directly also $L = 1$ because this will also be used in the proof for the step of induction.

Suppose that (10) holds for L. Then for $L + 1$, by Leibniz rule, the LHS is

$$C_{\mu\rho}^\gamma \big(\mathbf{C}^{L+1}\big)_\nu^\rho + \mathbf{C}_\nu^\rho \mathbf{C}_\mu^* \delta_\rho \big(\mathbf{C}^{L+1}\big)_*^\gamma + \mathbf{C}_\nu^\rho C_{\mu\rho}^* \big(\mathbf{C}^L\big)_*^\gamma - (\mu \leftrightarrow \nu).$$

According to (9) the term $\mathbf{C}_\nu^\rho C_{\mu\rho}^* (\mathbf{C}^L)_*^\gamma - (\mu \leftrightarrow \nu)$ contributes to exactly one half of required RHS. Hence it is sufficient to prove that

$$C_{\mu\rho}^\gamma \big(\mathbf{C}^{L+1}\big)_\nu^\rho + \mathbf{C}_\nu^\rho \mathbf{C}_\mu^* \delta_\rho \big(\mathbf{C}^L\big)_*^\gamma - (\mu \leftrightarrow \nu) = C_{\mu\nu}^\sigma \big(\mathbf{C}^{L+1}\big)_\sigma^\gamma. \tag{11}$$

This equation is then proved by induction. Suppose it holds for L, for all γ. Then multiply by $\mathbf{C}_\gamma^\tau$ and sum over γ to obtain $C_{\mu\nu}^\sigma (\mathbf{C}^{L+2})_\sigma^\tau$ at RHS and

$$\mathbf{C}_\gamma^\tau C_{\mu\rho}^\gamma \big(\mathbf{C}^{L+1}\big)_\nu^\rho + \mathbf{C}_\nu^\rho \mathbf{C}_\mu^* \delta_\rho \big(\mathbf{C}^{L+1}\big)_*^\tau - \mathbf{C}_\nu^\rho \big(\mathbf{C}^{L+1}\big)_\mu^* C_{*\rho}^\tau - (\mu \leftrightarrow \nu)$$

at LHS (we used the Leibniz rule again). Using the antisymmetrization in $(\mu \leftrightarrow \nu)$ and renaming some dummy indices we obtain

$$\big(C_{\mu\rho}^* \mathbf{C}_*^\tau + \mathbf{C}_\mu^* C_{\rho *}^\tau\big)\big(\mathbf{C}^{L+1}\big)_\nu^\rho + \mathbf{C}_\nu^\rho \mathbf{C}_\mu^* \delta_\rho \big(\mathbf{C}^{L+1}\big)_*^\tau - (\mu \leftrightarrow \nu).$$

Using (6) we can sum inside the brackets to obtain

$$C_{\mu *}^\tau \big(\mathbf{C}^{L+2}\big)_\nu^* + \mathbf{C}_\nu^\rho \mathbf{C}_\mu^* \delta_\rho \big(\mathbf{C}^{L+1}\big)_*^\tau - (\mu \leftrightarrow \nu)$$

as required. □

Corollary. *If $A_0 = 1$ and $A_1 = 1/2$, $A_{2K+1} = 0$ for $K = 1, \dots, K_0 - 1$ and the relation* (5) *holds for N odd where $N = 2K + 1$ with $K < K_0$, then the relation* (5) *also holds for $N = 2K_0 + 1$.*

Corollary. *In particular, relation* (5) *holds for N odd if $A_K = (-1)^K B_K / K!$.*

The even case will require much longer calculation. Before that, observe that $K_{I,N-I} = K_{N-I,I}$, and regarding that for $N > 2$, the term $A_1 A_{N-1} = 0$, hence all odd-label terms are zero, hence Eq. (8) for even $N \geqslant 4$ reads

$$-A_N (M_0 + M_1 + \cdots + M_{N-1}) = \sum_{k=1}^{N/2} A_{2k} A_{N-2k} K_{2k,N-2k}. \tag{12}$$

5. Hierarchy of formulas and the basis of identities

Throughout this section the order $N \geqslant 2$.

5.1 *(Z-tensors, $\hat{Z}$-tensors).* For $1 \geqslant \gamma, \mu, \nu \leqslant n$, $\mu \neq \nu$ define the (components of) Z-tensor

$$\left(Z^{l,m,k}\right)^{\gamma}_{\mu\nu} := \left(\mathbf{C}^{l}\right)^{*}_{\mu}\left(\mathbf{C}^{m}\right)^{*}_{\nu} C^{*}_{**}\left(\mathbf{C}^{k}\right)^{\gamma}_{*} - (\mu \leftrightarrow \nu), \tag{13}$$

where $l, m, k \geqslant 0$, $l + m + k + 1 = N$. Recall that C^{r}_{su} are either the structure constants of a Lie algebra of rank n as a $\boldsymbol{k}$-module, or, in the universal case, the generators of $\boldsymbol{k}[\mathcal{C}_n]$. However, the RHS makes also sense when the C^{r}_{su}, for $r, s, u = 1, \ldots, n$, are simply the n^3 generators of a free commutative $\boldsymbol{k}$-algebra (no Jacobi, no antisymmetry). In that case, the LHS will be denoted $(\hat{Z}^{l,m,k})^{\gamma}_{\mu\nu}$.

From now on, whenever μ, ν, γ are of no special importance, we write simply $Z^{l,m,k}$, and $\hat{Z}^{l,m,k}$, i.e. γ, μ, ν will be skipped from the notation whenever they are clear from the context. As before, $*$s are dummy indices, and upper $*$s are contracted to lower stars pairwise in left-to-right order. $(Z^{l,m,k})^{\gamma}_{\mu\nu}$ are components of a rank 3 tensor, antisymmetric in lower indices. Each of the two summands is a contraction of $l + m + k$ $\mathbf{C}$s and one C. Such expressions appear in our analysis when $N = l + m + k + 1$. By obvious combinatorial arguments, this tensor T is contraction-connected (the copies of the generators (Cs and ∂s) involved cannot be separated into two disjoint subsets without a contraction involving elements in different subsets). In a universal case, the components of this tensor lie in $\boldsymbol{k}[\mathcal{C}_n][\partial^1, \ldots, \partial^n]$. The Z-tensors may be called also "star-tensor," what points to a useful graphical notation in which the three branches $\mathbf{C}^l$, $\mathbf{C}^m$, $\mathbf{C}^k$ are drawn respectively left, down and right from the central "node" C, attached by lines denoting contractions to the three indices of C.

5.2 *(Special cases of Z-tensors: b_i, M_j).* Any of l, m, k may take value zero: $(\mathbf{C}^0)^i_j = \delta^i_j$ is then the Kronecker tensor. This mean that a "branch" was cut and we have monomials in the tensor which are linear "M-" and "b-" chains denoted $M_0, \ldots, M_{N-1}, b_0, b_1, \ldots, b_{N-1}$, where for all $0 \leqslant k \leqslant N-1$,

$$\begin{aligned} b_k &:= Z^{k,N-k-1,0} = \left(\mathbf{C}^{k}\right)^{*}_{\nu} C^{\gamma}_{**}\left(\mathbf{C}^{N-k-1}\right)^{*}_{\mu} - (\mu \leftrightarrow \nu), \\ M_k &:= Z^{k,0,N-k-1} = \left(\mathbf{C}^{k}\right)^{*}_{\mu} C^{*}_{*\nu}\left(\mathbf{C}^{N-k}\right)^{\gamma}_{*} - (\mu \leftrightarrow \nu). \end{aligned}$$

Then

$$\begin{aligned} M_0 &= Z^{0,0,N-1} = C^{*}_{\mu\nu}\left(\mathbf{C}^{N-1}\right)^{\gamma}_{*} - (\mu \leftrightarrow \nu), \\ M_{N-1} &= Z^{N-1,0,0} = \left(\mathbf{C}^{N-1}\right)^{*}_{\mu} C^{\gamma}_{*\nu} - (\mu \leftrightarrow \nu) = b_0 = b_{N-1}, \\ K_{I,N-I} &= \left[\delta_{\rho}\left(\mathbf{C}^{I}\right)^{\gamma}_{\mu}\right]\left(\mathbf{C}^{N-I}\right)^{\rho}_{\nu} - \left[\delta_{\rho}\left(\mathbf{C}^{I}\right)^{\gamma}_{\nu}\right]\left(\mathbf{C}^{N-I}\right)^{\rho}_{\mu} \\ &= \sum_{l=0}^{I-1}\left(\mathbf{C}^{l}\right)^{*}_{\mu} C^{*}_{*\rho}\left(\mathbf{C}^{I-l-1}\right)^{\gamma}_{*}\left(\mathbf{C}^{N-I}\right)^{\rho}_{n} u - (\mu \leftrightarrow \nu) \\ &= \sum_{l=0}^{I-1} Z^{l,N-I,I-l-1}. \end{aligned} \tag{14}$$

5.3 *(Spaces of Z-tensors).* The $\boldsymbol{k}$-span of all Zs is denoted $\mathcal{Z}_N = \mathcal{Z}^{\gamma}_{\boldsymbol{k},N,\mu,\nu} \subset \boldsymbol{k}[\mathcal{C}_n][\partial^1,\dots,\partial^n]$. Similarly, the $\boldsymbol{k}$span of all $\hat{Z}$s is denoted by $\hat{\mathcal{Z}}^{\gamma}_{\boldsymbol{k},N,\mu,\nu}$. First of all we need

Lemma. (*Before we quotient out by Jacobi identities and antisymmetry*)*, if n is sufficiently big, all $\hat{Z}^{l,m,k}$ are $\boldsymbol{k}$-linearly independent.*

This lemma is in the setting of the free polynomial algebra on C^r_{su} (tensored by the symmetric algebra in ∂^is), hence finding the monomial summands in $\hat{Z}^{l,m,k}$, which comprise a part of the standard basis, and which are not summands in any other $\hat{Z}^{l',m',k'}$, is easy to do, if there are sufficiently many distinct indices to choose from, say, one distinct from each contraction. On the other extreme, if $n=1$, all contractions involve the same index, and hence there are degeneracies. We leave more precise argument (proof of the lemma) as a combinatorial exercise for the reader.

Clearly, $(\mathcal{Z}_N)^{\gamma}_{\mu\nu} = \hat{\mathcal{Z}}_N/(J_N + J'_N)$, where J_N, J'_N are the two submodules defined as follows. J_N consists of all elements of the form $\sum_{ijk} r^p_{ijk}(Q^{ijk}_p)^{\gamma}_{\mu\nu}$ where $r^p_{ijk} = C^*_{ij}C^p_{*k} + C^*_{jk}C^p_{*i} + C^*_{ki}C^p_{*j}$, and Q^{ijk}_p is a tensor involving $(N-2)$ Cs and $(N-1)$ ∂s (those numbers are fixed, because both the Jacobi identities and the antisymmetry are *homogeneous* relations), and of external $GL_n(\boldsymbol{k})$-covariance $()^{\gamma}_{\mu\nu}$. Contraction-connected tensors are in generic case linearly independent from disconnected, hence by counting free and contracted indices, we observe that in each monomial involved in Q^{ijk}_p, at least one of the indices i, j, k, is contracted to a ∂-variable. Similarly, using J'_N, one handles the antisymmetry.

Our strategy to determine the structure of $(\mathcal{Z}_N)^{\gamma}_{\mu\nu}$ is as follows: we start with $\hat{\mathcal{Z}}_N$ and then determine the submodule of relations $J_N + J'_N$, trying to eliminate superfluous generators to the point where algebraic analysis will become explicit enough. We notice first, that the terms where r_{ijk} is coupled to ∂^i and ∂^j simultaneously are superfluous: by antisymmetry (using J'_N) this will be the contraction of a symmetric and antisymmetric tensor in one of the summands, and the two others are equal in $\hat{\mathcal{Z}}_N/J'_N$. Similarly, with other double contractions of r_{ijk} with ∂. In other words, Jacobi identities always come in the covariant combination obtained by contracting exactly one of the three indices to ∂, as in (6). In degree N, exactly the following relations are of that Jacobi type:

$$Z^{l,m+1,k} = Z^{l,m,k+1} - Z^{l+1,m,k}. \tag{15}$$

By this identity we can do recursions in l, m or k. When l or m is zero Zs are Ms. Hence M-chains span the whole $\mathcal{Z}_N$ (but they are not independent). The antisymmetry in lower indices of C^i_{jk} is nontrivial only if neither of the two indices are contracted to ∂s. But that means, that this is the central node of the star-tensor, and the interchange of the two indices may be traded for the interchange of nodes of μ and ν branch, what results in the identities

$$Z^{l,m,k} = Z^{m,l,k} \tag{16}$$

thereafter called *symmetries*. In particular, $b_k = b_{N-k-1}$.

In addition to $\mathcal{Z}^{\gamma}_{\boldsymbol{k}N\mu\nu}$ we introduce also spaces $\mathcal{F}^{\gamma}_{\boldsymbol{k}N}$ and $\mathcal{ZF}_{\boldsymbol{k}N}$ as follows: $\mathcal{F}_{\boldsymbol{k},N}$ is the free $\boldsymbol{k}$-module generated by the symbols $F^{l,m,k}$ where $l, m, k \geqslant 0$ and $l+m+k+1 = N$; furthermore, $\mathcal{ZF}_{\boldsymbol{k}N} := \mathcal{F}_{\boldsymbol{k}N}/I$, where I is the submodule generated by symmetries $F^{l,m,k} - F^{m,l,k}$ and relations $F^{l,m+1,k} + F^{l+1,m,l} - F^{l,m,k+1}$. Image of $F^{l,m,k}$ in $\mathcal{ZF}_{\boldsymbol{k}N}$ will be denoted by $Z^{l,m,k}_F$.

The summary of the above discussion may be phrased as follows:

5.4. Proposition. *Let $N \geqslant 1$, n sufficiently big, $1 \leqslant \mu, \nu, \gamma \leqslant n$ and $\mu \neq \nu$. The correspondence $Z_F^{l,m,k} \mapsto (Z^{l,m,k})^{\gamma}_{\mu\nu}$ extends to a $\boldsymbol{k}$-module isomorphism $\mathcal{ZF}_{\boldsymbol{k}N} \cong \mathcal{Z}^{\gamma}_{\boldsymbol{k}N\mu\nu}$.*

Thus from now on, we may work with the presentation for $\mathcal{Z}^{\gamma}_{\boldsymbol{k},N,\mu,\nu}$. Using (15) we can do recursions to express Zs in terms of special cases when one of the labels is zero: namely Ms or bs. We can choose a distinguished way to do recursion, e.g. in each step lower m in each monomial. One ends with Z expressed in terms of Ms in a distinguished way, and each symmetry will be an identity between Ms, and there are no other identities in $\mathcal{Z}_N$. If we mix various recursions, this is the same as doing the algorithm of distinguished recursion, but at some steps intercepted by applying a symmetry.

5.5. Lemma. *The distinguished way of recursion, always lowering m, yields*

$$Z^{l,m,k} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} M_{l+j}. \tag{17}$$

The alternative way of recursion, always lowering k, yields

$$Z^{l,m,k} = \sum_{j=0}^{k} \binom{k}{j} b_{l+j}, \tag{18}$$

$$M_{N-k-1} = \sum_{i=0}^{k} \binom{k}{i} b_i. \tag{19}$$

Moreover,

$$M_0 + M_1 + \cdots + M_{N-1} = \delta_{N,\mathrm{odd}} \binom{N}{\frac{N+1}{2}} b_{(N-1)/2} + \sum_{i=0}^{\lfloor N/2 \rfloor - 1} \binom{N+1}{i+1} b_i \tag{20}$$

and for any N even and $1 \leqslant I \leqslant N/2$,

$$K_{I,N-I} = \sum_{i=0}^{I-1} \binom{I}{i} b_i. \tag{21}$$

Proof. The first two formulas follow by easy induction. Too see (20) we use (18):

$$M_{M-k-1} = Z^{N-k-1,0,k} = \sum_{j=0}^{k} \binom{k}{j} b_{N-k-1+j} = \sum_{i=0}^{k} \binom{k}{i} b_{N-i-1} = \sum_{i=0}^{k} \binom{k}{i} b_i.$$

To see (20), notice that from (19) we have

$$\begin{aligned}\sum_{k=0}^{N-1} M_{N-k-1} &= b_0 + (b_0 + b_1) + (b_0 + 2b_1 + b_2) + \cdots \\ &= Nb_0 + \binom{N}{2} b_1 + \cdots + b_{N-1} = \sum_{i=0}^{N-1} \binom{N}{i+1} b_i \\ &= \delta_{N,\mathrm{odd}} \binom{N}{(N-1)/2} b_{(N-1)/2} + \sum_{i=0}^{\lfloor N/2 \rfloor - 1} \left(\binom{N}{i+1} + \binom{N}{N-i} \right) b_i\end{aligned}$$

with (20) immediately. To prove (21) we proceed as follows

$$\begin{aligned}K_{I,N-I} &\overset{(14)}{=} \sum_{l=0}^{I-1} Z^{l,N-I,I-l-1} = \sum_{j=0}^{I-1} Z^{I-j-1,N-I,j} \\ &= b_{I-1} + (b_{I-1} + b_{I-2}) + (b_{I-1} + 2b_{I-2} + b_{I-3}) + \cdots \\ &\quad + \big(b_{I-1} + \cdots + (I-1)b_1 + b_0\big) \\ &= Ib_{I-1} + \binom{I}{2} b_{I-2} + \cdots + b_0 = \sum_{j=1}^{I} \binom{I}{j} b_{I-j} = \sum_{i=0}^{I-1} \binom{I}{i} b_i. \qquad \square\end{aligned}$$

5.6. This result enables us that we can effectively do all computations with Z-tensors either in terms of bs or in terms of Ms. However, both sets of variables have internal linear dependences which we will now study.

First the case of bs, which is much simpler. To study the relations among the relations we introduce $\boldsymbol{k}$-module $\mathcal{F}_{b\boldsymbol{k}N}$ as the free module on N symbols $\bar{b}_0, \ldots, \bar{b}_{N-1}$ modulo the relations $\bar{b}_i = \bar{b}_{N-i-1}$.

The relations (18) will be now taken as the definitions of Z-variables. More precisely, define $Z_b^{l,m,k} = \sum_{j=0}^{k} \bar{b}_{l+j} \in \mathcal{F}_{b\boldsymbol{k}N}$. The only relations among $Z_b^{l,m,k}$ are symmetries $X_{b,k}^{(s)} := (Z_b)^{l,l+s,N-2l-s-1} - (Z_b)^{l+s,l,N-2l-s-1}$. Now

$$X_{b,k}^{(s)} = \sum_{j=0}^{N-2l-s-1} \bar{b}_{l+j} - \sum_{i=0}^{N-2l-s-1} \bar{b}_{l+s+i} = \sum_{j=0}^{N-2l-s-1} (\bar{b}_{l+j} - \bar{b}_{N-l-j-1}),$$

where we replaced i by $j = N - 2l - s - 1 - i$ in the second sum. Thus every symmetry is a linear combination of the $\lfloor N/2 \rfloor$ relations $\bar{b}_i - \bar{b}_{N-i-1}$, which are linearly independent in $\mathcal{F}_{b\boldsymbol{k}N}$. This result, together with the way definitions were set implies

5.7. Theorem. *The correspondence $Z_F^{k,l,m} \mapsto Z_b^{k,l,m}$ extends to a unique isomorphism*

$$\mathcal{F}_{\boldsymbol{k}N} \xrightarrow{\cong} \mathcal{F}_{b\boldsymbol{k}N} / \big\langle \bar{b}_i - \bar{b}_{N-i-1},\ i = 0, \ldots, \lfloor (N-1)/2 \rfloor \big\rangle.$$

Consequently, using 9.4 we get,

Corollary. *For sufficiently large n, $\mathcal{ZF}_{\boldsymbol{k}N}$ is canonically isomorphic to $\mathcal{F}_{b\boldsymbol{k}N}/\langle\bar{b}_i - \bar{b}_{N-i-1}\rangle$. In particular,* $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathbb{Q}N\gamma\mu\nu} = N - \lfloor N/2\rfloor = \lfloor (N+1)/2\rfloor = \lceil N/2\rceil$.

In plain words, there are no relations among the b_is except (consequences of) the $\lfloor N/2\rfloor$ symmetries for b-chains: $b_i = b_{N-i-1}$. The remainder of this section will be dedicated to a much harder analogue of above calculation concerning M-variables (not used in further sections).

5.8. Let $\mathcal{F}_{M\boldsymbol{k}N}$ be the free $\boldsymbol{k}$-module on $N-1$ symbols $\bar{M}_i$, $i = 0, \dots, N-1$.
Consider the "special symmetries"

$$X_k := Z_M^{k,k+1,N-2k-2} - Z_M^{k+1,k,N-2k-2}, \quad k = 0, 1, \dots, \lfloor (N-1)/2 \rfloor, \tag{22}$$

where $Z_M^{k,l,m}$ is defined to mimic $Z^{k,l,m}$ expressed in terms of Ms, using the distinguished recursion lowering m, i.e. by the following version of (17):

$$Z_M^{k,l,m} := \sum_{j=0}^{m} (-1)^j \binom{m}{j} \bar{M}_{l+j},$$

$$X_k = \bar{M}_k + \sum_{j=0}^{k} (-1)^{j+1} \left[\binom{k}{j} + \binom{k+1}{j+1} \right] \bar{M}_{k+j+1}, \quad k \leqslant N/2 - 1,$$

$$X_0 = \bar{M}_0 - 2\bar{M}_1 = \bar{M}_0 \uparrow (1, -2),$$

$$X_1 = \bar{M}_1 - 3\bar{M}_2 + 2\bar{M}_3 = \bar{M}_1 \uparrow (1, -3, 2),$$

$$X_2 = \bar{M}_2 - 4\bar{M}_3 + 5\bar{M}_4 - 2\bar{M}_5 = \bar{M}_2 \uparrow (1, -4, 5, -2),$$

$$X_3 = \cdots = \bar{M}_3 \uparrow (1, -5, 9, -7, 2),$$

$$X_4 = \cdots = \bar{M}_4 \uparrow (1, -6, 14, -16, 9, -2),$$

and in general

$$X_k = \bar{M}_k \uparrow \left(1, -k, \dots, (-1)^{k-1}\left(\binom{k}{2} + \binom{k+1}{2}\right), (-1)^k(2k+1), (-1)^{k+1}2\right). \tag{23}$$

Our aim now is to show that all other symmetries are expressed in terms of X_k. For $s \geqslant 0$, these formulas generalize to

$$X_j^{(s)} := Z_M^{j,j+s,*} - Z_M^{j+s,j,*} = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} (-1)^k \binom{s-k-1}{k} X_{j+k}, \tag{24}$$

where $2j + s + * = N$ and $\lfloor\ \rfloor$ denotes the greatest integer part. In particular,

$$X_j^{(0)} = 0, \qquad X_j^{(1)} = X_j^{(2)} = X_j,$$
$$X_j^{(3)} = X_j - X_{j+1},$$
$$X_j^{(4)} = X_j - 2X_{j+1},$$
$$X_j^{(5)} = X_j - 3X_{j+1} + X_{j+2},$$
$$X_j^{(6)} = X_j - 4X_{j+1} + 3X_{j+2},$$
$$X_j^{(7)} = X_j - 5X_{j+1} + 6X_{j+2} - X_{j+3},$$
$$X_j^{(8)} = X_j - 6X_{j+1} + 10X_{j+2} - 4X_{j+3},$$
$$X_j^{(9)} = X_j - 7X_{j+1} + 15X_{j+2} - 10X_{j+3} + X_{j+4}$$

as it is easy to verify directly. Formula (24) in general follows from a cumbersome induction. Alternatively, by (17) and (22) Eq. (24) is equivalent to the following statement in the free $\boldsymbol{k}$-module on $\bar{M}$-symbols:

$$\begin{aligned}
&\sum_{i=0}^{j+s}(-1)^i\binom{j+s}{i}\bar{M}_{j+i} - \sum_{i=0}^{j}(-1)^i\binom{j}{i}\bar{M}_{j+s+i} \\
&\quad = \sum_{k=0}^{[\frac{s-1}{2}]}(-1)^k\binom{s-k-1}{k}\left(\sum_{i=0}^{j+k+1}(-1)^i\binom{j+k+1}{i}\bar{M}_{j+k+i}\right. \\
&\qquad \left. - \sum_{i=0}^{j+k}(-1)^i\binom{j+k}{i}\bar{M}_{j+k+i+1}\right).
\end{aligned}$$

Equate the coefficients in front of $(-1)^i\bar{M}_{j+i}$ to obtain that (24) is equivalent to the assertion that, for all i,

$$\binom{j+s}{i} - (-1)^s\binom{j}{i-s} = \sum_{k=0}^{\lfloor\frac{s-1}{2}\rfloor}\binom{s-k-1}{k}\left(\binom{j+k+1}{i-k} + \binom{j+k}{i-k-1}\right), \tag{25}$$

with the convention $\binom{a}{b} = 0$ whenever b falls out of the range $0 \leqslant b \leqslant a$. E.g. if $s = 5$, $j = 4$, the equality reads $126 = 126$ for $i = 4$ and $88 = 88$ for $i = 6$.

5.9. Proposition. *Equation* (25) *holds for all integers* $i > 0$, $j > 0$, $s > 0$.

Proof. Label Eq. (25) with the triple (j, s, i). Roughly speaking, we do the induction on s, however being a bit careful with the choice of j and i in the induction step (it seems that the direct implication $(j, s, i) \Rightarrow (j, s+1, i)$ is far too complex to be exhibited).

If $i = 1$, then (25) is the tautology $j + s = j + s$. If $s = 1, i > 0, j > 0$, Eq. (25) becomes the tautology $\binom{j+1}{i} + \binom{j}{i-1} = \binom{j+1}{i} + \binom{j}{i-1}$, and for $s = 2$ the simple identity $\binom{j+2}{i} - \binom{j}{i-2} = \binom{j+1}{i} + \binom{j}{i-1}$. If $i \geqslant 2$, then adding LHS for (j, s, i) and for $(j+1, s-1, i-1)$ together, we get $\binom{j+s}{i}+\binom{j+s}{i-1}-(-1)^{s-1}(\binom{j+1}{i-s}-\binom{j}{i-s})=\binom{j+s+1}{i}-(-1)^{s+1}\binom{j}{i-s-1}$, what is the LHS of $(j,s+1,i)$. When adding the RHS of (j,s,i) and of $(j+1,s-1,i-1)$ add pairwise the summands for label k in (j,s,i) and those for label $k-1$ in $(j+1,s-1,i-1)$: for such combination of ks the expression in the brackets on the RHS is identical and the prefactors involving s add nicely.

The identity (25) therefore holds for all $j>0, s>0, i>0$. $\square$

We have thus proved

5.10. Theorem. *Equation* (24) *holds for all $s>0$. In particular, $X_j^{(s)}$ belong to the $\boldsymbol{k}$-linear span of special symmetries $X_0, X_1, \ldots, X_{\lfloor (N-1)/2\rfloor}$. The formulas for $X_j = X_j^{(1)}$ show that they are $\boldsymbol{k}$-linearly independent in $\mathcal{F}_{M\boldsymbol{k}N}$. The correspondence $Z_F^{k,l,m} \mapsto Z_M^{k,l,m}$ extends to a well-defined isomorphism of $\boldsymbol{k}$-modules $\mathcal{F}_{\boldsymbol{k}N} \to \mathcal{F}_{M\boldsymbol{k}N}/\langle X_k, k=1,\ldots,\lfloor (N-1)/2\rfloor\rangle$.*

This gives another way to see the dimension of the space of Z-tensors over a field: $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathbb{Q}N\gamma\mu\nu} = \#\bar{M}_i - \#X_k = N - \lfloor N/2\rfloor = \lfloor (N+1)/2\rfloor$. With respect to the obvious natural $GL_n(\boldsymbol{k})$-actions, the isomorphism in the theorem are also $GL_n(\boldsymbol{k})$-equivariant.

6. Formula involving derivatives of coth$(x/2)$

We return to proving (12) for Ansatz $A_K = B_K/K!$ for $N \geqslant 4$ even. This reads

$$\beta_N K_{0,N} + \sum_{k=1}^{N/2} \beta_{2k}\beta_{N-2k} K_{2k,N-2k} = 0, \quad \text{where } \beta_{2k} := \frac{B_{2k}}{(2k)!}. \tag{26}$$

Let $I \leqslant N/2-1$. Using (20) and (21) and replacing $b_i = b_{N-i-1}$ for $i \geqslant N/2$ we get

$$\sum_{i=0}^{N/2-1} \binom{N+1}{i+1} \beta_N b_i + \sum_{k=0}^{N/2} \sum_{i=0}^{2k-1} \binom{2k}{i} \beta_{2k}\beta_{N-2k} b_i = 0.$$

Now we recall that not all b_is are independent. In the case of $\mathfrak{g} = su(2)$ one has $b_i = 0$ for $1 \leqslant i \leqslant N/2-1$ and one has only the terms with b_0. In the generic/universal case, the $M_{N-1} = b_0, b_1, \ldots, b_{N/2-1}$s are independent, hence we need $\alpha_i = 0$ for all i. But it is easy to replace b_i by b_{N-i-1} whenever $i \geqslant N/2$ and then we get an equation of the form $\sum_{i=0}^{N/2-1} \alpha_i b_i = 0$, where $\alpha_0, \ldots, \alpha_{N/2-1}$ are rational numbers depending on N and i. Thus in universal case, (26) boils down to $\alpha_i = 0$ for all $0 \leqslant i \leqslant N/2-1$. If $i < N/2$, then the coefficient in front of b_i in the second (double) sum is

$$\sum_{k=\lfloor \frac{i}{2}\rfloor+1}^{N/2-1} \beta_{2k}\beta_{N-2k}\binom{2k}{i} + \sum_{l=\lfloor \frac{N-i-1}{2}\rfloor+1}^{N/2-1} \beta_{2l}\beta_{N-2l}\binom{2l}{N-i-1}.$$

In these two sums, it will be convenient to count the summation index from $N/2$ downwards ($k \mapsto N/2 - k$ etc.). Thus, (26) is equivalent to

$$\binom{N+1}{i+1}\beta_N + \sum_{k=1}^{N/2-1-\lfloor \frac{i}{2} \rfloor} \beta_{2k}\beta_{N-2k}\binom{N-2k}{i} + \sum_{l=1}^{\lfloor \frac{i}{2} \rfloor} \beta_{2l}\beta_{N-2l}\binom{N-2l}{i-2l+1} = 0$$

for all $0 \leqslant i \leqslant N/2 - 1$. Now we subtract $\binom{N}{i}\beta_N = \binom{N}{i}\beta_N\beta_0$ from the first summand and absorb it into the first sum, as the new $k = 0$ summand. Then, the remainder of the first summand is $\binom{N}{i+1}\beta_N$ what can be absorbed into the second sum, as the new $l = 0$ summand. We obtain

$$\sum_{k=0}^{N/2-1-\lfloor \frac{i}{2} \rfloor} \beta_{2k}\beta_{N-2k}\binom{N-2k}{i} + \sum_{l=0}^{\lfloor \frac{i}{2} \rfloor} \beta_{2l}\beta_{N-2l}\binom{N-2l}{i-2l+1} = 0. \tag{27}$$

The generating function for the even Bernoulli numbers is

$$f(x) := (x/2)\coth(x/2) = \sum_{J=0}^{\infty} \frac{B_{2J}}{(2J)!}x^{2J} = \sum_{J=0}^{\infty} \beta_{2J}x^{2J}. \tag{28}$$

It can be easily checked that Eq. (27) follows from the following functional equation for f:

$$\frac{1}{i!}ff^{(i)} + \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \beta_{2k}\frac{xf^{(i-2k+1)}}{(i-2k+1)!} - \delta_{i,\,\mathrm{even}}\beta_i f = 0, \tag{29}$$

where $\delta_{i,\,\mathrm{even}} = 1$ if i is even and vanishes otherwise. To see this we write the whole LHS as a power series expansion in x and then for fixed N even number with $N > i$ we equate the coefficient in front of x^{N-i} with zero. So obtained equation multiply with $(N-i)!$ and we obtain (27). The summand with $\delta_{i,\,\mathrm{even}} = 1$ was added to offset the extension of the first sum in (27) to $k = N/2 - i/2$, for i even, which is needed to make the correspondence of that sum with $\frac{1}{i!}ff^{(i)}$.

The rest of the section is devoted to the proof of the functional equation (29).

Denote by $g := \coth(x/2)$ and $g^{(j)} := \frac{d^j}{dx}\coth(x/2)$. Then $f = xg/2$ and Eq. (29) becomes

$$\left[\frac{1}{2}\frac{1}{i!}gg^{(i)} + \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \beta_{2k}\frac{g^{(i-2k+1)}}{(i-2k+1)!}\right]x^2 + \left[\frac{1}{2}\frac{1}{(i-1)!}gg^{(i)} + \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \beta_{2k}\frac{g^{(i-2k)}}{(i-2k)!} - \delta_{i,\,\mathrm{even}}\beta_i g\right]x$$
$$= 0.$$

If we denote the expression in brackets in front of x by I_i then the expression in brackets in front of x^2 equals I_{i+1} (the $\delta_{i,\,\mathrm{even}}$-term effectively cancels the $k = i/2$ summand). Note that I_i and I_{i+1} are hyperbolic functions, and x and x^2 are linearly independent under the action by multiplication of the algebra of all hyperbolic functions on the space of all functions of one variable. Thus we have the equality *only if* $I_i = I_{i+1} = 0$. We will show $I_i = 0$ for all i by induction. We rephrase this as the following statement:

Suppose $i \geqslant 2$. Then the following identity holds:

$$\boxed{\tfrac{i}{2}\coth(x/2)\tfrac{d^{i-1}}{dx}\coth(x/2)+\textstyle\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}\tfrac{d^{i-2k}}{dx}\coth(x/2)=0.}$$

To simplify notation denote $g^{(j)} := \frac{d^j}{dx}\coth(x/2)$. Then we can rewrite the statement as

$$\frac{i}{2}gg^{(i-1)}+\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}g^{(i-2k)}=0. \tag{30}$$

In fact, this series of equations (as well as the proof below) holds also for $\mathrm{th}(x/2)$, and, more generally, for $g=\frac{Ae^{x/2}+e^{-x/2}}{Ae^{x/2}-e^{-x/2}}$ where A is any constant.

Proof. *Basis of induction.* Substituting $\coth(x/2)$ for g observe that $2g'+g^2-1=0$, hence $g''+gg'=0$, what is our identity (30) for $i=2$.

The step of induction. Suppose the identity holds for i and act with d/dx to the whole identity to obtain

$$(i/2)g'g^{(i-1)}+(i/2)gg^{(i)}+\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}g^{(i-2k+1)}=0.$$

Now replace g' by $(1/2)(1-g^2)$. By the induction hypothesis, we may also replace $(-i/4)g^2g^{(i-1)}$ by

$$gg^{(i)}/2+(g/2)\sum_{s=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2s}B_{2s}g^{(i-2s)},$$

thus obtaining

$$\frac{i+1}{2}gg^{(i)}+(i/4)g^{(i-1)}+\frac{1}{2}\sum_{s=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2s}B_{2s}gg^{(i-2s)}+\sum_{k=0}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}g^{(i-2k+1)}=0.$$

Further replace $gg^{(i-2s)}/2$ by the sum expression given by the induction hypothesis to obtain

$$\begin{aligned}&\frac{i+1}{2}gg^{(i)}+(i/4)g^{(i-1)}-\sum_{s=1}^{\lfloor\frac{i-1}{2}\rfloor}\frac{1}{i-2s+1}\binom{i}{2s}B_{2s}\sum_{r=0}^{\lfloor\frac{i-2s}{2}\rfloor}\binom{i-2s+1}{2r}B_{2r}g^{(i-2s-2r+1)}\\&\quad+g^{(i+1)}+\sum_{k=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}g^{(i-2k+1)}=0.\end{aligned}$$

Since the condition $r \leqslant \lfloor\frac{i-2s}{2}\rfloor$, is equivalent to $r+s \leqslant \lfloor\frac{i}{2}\rfloor$, we have

$$\frac{i+1}{2}gg^{(i)}+g^{(i+1)}+(i/4)g^{(i-1)}-\sum_{l=1}^{\lfloor\frac{i}{2}\rfloor}\sum_{s=1}^{l}\frac{i!}{(i-2l+1)!}\frac{B_{2s}}{(2s)!}\frac{B_{2l-2s}}{(2l-2s)!}g^{(i-2l+1)}$$
$$+\sum_{k=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}g^{(i-2k+1)}=0.$$

Using the identity

$$\sum_{s=1}^{l}\frac{B_{2s}}{(2s)!}\frac{B_{2l-2s}}{(2l-2s)!}=\frac{-B_{2l}}{(2l-1)!}+\frac{1}{4}\delta_{l,1},$$

valid for $l>0$ (proof: use the obvious identity $xf'=f-f^2+(x/2)^2$ for the generating function f in (28)) and $B_2=1/12$, we then obtain

$$\frac{i+1}{2}gg^{(i)}+g^{(i+1)}+\sum_{l=1}^{\lfloor\frac{i}{2}\rfloor}\binom{i}{2l-1}B_{2l}g^{(i-2l+1)}+\sum_{k=1}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{2k}B_{2k}g^{(i-2k+1)}=0.$$

Using $\binom{i}{2k}+\binom{i}{2k-1}=\binom{i+1}{2k}$ we recover Eq. (30) with i replaced by $i+1$. This finishes the proof. □

7. Prerequisites on formal group schemes in functorial approach

In the next three sections we present an invariant derivation of our main formula by the first author (N. Durov), valid over arbitrary rings containing $\mathbb{Q}$. The idea of this proof is the following. Given an n-dimensional complex Lie algebra $\mathfrak{g}$, we can always find a complex Lie group G with Lie algebra equal to $\mathfrak{g}$. Elements $X\in\mathfrak{g}$ correspond to right-invariant vector fields X_G on G, and clearly $[X,Y]_G=[X_G,Y_G]$, so we get an embedding $\mathfrak{g}\to\mathrm{Vect}(G)$ of $\mathfrak{g}$ into the Lie algebra $\mathrm{Vect}(G)$ of vector fields on G. If $e\in U\subset G$, $U\to\mathbb{C}^n$ is a coordinate neighborhood of the identity of G, we get an embedding $\mathfrak{g}\to\mathrm{Vect}(U)$, $X\mapsto X_G|_U$, and these vector fields can be expressed in terms of these coordinates, i.e. they can be written as some differential operators in n variables with analytic coefficients. Basically we obtain an embedding of $\mathfrak{g}$ into some completion of the Weyl algebra in $2n$ generators. Of course, this embedding depends on the coordinate chart; a natural choice would be to take the chart given by the exponential map $\exp:\mathfrak{g}\to G$. The embedding thus obtained, when written in coordinates, turns out to be exactly the one defined by the main formula of this paper. However, we would like to deduce such a formula over any ring containing $\mathbb{Q}$, where such complex-analytic arguments cannot be used. We proceed by replacing in this argument all Lie groups and complex manifolds by their analogues in formal geometry – namely, formal groups and formal schemes. This determines the layout of the next three sections.

Section 7 is dedicated to some generalities on formal schemes. Most notions and notations are variants of those developed in [SGA3] for the case of group schemes. We develop a similar functorial formalism for formal schemes, suited for our purpose, without paying too much attention to representability questions. Our exposition differs from most currently used approaches since we never require our rings to be noetherian. Besides, this section contains the construction of a formal group with given Lie algebra as well as some computations of tangent spaces and vector fields.

Section 8 contains some generalities on Weyl algebras and completed Weyl algebras; the aim here is to develop some invariant descriptions of well-known mathematical objects, valid over any commutative base ring and for any finitely generated projective module. Besides, we establish isomorphisms between Lie algebras of derivations of (completed) symmetric algebras and some Lie subalgebras of the (completed) Weyl algebras.

Finally, in Section 9 we deal with some questions of pro-representability, and use the results of the previous two sections to obtain and prove an invariant version of the main formula (cf. 9.13). Of course, explicit computation in terms of a chosen base of the Lie algebra $\mathfrak{g}$ and its structural constants gives us again the formula already proved in the first part of this paper by other methods.

So we proceed with an exposition of our functorial approach to formal schemes.

7.1. Fix a base ring $\boldsymbol{k} \supset \mathbb{Q}$. We will use the category $\mathcal{P}$ defined as follows:

$$\operatorname{Ob}\mathcal{P} := \big\{(R, I) \bigm| I \subset R\text{—a nilpotent ideal in a commutative ring } R\big\},$$
$$\operatorname{Hom}_{\mathcal{P}}\big((R, I), (R', I')\big) := \big\{\text{ring homomorphisms } \varphi : R \to R' \bigm| \varphi(I) \subset I'\big\}.$$

Category $\mathcal{P}$ has pushouts (amalgamated sums) constructed as follows:

$$\begin{array}{ccc} (R, I) & \longrightarrow & (R', I') \\ \downarrow & & \downarrow \\ (R'', I'') & \longrightarrow & \big(R' \otimes_R R'', I'(R' \otimes_R R'') + I''(R' \otimes_R R'')\big). \end{array}$$

There is a distinguished object $\boldsymbol{k} = (\boldsymbol{k}, 0)$ in $\mathcal{P}$. We will consider the category ${}_{\boldsymbol{k}\backslash}\mathcal{P}$ of morphisms in $\mathcal{P}$ with source $\boldsymbol{k}$. Sometimes we denote (R, I) simply by R and I is assumed; then we denote I by I_R. When we write tensor products in ${}_{\boldsymbol{k}\backslash}\mathcal{P}$, they are usually understood as amalgamated sums just described, i.e. $I_{R'\otimes R''} = I_{R'} \cdot (R' \otimes R'') + I_{R''} \cdot (R' \otimes R'')$.

7.2. Our basic category of interest will be $\mathcal{E} := \mathbf{Funct}({}_{\boldsymbol{k}\backslash}\mathcal{P}, \text{Sets})$ whose objects will be called presheaves (of sets) and should be viewed as 'formal varieties'; $\text{Grp}(\mathcal{E}) = \mathcal{E}^{\text{Grp}} := \mathbf{Funct}({}_{\boldsymbol{k}\backslash}\mathcal{P}, \text{Grp})$ is the category of group objects in $\mathcal{E}$. Its objects should be viewed as presheaves of groups or 'formal group schemes'.

For any (R', I') denote by $\operatorname{Spf}(R', I')$ the corresponding representable functor $(R, I) \mapsto \operatorname{Hom}_{\mathcal{P}}((R', I'), (R, I))$; if $I' = 0$, $\operatorname{Spf}(R', I')$ is also denoted by $\operatorname{Spec} R'$.

Consider the following examples of rings and groups in $\mathcal{E}$ that might clarify the relationship with other approaches to formal groups:

$$\begin{aligned} &\mathscr{O} : (R, I) \mapsto R && \text{(a ring in } \mathcal{E}\text{)},\\ &\hat{\mathbb{G}}_a : (R, I) \mapsto I && \text{(a group in } \mathcal{E}\text{)},\\ &\hat{\mathbb{G}}_m : (R, I) \mapsto (1 + I)^{\times} && \text{(a group in } \mathcal{E}\text{)}. \end{aligned}$$

Only the first of these examples will be used in the sequel.

For any $\boldsymbol{k}$-module M consider the $\mathscr{O}$-modules

$$\mathbf{W}(M):(R,I)\mapsto R\otimes_{\boldsymbol{k}} M,\qquad \mathbf{W}(M)\supset\mathbf{W}^{\omega}(M):(R,I)\mapsto I\cdot(R\otimes_{\boldsymbol{k}} M).$$

If M is free or projective, $\mathbf{W}^{\omega}(M):(R,I)\mapsto I\otimes_{\boldsymbol{k}} M$; in this case one should think of $\mathbf{W}(M)$ as "vector space M considered as a manifold," and of $\mathbf{W}^{\omega}(M)\subset\mathbf{W}(M)$ as "formal neighborhood of zero in $\mathbf{W}(M)$."

7.3. Given a morphism $\boldsymbol{k}\xrightarrow{\pi}\boldsymbol{k}'$ in $\mathcal{P}$, it induces a functor $\pi_!:{}_{\boldsymbol{k}'\backslash}\mathcal{P}\to{}_{\boldsymbol{k}\backslash}\mathcal{P}$, $\varphi\mapsto\varphi\circ\pi$, hence the restriction or base change functors $\pi^*:\mathcal{E}=\mathbf{Funct}({}_{\boldsymbol{k}\backslash}\mathcal{P},\mathrm{Sets})\to\mathbf{Funct}({}_{\boldsymbol{k}'\backslash}\mathcal{P},\mathrm{Sets})=:\mathcal{E}_{\boldsymbol{k}'}$, $F\mapsto F\circ\pi_!$. Functor $\pi^*(F)$ is usually denoted by $F|_{\boldsymbol{k}'}$ or $F_{(\boldsymbol{k}')}$. Functor $\pi^*:\mathcal{E}\to\mathcal{E}_{\boldsymbol{k}'}$ is exact and has a right adjoint $\pi_*:\mathcal{E}_{\boldsymbol{k}'}\to\mathcal{E}$ computed as follows: for any $F:{}_{\boldsymbol{k}'\backslash}\mathcal{P}\to\mathrm{Sets}$ we define $\pi_*F:{}_{\boldsymbol{k}\backslash}\mathcal{P}\to\mathrm{Sets}$ by $\pi_*F:R\mapsto F(R\otimes_{\boldsymbol{k}}\boldsymbol{k}')$; tensor products are understood as coproducts in $\mathcal{P}$ (as explained above in 7.1); $\_\mapsto\_\otimes_{\boldsymbol{k}}\boldsymbol{k}'$ is a functor ${}_{\boldsymbol{k}\backslash}\mathcal{P}\to{}_{\boldsymbol{k}'\backslash}\mathcal{P}$.

Notation. π_*F is usually denoted by $R_{\boldsymbol{k}'/\boldsymbol{k}}(F)$ ("*Weil scalar restriction*") or $\prod_{\boldsymbol{k}'/\boldsymbol{k}}F$.

7.4. Projective limits (e.g. direct products, fibered products, kernels) are computed in $\mathcal{E}$, $\mathcal{E}^{\mathrm{Grp}}$ etc. componentwise, e.g. $F\times G:R\mapsto F(R)\times G(R)$, for $F,G\in\operatorname{Ob}\mathcal{E}$. Category $\mathcal{E}$ is in fact a closed Cartesian category. In particular, it has inner homs: given $F,G\in\operatorname{Ob}\mathcal{E}$, the inner hom is the presheaf

$$\mathbf{Hom}(F,G):R\mapsto\operatorname{Hom}_{\mathcal{E}_R}(F|_R,G|_R)$$

with the characteristic property $\operatorname{Hom}_{\mathcal{E}}(F,\mathbf{Hom}(G,H))\cong\operatorname{Hom}_{\mathcal{E}}(F\times G,H)$. There are also canonical maps $\mathbf{Hom}(F,G)\times F\to G$, $\mathbf{Hom}(F,\mathbf{Hom}(G,H))\cong\mathbf{Hom}(F\times G,H)$ and so on. If F and G are presheaves of groups, one defines similarly $\mathbf{Hom}_{\mathrm{Grp}}(F,G)\subset\mathbf{Hom}(F,G)$. There is an obvious subfunctor $\mathbf{Isom}(F,G)\subset\mathbf{Hom}(F,G)$ and the special cases $\mathbf{End}(F):=\mathbf{Hom}(F,F)$ and $\mathbf{Aut}(F):=\mathbf{Isom}(F,F)$.

Functor $\mathbf{Aut}(F)$ is a group in $\mathcal{E}$; any group homomorphism $G\xrightarrow{\rho}\mathbf{Aut}(F)$ gives a group action $\nu:G\times F\to F$ and conversely.

We have the global sections functor $\Gamma:\mathcal{E}\to\mathrm{Sets}$, $F\mapsto F(\boldsymbol{k})$ which is exact and satisfies $\Gamma(\mathbf{Hom}(F,G))\cong\operatorname{Hom}_{\mathcal{E}}(F,G)$.

7.5 *(Tangent spaces).* Consider the dual numbers algebra $\boldsymbol{k}[\varepsilon]=\boldsymbol{k}[T]/(T^2)$. In our setup, we have in fact two different versions of this: $\boldsymbol{k}[\varepsilon]:=(\boldsymbol{k}[\varepsilon],(\varepsilon))$ and $\boldsymbol{k}[\varepsilon]^{\omega}:=(\boldsymbol{k}[\varepsilon],0)$. Notice also the following canonical morphisms in $\mathcal{P}$: $\boldsymbol{k}[\varepsilon]^{\omega}\xrightarrow{\nu}\boldsymbol{k}[\varepsilon]$, inclusions $\boldsymbol{k}\overset{i}{\hookrightarrow}\boldsymbol{k}[\varepsilon]^{\omega}$, $\boldsymbol{k}\hookrightarrow\boldsymbol{k}[\varepsilon]$ and the projections $p:\boldsymbol{k}[\varepsilon]^{\omega}$ and $\boldsymbol{k}[\varepsilon]\to\boldsymbol{k}$.

For any functor $F\in\operatorname{Ob}\mathcal{E}$ consider two new functors $TF:=\prod_{\boldsymbol{k}[\varepsilon]/\boldsymbol{k}}(F|_{\boldsymbol{k}[\varepsilon]})$ and $T^{\omega}F:=\prod_{\boldsymbol{k}[\varepsilon]^{\omega}/\boldsymbol{k}}(F|_{\boldsymbol{k}[\varepsilon]^{\omega}})$. Then $TF:R\mapsto F(\boldsymbol{k}[\varepsilon]\otimes_{\boldsymbol{k}}R)$, $T^{\omega}F:R\mapsto F(\boldsymbol{k}[\varepsilon]^{\omega}\otimes_{\boldsymbol{k}}R)$, or, more precisely, $TF:(R,I)\mapsto F((R[\varepsilon],I+R\varepsilon))$, $T^{\omega}F:(R,I)\mapsto F((R[\varepsilon],I+I\varepsilon))$.

Notation. $R[\varepsilon]:=\boldsymbol{k}[\varepsilon]\otimes_{\boldsymbol{k}}R$, $R[\varepsilon]^{\omega}:=\boldsymbol{k}[\varepsilon]^{\omega}\otimes_{\boldsymbol{k}}R$ where tensor products are understood as in 7.1.

There is a canonical map $T^{\omega}F\xrightarrow{\nu_*}TF$, as well as maps $\pi:=p_*:TF\to F$ and $s:=i_*:F\to T^{\omega}F\to TF$. (One should think of F as a "manifold," TF—its tangent bundle, $\pi:TF\to F$ its structural map, $s : F \to TF$—zero section, $T^\omega F \hookrightarrow TF$—formal neighborhood of zero section in TF.)

If G is a group, then TG and $T^\omega G$ are also groups (in $\mathcal{E}$) and π, ν, s—group homomorphisms; we define $\mathbf{Lie}(G)$ (respectively $\mathbf{Lie}^\omega(G)$) to be the kernel of $TG \xrightarrow{\pi} G$ (respectively $T^\omega G \xrightarrow{\pi} G$):

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Lie}^\omega(G) \arrow[r] \arrow[d] & T^\omega G \arrow[r, shift left, "\pi"] \arrow[d] & G \arrow[l, shift left, "s"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{Lie}(G) \arrow[r] & TG \arrow[r, shift left, "\pi"] & G \arrow[l, shift left, "s"] \arrow[r] & 0. \end{tikzcd}$$

$\mathbf{Lie}^\omega(G), \mathbf{Lie}(G)$ are clearly $\mathscr{O}$-modules; if G is a "good" presheaf of groups (cf. [SGA3, Ch. I]), e.g. pro-representable, $\mathbf{Lie}^\omega(G)$ and $\mathbf{Lie}(G)$ have natural $\mathscr{O}$-Lie algebra structures in $\mathcal{E}$.

Put $\mathfrak{g} := \Gamma\,\mathbf{Lie}(G)$; this is a Lie algebra over $\boldsymbol{k}$, hence by adjointness of functors $\mathbf{W} \dashv \Gamma$ we have a canonical map $\mathbf{W}(\mathfrak{g}) \to \mathbf{Lie}(G)$. In most interesting situations this map is an isomorphism, and it maps $\mathbf{W}^\omega(\mathfrak{g}) \subset \mathbf{W}(\mathfrak{g})$ into $\mathbf{Lie}^\omega(G) \subset \mathbf{Lie}(G)$; then we identify $\mathbf{Lie}(G)$ with $\mathbf{W}(\mathfrak{g})$ and $\mathbf{Lie}^\omega(G)$ with $\mathbf{W}^\omega(\mathfrak{g})$.

7.6. For any $F \in \operatorname{Ob}\mathcal{E}$ we define $\mathbf{Vect}(F) := \mathbf{Hom}_F(F, TF) : R \mapsto \{\varphi \in \operatorname{Hom}_{\mathcal{E}}(F|_R, TF|_R) : \pi|_R \circ \varphi = \mathrm{id}_{F|_R}\}$ (sections of TF over F); and similarly $\mathbf{Vect}^\omega(F) := \mathbf{Hom}_F(F, T^\omega F) \to \mathbf{Vect}(F)$; elements of $\mathbf{Vect}(F)(R)$ are "vector fields" on F defined over R.

7.7. Short exact sequence of groups

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Lie}(G) \arrow[r] & TG \arrow[r, shift left, "\pi"] & G \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

splits; so any $\bar{x} \in TG(R)$ can be written in form $s(g) \cdot X$ where $X \in \mathbf{Lie}(G)(R)$ and $g \in G(R)$; this decomposition is unique since necessarily $g = \pi(\bar{x})$, $X = s(g)^{-1} \cdot \bar{x}$. This splitting gives us an isomorphism $TG \xrightarrow{\sim} G \times \mathbf{Lie}(G)$, and similarly $T^\omega G \xrightarrow{\sim} G \times \mathbf{Lie}^\omega(G)$; in interesting situations $\mathbf{Lie}(G) \cong \mathbf{W}(\mathfrak{g})$ and $\mathbf{Lie}^\omega(G) \cong \mathbf{W}^\omega(\mathfrak{g})$, hence $TG \cong G \times \mathbf{W}(\mathfrak{g})$ and $T^\omega G \cong G \times \mathbf{W}^\omega(\mathfrak{g})$.

Recall that TG is a group, hence G acts on TG (say, from the left) by means of $G \xrightarrow{s} TG$; hence G acts also on $\mathbf{Vect}(G) = \mathbf{Hom}_G(G, TG) \cong \mathbf{Hom}_G(G, G \times \mathbf{W}(\mathfrak{g})) \cong \mathbf{Hom}(G, \mathbf{W}(\mathfrak{g}))$. Constant maps of $\operatorname{Hom}(G, \mathbf{W}(\mathfrak{g}))$ correspond to left-invariant vector fields under this identification.

Right-invariant vector fields give us another isomorphism $TG \xrightarrow{\sim} \mathbf{Lie}(G) \times G$, corresponding to decomposition $\bar{x} = X \cdot s(g)$.

7.8 *(Tangent spaces of $\mathbf{W}^\omega(M)$ and $\mathbf{W}(M)$).* Let M be a $\boldsymbol{k}$-module. Then for any $R = (R, I) \in \operatorname{Ob}({}_{\boldsymbol{k}\backslash}\mathcal{P})$ we have

$$\mathbf{W}^\omega(M)(R) = I \cdot (R \otimes_{\boldsymbol{k}} M) \subset \mathbf{W}(M)(R) = R \otimes_{\boldsymbol{k}} M,$$

$$\mathbf{W}(M)\bigl(R[\varepsilon]^\omega\bigr) = \boldsymbol{k}[\varepsilon] \otimes_{\boldsymbol{k}} M = \mathbf{W}(M)(R) \oplus \varepsilon \cdot \mathbf{W}(M)(R),$$

$$\mathbf{W}^{\omega}(M)\big(R[\varepsilon]\big) = I\cdot(R\otimes_{\boldsymbol{k}} M) + \varepsilon R\otimes_{\boldsymbol{k}} M = \mathbf{W}^{\omega}(M)(R)\oplus\varepsilon\cdot\mathbf{W}(M)(R),$$
$$\mathbf{W}^{\omega}(M)\big(R[\varepsilon]^{\omega}\big) = \mathbf{W}^{\omega}(M)(R)\oplus\varepsilon\cdot\mathbf{W}^{\omega}(M)(R).$$

From this we get the following four *split* exact sequences of abelian groups in $\mathcal{E}$:

$$0\to\mathbf{W}(M)\xrightarrow{\cdot\varepsilon}T\mathbf{W}(M)\to\mathbf{W}(M)\to 0,$$
$$0\to\mathbf{W}(M)\xrightarrow{\cdot\varepsilon}T^{\omega}\mathbf{W}(M)\to\mathbf{W}(M)\to 0,$$
$$0\to\mathbf{W}(M)\xrightarrow{\cdot\varepsilon}T\mathbf{W}^{\omega}(M)\to\mathbf{W}^{\omega}(M)\to 0,$$
$$0\to\mathbf{W}^{\omega}(M)\xrightarrow{\cdot\varepsilon}T^{\omega}\mathbf{W}^{\omega}(M)\to\mathbf{W}^{\omega}(M)\to 0.$$

We deduce from these sequences canonical isomorphisms

$$T\mathbf{W}^{\omega}(M)\cong\mathbf{W}^{\omega}(M)\times\mathbf{W}(M),\qquad T^{\omega}\mathbf{W}(M)\cong\mathbf{W}(M)\times\mathbf{W}(M)$$

and so on.

7.9 *(Formal groups with given Lie algebra; exponential map). From now on we assume $\boldsymbol{k}\supset\mathbb{Q}$.* Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{k}$, projective and finitely generated as a $\boldsymbol{k}$-module (free of finite rank suffices for most applications). Denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping $\boldsymbol{k}$-algebra, and by $\mathcal{U}_i(\mathfrak{g})$ its increasing filtration. The PBW theorem implies that $\mathcal{U}_i(\mathfrak{g})/\mathcal{U}_{i-1}(\mathfrak{g})\cong S^i(\mathfrak{g})$. The diagonal map $\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ induces the comultiplication $\Delta:\mathcal{U}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})\cong\mathcal{U}(\mathfrak{g})\otimes_{\boldsymbol{k}}\mathcal{U}(\mathfrak{g})$, and the map $\mathfrak{g}\to 0$ induces the counit $\boldsymbol{\eta}:\mathcal{U}(\mathfrak{g})\to\mathcal{U}(0)=\boldsymbol{k}$. Thus $\mathcal{U}(\mathfrak{g})$ is a bialgebra and even a Hopf algebra. For any $(R,I)/\boldsymbol{k}$ we have a canonical isomorphism $\mathcal{U}(\mathfrak{g})_{(R)}=R\otimes_{\boldsymbol{k}}\mathcal{U}(\mathfrak{g})\xleftarrow{\sim}\mathcal{U}(\mathfrak{g}_{(R)})$, and $I\cdot\mathcal{U}(\mathfrak{g})_{(R)}$ is a nilpotent two-sided ideal in this ring. Now consider two formal groups

$$\mathbf{Exp}_{+}(\mathfrak{g}):(R,I)\mapsto\big\{\alpha\in I\cdot\mathcal{U}(\mathfrak{g})_{(R)}\;\big|\;\Delta(\alpha)=1\otimes\alpha+\alpha\otimes 1,\ \boldsymbol{\eta}(\alpha)=0\big\},$$
$$G:=\mathbf{Exp}_{\times}(\mathfrak{g}):(R,I)\mapsto\big\{\alpha\in 1+I\cdot\mathcal{U}(\mathfrak{g})_{(R)}\;\big|\;\Delta(\alpha)=\alpha\otimes\alpha,\ \boldsymbol{\eta}(\alpha)=1\big\}.$$

Group operation on $\mathbf{Exp}_{\times}(\mathfrak{g})$ is induced by the multiplication of $\mathcal{U}(\mathfrak{g})_{(R)}$, and that of $\mathbf{Exp}_{+}(\mathfrak{g})$ is determined by the requirement that $\exp:\mathbf{Exp}_{+}(\mathfrak{g})\to\mathbf{Exp}_{\times}(\mathfrak{g})$ be a *group* isomorphism where $\exp:\alpha\mapsto\sum_{n\geqslant 0}\frac{\alpha^n}{n!}$ is defined by the usual exponential series; it makes sense for $\alpha\in I\cdot\mathcal{U}(\mathfrak{g})_{(R)}$ since $I\cdot\mathcal{U}(\mathfrak{g})_{(R)}$ is a nilpotent ideal and $\mathbb{Q}\subset R$; it is well known [Bourbaki, Chapter II] that $\exp_{(R,I)}:\mathbf{Exp}_{+}(\mathfrak{g})(R,I)\xrightarrow{\sim}\mathbf{Exp}_{\times}(\mathfrak{g})(R,I)$ is an isomorphism (i.e. it is bijective—inverse is given by the log, and $\Delta(\alpha)=1\otimes\alpha+\alpha\otimes 1$, $\boldsymbol{\eta}(\alpha)=0$ iff $\Delta(\exp(\alpha))=\exp(\alpha)\otimes\exp(\alpha)$, $\boldsymbol{\eta}(\exp(\alpha))=1$). Recall [Bourbaki, Chapter II] that the *Campbell–Hausdorff series* is a formal Lie power series $H(X,Y)$ in two variables X, Y with rational coefficients, defined by the formal equality $\exp(H(X,Y))=\exp(X)\exp(Y)$ in the Magnus algebra $\hat{A}_{\mathbb{Q}}(X,Y)=\hat{U}_{\mathbb{Q}}(L(X,Y))$. This implies that the group law on $\mathbf{Exp}_{+}(\mathfrak{g})(R,I)$ is given by $H(X,Y)$, i.e. $\alpha\star\beta=H(\alpha,\beta)$. [Since α and β lie in a nilideal, $H(\alpha,\beta)$ is a finite sum.]

There is also a canonical map $\nu:\mathbf{W}^{\omega}(\mathfrak{g})\to\mathbf{Exp}_{+}(\mathfrak{g})$ defined as follows: $\nu_R:\mathbf{W}^{\omega}(\mathfrak{g})(R)=I_R\cdot(R\otimes_{\boldsymbol{k}}\mathfrak{g})=I_R\cdot\mathfrak{g}_{(R)}\to\mathbf{Exp}_{+}(\mathfrak{g})(R)=\{\alpha\in I\cdot\mathcal{U}(\mathfrak{g})_{(R)}\mid\Delta(\alpha)=1\otimes\alpha+\alpha\otimes 1,\ \boldsymbol{\eta}(\alpha)=0\}$ is induced by the canonical embedding $\mathfrak{g}_{(R)}\hookrightarrow\mathcal{U}(\mathfrak{g}_{(R)})\cong\mathcal{U}(\mathfrak{g})_{(R)}$. Actually, ν is an *isomorphism*:

In [Bourbaki, II], it is proved that for $R \supset \mathbb{Q}$ all primitive elements of $\mathcal{U}(\mathfrak{g})_{(R)}$ come from $\mathfrak{g}_{(R)}$. By the PBW theorem, $\mathcal{U}(\mathfrak{g}_{(R)})/\mathfrak{g}_{(R)}$ is flat, hence $I \cdot \mathcal{U}(\mathfrak{g}_{(R)}) \cap \mathfrak{g}_{(R)} = I \cdot \mathfrak{g}_{(R)}$, hence all ν_R are isomorphisms, i.e. ν is an isomorphism:

$$\begin{tikzcd}
\mathbf{W}^{\omega}(\mathfrak{g}) \arrow[rr, "\nu", "\sim"'] \arrow[dr, "\sim", "\exp'"'] & & \mathbf{Exp}_{+}(\mathfrak{g}) \arrow[dl, "\sim"', "\exp"] \\
& G = \mathbf{Exp}_{\times}(\mathfrak{g}). &
\end{tikzcd}$$

We put $\exp' := \exp \circ \nu : \mathbf{W}^{\omega}(\mathfrak{g}) \xrightarrow{\sim} G = \mathbf{Exp}_{\times}(\mathfrak{g})$ and call $\exp'$ *the exponential map* for $\mathfrak{g}$. Note that in the diagram above only exp is a *group* isomorphism.

7.10 *(Lie algebra and tangent space computations for the exponential map).* Now we are going to compute the maps $T(\nu)$, $T(\exp)$ and $T(\exp')$ from the previous diagram as well as $T(\mathbf{W}^{\omega}(\mathfrak{g}))$, $T(\mathbf{Exp}_{+}(\mathfrak{g}))$ and $T(\mathbf{Exp}_{\times}(\mathfrak{g}))$. In particular, we shall prove $\mathbf{Lie}(G) \cong \mathbf{W}(\mathfrak{g})$ and $\mathbf{Lie}^{\omega}(G) \cong \mathbf{W}^{\omega}(\mathfrak{g})$.

7.10.1. We know $T(\mathbf{W}^{\omega}(\mathfrak{g})) \cong \mathbf{W}^{\omega}(\mathfrak{g}) \times \mathbf{W}(\mathfrak{g})$ and $T^{\omega}(\mathbf{W}^{\omega}(\mathfrak{g})) \cong \mathbf{W}^{\omega}(\mathfrak{g}) \times \mathbf{W}^{\omega}(\mathfrak{g})$ with the first projection as the structural map $T(\mathbf{W}^{\omega}(\mathfrak{g})) \to \mathbf{W}^{\omega}(\mathfrak{g})$ (respectively).

7.10.2. We compute $\mathbf{Lie}(\mathbf{Exp}_{+}(\mathfrak{g}))(R)$ by definition (here $R = (R, I)$):

$$\begin{aligned}
\mathbf{Lie}\big(\mathbf{Exp}_{+}(\mathfrak{g})\big)(R) &= \mathrm{Ker}\big(\mathbf{Exp}_{+}(\mathfrak{g})\big(R[\varepsilon]\big) \to \mathbf{Exp}_{+}(\mathfrak{g})(R)\big) \\
&= \big\{\alpha\varepsilon \in \mathcal{U}(\mathfrak{g})_{(R[\varepsilon])} \bigm| \Delta(\alpha\varepsilon) = \alpha\varepsilon \otimes 1 + 1 \otimes \alpha\varepsilon,\ \boldsymbol{\eta}(\alpha\varepsilon) = 0\big\} \\
&\cong \mathbf{W}(\mathfrak{g})(R),
\end{aligned}$$

hence $\mathbf{Lie}(\mathbf{Exp}_{+}(\mathfrak{g})) = \mathbf{W}(\mathfrak{g})$, and similarly $\mathbf{Lie}^{\omega}(\mathbf{Exp}_{+}(\mathfrak{g})) = \mathbf{W}^{\omega}(\mathfrak{g})$.

For any formal group, hence for $\mathbf{Exp}_{+}(\mathfrak{g})$, we have (cf. 7.7) $T(\mathbf{Exp}_{+}(\mathfrak{g})) \cong \mathbf{Exp}_{+}(\mathfrak{g}) \times \mathbf{Lie}(\mathbf{Exp}_{+}(\mathfrak{g})) \cong \mathbf{Exp}_{+}(\mathfrak{g}) \times \mathbf{W}(\mathfrak{g})$, and similarly for $T^{\omega}(\mathbf{Exp}_{+}(\mathfrak{g}))$.

7.10.3. Since $\exp : \mathbf{Exp}_{+}(\mathfrak{g}) \to G = \mathbf{Exp}_{\times}(\mathfrak{g})$ is an isomorphism, we can expect similar descriptions for $\mathbf{Lie}(G)$ and $\mathbf{Lie}^{\omega}(G)$. One can also check directly $\mathbf{Lie}(G)(R) = \{1 + \alpha\varepsilon \in \mathcal{U}(\mathfrak{g})_{(R[\varepsilon])} \mid \Delta(1 + \alpha\varepsilon) = (1 + \alpha\varepsilon) \otimes (1 + \alpha\varepsilon),\ \boldsymbol{\eta}(1 + \alpha\varepsilon) = 1\} \cong \mathbf{W}(\mathfrak{g})(R)$ and similarly for $\mathbf{Lie}^{\omega}(G)$. Note that $\mathbf{Lie}(\exp) : \mathbf{Lie}(\mathbf{Exp}_{+}(\mathfrak{g})) \to \mathbf{Lie}(\mathbf{Exp}_{\times}(\mathfrak{g}))$, $\alpha\varepsilon \mapsto 1 + \alpha\varepsilon$ is an isomorphism even if $\boldsymbol{k}$ does not contain $\mathbb{Q}$. If we identify $\mathbf{Lie}(\mathbf{Exp}_{+}(\mathfrak{g}))$ and $\mathbf{Lie}(G)$ with $\mathbf{W}(\mathfrak{g})$, $\mathbf{Lie}(\exp)$ is identified with the identity map, and similarly for $\mathbf{Lie}^{\omega}$; hence we get

$$\begin{tikzcd}
T(\mathbf{Exp}_{+}(\mathfrak{g})) \arrow[r, "\sim"] \arrow[d, "T(\exp)"'] & \mathbf{Exp}_{+}(\mathfrak{g}) \times \mathbf{W}(\mathfrak{g}) \arrow[r] \arrow[d, "\exp \times \mathrm{id}"'] & \mathbf{Exp}_{+}(\mathfrak{g}) \arrow[d, "\exp"] \\
TG \arrow[r, "\sim"] & G \times \mathbf{W}(\mathfrak{g}) \arrow[r] & G.
\end{tikzcd} \tag{31}$$

7.10.4. One checks that our identifications $\mathbf{Lie}(\mathbf{Exp}_{+}(\mathfrak{g})) \cong \mathbf{Lie}(G) \cong \mathbf{W}(\mathfrak{g})$ and their analogues for $\mathbf{Lie}^{\omega}$ are compatible with the original Lie algebra structure on $\mathfrak{g}$. Indeed, consider two elements $1 + X\varepsilon$, $1 + Y\eta$ in $G(R[\varepsilon, \eta]/(\varepsilon^2, \eta^2))$; then by definition

$$\begin{aligned}1+[X,Y]_{\mathbf{Lie}(G)}\cdot\varepsilon\eta &= (1+X\varepsilon)(1+Y\eta)(1+X\varepsilon)^{-1}(1+Y\eta)^{-1}\\ &= (1+X\varepsilon+Y\eta+XY\varepsilon\eta)(1+X\varepsilon+Y\eta+YX\varepsilon\eta)^{-1}\\ &= 1+(XY-YX)\varepsilon\eta = 1+[X,Y]_{\mathfrak g}\cdot\varepsilon\eta.\end{aligned}$$

7.10.5. Consider now $\nu:\mathbf{W}^{\omega}(\mathfrak g)\to\mathbf{Exp}_+(\mathfrak g)$ (this is an isomorphism, but not an isomorphism of groups):

$$\begin{tikzcd}
T(\mathbf{W}^{\omega}(\mathfrak g)) \arrow[r, "\sim"] \arrow[d, "T(\nu)"'] & \mathbf{W}^{\omega}(\mathfrak g)\times\mathbf{W}(\mathfrak g) \arrow[r] \arrow[d, dashed, "\nu'"] & \mathbf{W}^{\omega}(\mathfrak g) \arrow[d, "\nu"] \\
T(\mathbf{Exp}_+(\mathfrak g)) \arrow[r, "\sim"] & \mathbf{Exp}_+(\mathfrak g)\times\mathbf{W}(\mathfrak g) \arrow[r, "\pi"] & \mathbf{Exp}_+(\mathfrak g). \arrow[l, bend right, dotted, "s"']
\end{tikzcd} \tag{32}$$

In this diagram we identify $\mathbf{Lie}(\mathbf{Exp}_+(\mathfrak g))$ with $\mathbf{W}(\mathfrak g)$ as explained above. We want to compute the map ν'. Let $R=(R,I)$, $X\in\mathbf{W}^{\omega}(\mathfrak g)=I\otimes_{\boldsymbol k}\mathfrak g$ and $Y\in\mathbf{W}(\mathfrak g)(R)=R\otimes_{\boldsymbol k}\mathfrak g$; the corresponding element of $T(\mathbf{W}^{\omega}(\mathfrak g))(R)=\mathbf{W}^{\omega}(\mathfrak g)(R[\varepsilon])$ is given by $X+Y\varepsilon$, and $T(\nu)_R(X+Y\varepsilon)=\nu_{R[\varepsilon]}(X+Y\varepsilon)=X+Y\varepsilon$ considered as a primitive element of $\mathcal U(\mathfrak g)_{(R[\varepsilon])}$. Clearly $\pi_R(X+Y\varepsilon)=X$, $s_R\pi_R(X+Y\varepsilon)=X$, and we want to find $Z\in\mathbf{Lie}(\mathbf{Exp}_+(\mathfrak g))(R)=\mathbf{W}(\mathfrak g)(R)$, such that $(X+Y\varepsilon)=X\star Z\varepsilon$ inside $T(\mathbf{Exp}_+(\mathfrak g))(R)$. Since $(-X)\star X=0$, we have $Z\varepsilon=(-X)\star(X+Y\varepsilon)$; classical formula for $H(X+Y,-X)$ mod $\deg_Y 2$ (or for $H(-X,X+Y)$) [Bourbaki, II 6.5.5] gives us $Z=\sum_{n\geqslant 0}\frac{(-\operatorname{ad}X)^n}{(n+1)!}(Y)$. Hence ν' is given by $\nu'_R:(X,Y)\mapsto(X,Z)$ with Z defined by the above formula.

Remark. If we consider the other canonical splitting of $T(\mathbf{Exp}_+(\mathfrak g))$ by right-invariant vector fields, we obtain almost the same formula for Z, but without the $(-1)^n$ factors: $Z=\sum_{n\geqslant 0}\frac{(\operatorname{ad}X)^n}{(n+1)!}(Y)$.

7.10.6. We now consider the exponential map $\exp'=\exp\circ\nu:\mathbf{W}^{\omega}(\mathfrak g)\to G$. By composing (31) and (32) we get the commutative diagram

$$\begin{tikzcd}
T(\mathbf{W}^{\omega}(\mathfrak g)) \arrow[r, "\sim"] \arrow[d, "T(\exp')"] & \mathbf{W}^{\omega}(\mathfrak g)\times\mathbf{W}(\mathfrak g) \arrow[r] \arrow[d, dashed, "\tau"] & \mathbf{W}^{\omega}(\mathfrak g) \arrow[d, "\exp'"] \\
TG \arrow[r, "\sim"] & G\times\mathbf{W}(\mathfrak g) \arrow[r, "\pi"] & G. \arrow[l, bend right, dotted, "s"']
\end{tikzcd} \tag{33}$$

The middle vertical map is given by $\tau:(X,Y)\mapsto(\exp(X),\sum_{n\geqslant 0}\frac{(-\operatorname{ad}X)^n}{(n+1)!}(Y))$. One can check this directly, without referring to 7.10.5 and the properties of Campbell–Hausdorff series. Indeed, if $\tau(X,Y)=(\exp(X),Z)$ for some $X\in\mathbf{W}^{\omega}(\mathfrak g)(R)$, $Y,Z\in\mathbf{W}(\mathfrak g)(R)$, we must have $\exp(X+Y\varepsilon)=\exp(X)(1+Z\varepsilon)$ inside $G(R[\varepsilon])\subset\mathcal U(\mathfrak g)_{(R[\varepsilon])}$. Since $\varepsilon^2=0$, we have $(X+Y\varepsilon)^n=X^n+\sum_{p+q=n-1}X^pYX^q\cdot\varepsilon$, hence $\exp(X)\cdot Z=\sum_{p,q\geqslant 0}\frac{X^pYX^q}{(p+q+1)!}$. One then checks directly that $Z=\sum_{n\geqslant 0}\frac{(-\operatorname{ad}X)^n}{(n+1)!}(Y)$ satisfies this equality. The argument like the one in the proof of [Bourbaki, II, 6.5.5] is better: one checks almost immediately that $(\operatorname{ad}X)(Z)=(1-e^{-\operatorname{ad}X})(Y)$, and then divides formally by $\operatorname{ad}X$, considering both sides as elements of the completed free Lie algebra in X and Y over $\mathbb Q$.

7.11 *(Formal completions).* Suppose we are given morphism $H \xrightarrow{\varphi} F$ in $\mathcal{E} = \mathcal{E}_{\boldsymbol{k}}$. In most cases of interest φ will be a monomorphism, i.e. H can be identified with a subfunctor of F. Denote the natural map $(R, I) \to (R/I, 0)$ by $\pi_{(R,I)}$. We say that H *is complete over* (or *in*) F if for any (R, I) in ${}_{\boldsymbol{k}\backslash}\mathcal{P}$ the following diagram is Cartesian

$$\begin{tikzcd} H(R,I) \arrow[r, "H(\pi_{(R,I)})"] \arrow[d, "\varphi_{(R,I)}"] & H(R/I,0) \arrow[d, "\varphi_{R/I}"] \\ F(R,I) \arrow[r, "F(\pi_{(R,I)})"] & F(R/I,0). \end{tikzcd} \tag{34}$$

Arbitrary morphism $H \xrightarrow{\varphi} F$ may be factored as $H \xrightarrow{u} \hat{F}_H \xrightarrow{\kappa} F$ where $\hat{F}_H \xrightarrow{\kappa} F$ is complete, and this *completion* $\hat{F}_H$ is universal in the sense that if $F' \xrightarrow{\kappa'} F$ is complete and $H \xrightarrow{u'} F'$ is such that $\kappa' \circ u' = \varphi$, then there is a unique map $F' \xrightarrow{\chi} \hat{F}_H$ such that $\chi \circ u' = u$ and $\kappa \circ \chi = \kappa'$. To this aim, define $\hat{F}_H(R, I)$ to be the fibered product of $F(R, I)$ and $H(R/I, 0)$ over $F(R/I, 0)$:

$$\begin{tikzcd} \hat{F}_H(R,I) \arrow[r] \arrow[d, "\kappa_{(R,I)}"] & H(R/I,0) \arrow[d, "\varphi_{R/I}"] \\ F(R,I) \arrow[r] & F(R/I,0). \end{tikzcd} \tag{35}$$

If $\varphi : H \to F$ is monic, then κ and u are also. $\hat{F}_H$ is called the *formal completion of F along H*. If F and H are groups (or $\mathcal{O}$-modules) and $H \to F$ a morphism of groups (respectively $\mathcal{O}$-modules) then $\hat{F}_H$ is also, and $H \xrightarrow{u} \hat{F}_H \xrightarrow{\kappa} F$ will be morphisms of such.

Example.

(a) For any $\boldsymbol{k}$-module M, $\mathbf{W}^\omega(M)$ is the formal completion of $\mathbf{W}(M)$ along $0 \subset \mathbf{W}(M)$. Indeed, the universality of (35) with $F = \mathbf{W}(M)$, $\hat{F}_H = \mathbf{W}^\omega(M)$, $H = 0$ is immediate.
(b) Suppose F is left-exact functor (e.g. pro-representable). Then $T^\omega F$ is the completion of TF along the zero section $s : F \to TF$.
(c) If G is a group in $\mathcal{E}$, left exact as a functor, then $\mathbf{Lie}^\omega(G)$ is the completion of $\mathbf{Lie}(G)$ along 0.

8. Weyl algebras

8.1. Let $\boldsymbol{k}$ be a commutative ring, M a $\boldsymbol{k}$-module, $\Phi : M \times M \to \boldsymbol{k}$ a bilinear form which is symplectic: $\forall x \in M$, $\Phi(x, x) = 0$ (this implies $\forall x, y \in M$, $\Phi(x, y) = -\Phi(y, x)$). *We do not require the nondegeneration.* Consider the category $\mathcal{C}_M^\Phi$, objects of which are pairs (A, λ_A) where A is an associative $\boldsymbol{k}$-algebra and $\lambda_A : M \to A$ is a $\boldsymbol{k}$-linear map, such that $\forall x, y \in M$, $[\lambda(x), \lambda(y)]_A = \lambda(x)\lambda(y) - \lambda(y)\lambda(x) = \Phi(x, y) \cdot 1$; morphisms $(A, \lambda_A) \to (B, \lambda_B)$ are just $\boldsymbol{k}$-algebra homomorphisms $f : A \to B$ compatible with λs: $\lambda_B = f \circ \lambda_A$.

Definition. The universal (i.e. initial) object of the category $\mathcal{C}_M^\Phi$ will be denoted by $(SW(M, \Phi), i_M)$ and it will be called the *symplectic Weyl algebra of* (M, Φ).

In particular, $SW(M) = SW(M, \Phi)$ is an associative $\boldsymbol{k}$-algebra and $i_M : M \to SW(M)$ is a $\boldsymbol{k}$-linear map such that $\forall x, y \in M$, $[i_M(x), i_M(y)] = \Phi(x, y)$.

Of course, $SW(M)$ exists, it may be constructed as a quotient of tensor algebra, namely $T(M)/I_\Phi$, where $T(M) = \bigoplus_{n\geqslant 0} M^{\otimes n}$ is the tensor algebra of M, and $I_\Phi \subset T(M)$ is the two-sided ideal of $T(M)$ generated by the elements of the form $x \otimes y - y \otimes x - \Phi(x, y) \cdot 1$ for all $x, y \in M$.

Note that if $\{x_\alpha\}$ is a system of generators of M, by bilinearity it is sufficient to require $[i_M(x_\alpha), i_M(x_\beta)] = \Phi(x_\alpha, x_\beta)$ in the definition of $\mathcal{C}_M^\Phi$ and $SW(M)$, and I_Φ is generated by $x_\alpha \otimes x_\beta - x_\beta \otimes x_\alpha - \Phi(x_\alpha, x_\beta)$.

Remark. Symplectic Weyl algebras are very similar to Clifford algebras, and, in some other respects, to the universal enveloping algebras of Lie algebras.

8.2 *(Basic properties of symplectic Weyl algebras).*

(a) $SW(0) = \boldsymbol{k}$, $SW(M, 0)$ is the symmetric algebra $S(M)$.
(b) (functoriality) Given a $\boldsymbol{k}$-linear map $f : (M, \Phi) \to (N, \Psi)$ with $\Psi \circ (f \times f) = f \circ \Phi$ there is a unique map $SW(f) : SW(M, \Phi) \to SW(N, \Psi)$ satisfying the obvious conditions.
(c) If $M = \boldsymbol{k}e$ is free of rank 1, then necessarily $\Phi = 0$ and $SW(M) \cong S(M) \cong \boldsymbol{k}[e]$.
(d) Functor $SW : (M, \Phi) \mapsto SW(M, \Phi)$ commutes with filtered inductive limits in (M, Φ).
(e) If M is an orthogonal direct sum of M_1 and M_2: $M = M_1 \oplus M_2$ and $\Phi = \Phi_1 \oplus \Phi_2$, then the canonical map $SW(M_1) \otimes_{\boldsymbol{k}} SW(M_2) \to SW(M_1 \oplus M_2)$ (induced by canonical maps $SW(M_i) \to SW(M_1 \oplus M_2)$ and multiplication in $SW(M_1 \oplus M_2)$) is an *isomorphism of $\boldsymbol{k}$-algebras* (this is immediate by checking that $SW(M_1) \otimes_{\boldsymbol{k}} SW(M_2)$ satisfies the universal property).
(f) $SW(M, \Phi)$ commutes with base change: For any $\boldsymbol{k}'/\boldsymbol{k}$ we have $SW(M_{(\boldsymbol{k}')}, \Phi_{(\boldsymbol{k}')}) = SW(M, \Phi)_{(\boldsymbol{k}')}$.

8.3 *(Filtration and $\mathbb{Z}/2\mathbb{Z}$-grading).* Consider the natural grading of the tensor algebra $T(M) = \bigoplus_{n\geqslant 0} T^n(M) := \bigoplus_{n\geqslant 0} M^{\otimes n}$ and the corresponding increasing filtration $T_n(M) = T_{\leqslant n}(M) := \bigoplus_{m\leqslant n} T^m(M)$; clearly $T_0(M) = \boldsymbol{k}$, $T_1(M) = \boldsymbol{k} \oplus M$. $T(M)$ has canonical supergrading: $T^+(M) := \bigoplus_n T^{2n}(M)$, $T^-(M) := \bigoplus_n T^{2n+1}(M)$. Since $I_\Phi \subset T(M)$ is generated by even elements $x \otimes y - y \otimes x - \Phi(x, y) \in T^+(M)$, $SW(M) := T(M)/I_\Phi$ inherits a canonical supergrading: $SW^\pm(M) = \pi(T^\pm(M))$, $SW(M) = SW^+(M) \oplus SW^-(M)$ where $\pi : T(M) \to SW(M)$ is the projection. There is also the canonical filtration (sometimes called "Bernstein" or "arithmetic") on $SW(M)$—the image under π of the filtration of $T(M)$, namely $SW_n(M) := \pi(T_n(M))$. We have a canonical surjective map $\mathrm{gr}(\pi) : T(M) = \mathrm{gr}_F(T(M)) \to \mathrm{gr}_F(SW(M))$. Clearly, $\mathrm{gr}_F(SW(M))$ is generated by the image of M in $\mathrm{gr}_F^1(SW(M))$, and for any two elements $x, y \in M \subset T_1(M)$, their images $\bar{x}$, $\bar{y}$ in $\mathrm{gr}_F^1(SW(M))$ commute since $xy - yx \in SW_1(M)$. This means that $\mathrm{gr}(\pi)$ factorizes through $T(M) \to S(M)$, hence we obtain surjective maps $S(M) \xrightarrow{\varphi} \mathrm{gr}_F(SW(M))$ and $S^n(M) \xrightarrow{\varphi^n} \mathrm{gr}_F^n(SW(M)) = SW_n(M)/SW_{n-1}(M)$.

8.4. Theorem. *If M is a flat $\boldsymbol{k}$-module (e.g. free or projective), then φ and $\varphi^n : S^n(M) \to SW_n(M)/SW_{n-1}(M)$ are isomorphisms for all $n \geqslant 0$.*

Proof. (a) First assume M free with a base $(e_\alpha)_{\alpha\in\Lambda}$. By Zermelo lemma we may assume that Λ is well-ordered set. Denote by x_α and z_α the images of e_α in $SW(M)$ and $S(M)$, respectively.

Consider the set of nondecreasing sequences $Seq = \{I = (i_1, \dots, i_n) \mid i_k \in \Lambda,\ i_1 \leqslant i_2 \leqslant \dots \leqslant i_n\}$ with notation $|I| = n$ whenever $I = (i_1, \dots, i_n)$. If $J = (i_2, \dots, i_n)$, we write $I = (i_1, J)$. Notation $\lambda \leqslant I$ means $\lambda \leqslant i_k$, for all k, and similarly $\lambda < I$. Now, for any $I = (i_1, \dots, i_n) \in Seq$ define $x_I := x_{i_1} \cdots x_{i_n} \in SW(M)$, $z_I := z_{i_1} \cdots z_{i_n} \in S(M)$. Clearly, $\{z_I\}$ form a basis of $S(M)$, $\{z_I\}_{|I|=n}$ form a basis of $S^n(M)$, and $\{x_I\}$ generate a $\boldsymbol{k}$-submodule in $SW(M)$ containing the image of M and closed under multiplication (to this aim check by induction on $|J|$ that $x_\lambda x_J$ lies in this submodule, and then by induction on $|I|$ that $x_I x_J$ lies there as well). Consequently, it is sufficient to show that the $\{x_I\}$ are linearly independent. To prove this we proceed as in the proof of PBW theorem in [Bourbaki, Chapter I]. We construct by induction a family of compatible bilinear maps $\rho_n : M \times S_n(M) \to S_{n+1}(M)$ enjoying the following properties:

- (A_n) $\rho_n(x_\lambda, z_I) = z_\lambda z_I$ if $\lambda \leqslant I$, $|I| \leqslant n$ (or $I = \emptyset$); $\rho_n(x_\lambda, z_I) \equiv z_\lambda z_I \bmod S_n(M)$ if $|I| = n$ for any $\lambda \in \Lambda$;
- (B_n) $\rho_n(x_\lambda, z_I) = \rho_{n-1}(x_\lambda, z_I)$ if $|I| \leqslant n-1$;
- (C_n) $\rho_n(x_\lambda, \rho_{n-1}(x_\mu, z_I)) = \rho_n(x_\mu, \rho_{n-1}(x_\lambda, z_I)) + \Phi(e_\lambda, e_\mu) \cdot z_I$ if $\lambda, \mu \in \Lambda$, $|I| \leqslant n-1$.

We see that such a family is uniquely defined: if $\lambda \leqslant I$, $\rho_n(x_\lambda, z_I)$ is defined by (A_n); if $|I| \leqslant n-1$ by (B_n); if $I = (\mu, J)$, $|J| = n-1$ and $\lambda > \mu$ by (C_n); then one checks directly that the maps ρ_n so defined do satisfy (A_n), (B_n) and (C_n).

This map $\rho : M \times S(M) \to S(M)$ induces a map $\tilde\rho : M \to \mathrm{End}_{\boldsymbol{k}}(S(M))$, such that $[\tilde\rho(e_\lambda), \tilde\rho(e_\mu)] = \Phi(e_\lambda, e_\mu)$; hence by universal property of Weyl algebras $\tilde\rho$ induces a map $\tilde\rho' : SW(M) \to \mathrm{End}_{\boldsymbol{k}}(S(M))$. One sees that the map $SW(M) \to S(M)$ defined by $x \mapsto \tilde\rho'(x)(1)$ maps x_I into z_I, and $\{z_I\}$ are linearly independent; hence $\{x_I\}$ are also independent.

(b) If M is flat, it can be written, by a classical result of Lazard, as a filtered inductive limit of free modules: $M = \varinjlim M_\alpha$, where all M_α are free; consider on each M_α the pullback Φ_α of the form Φ with respect to the natural map $M_\alpha \to M$; then $SW(M) = \varinjlim SW(M_\alpha)$; everything commutes with $\varinjlim$ and $S^n(M_\alpha) \to \mathrm{gr}^n(SW(M_\alpha))$ are isomorphisms, hence the same holds for M. $\square$

8.5 *(Consequences).*

- (a) If M is flat then all $SW_n(M)/SW_{n-1}(M) \cong S^n(M)$ are flat, hence also $SW_n(M)$ (by induction on n) and $SW(M) = \varinjlim SW_n(M)$ are flat.
- (b) If M is projective, all $SW_n(M)/SW_{n-1}(M) \cong S^n(M)$ are projective, hence $SW_{n-1}(M)$ has a complement K_n in $SW_n(M)$; $K_n \cong SW_n(M)/SW_{n-1}(M)$ is projective, hence $SW_n(M) = \bigoplus_{m \leqslant n} K_m$ and $SW(M) = \bigoplus_{m \geqslant 0} K_m$ are projective as well.
- (b′) If M is projective and finitely generated, the same can be said about all $SW_n(M)/SW_{n-1}(M)$ and all $SW_n(M)$ [but *not* about $SW(M)$!].
- (c) If M is free then the same is true for all $SW_n(M)/SW_{n-1}(M)$, all $SW_n(M)$ and for $SW(M)$.
- (c′) If M is free and of finite rank then the same can be said about all $SW_n(M)/SW_{n-1}(M)$ and all $SW_n(M)$.
- (d) If M is flat, then maps $\boldsymbol{k} \to SW(M)$ and $M \to SW(M)$ are injective, so $\boldsymbol{k}$ and M can be identified with their images in $SW(M)$; moreover, $\boldsymbol{k} \oplus M \to SW(M)$ is injective, and its image is $SW_1(M)$.

8.6 *(Convolution).* Fix a symplectic $\boldsymbol{k}$-module (M, Φ), and let $i = i_M : M \to SW(M)$ be the canonical map. Given $u \in M^* = \mathrm{Hom}_{\boldsymbol{k}}(M, \boldsymbol{k})$, the *convolution map* $D_u : SW(M) \to SW(M)$ is a $\boldsymbol{k}$-derivation restricting to u on the image of M. In other words, we require (0) $D_u(\boldsymbol{k}\cdot 1)=0$; (1) $\forall\alpha,\beta\in SW(M)$, $D_u(\alpha\cdot\beta)=D_u(\alpha)\cdot\beta+\alpha\cdot D_u(\beta)$; (2) $\forall x\in M$, $D_u(i(x))=u(x)\cdot 1$. These conditions imply $D_u(x_1x_2\cdots x_n)=\sum_{i=1}^n u(x_i)\cdot x_1\cdots\hat{x}_i\cdots x_n$ for all $n\geqslant 1$ and for all $x_1,\dots,x_n\in i(M)$. This implies the uniqueness of D_u. To show the existence, consider the dual numbers $SW(M)[\varepsilon]:=\boldsymbol{k}[\varepsilon]\otimes_{\boldsymbol{k}} SW(M)=SW(M)\oplus SW(M)\varepsilon$, and the $\boldsymbol{k}$-linear map $i_u: M\to SW(M)[\varepsilon]$ given by $i_u: x\mapsto i(x)+u(x)\varepsilon$. One checks directly $[i_u(x),i_u(y)]=[i(x),i(y)]=\Phi(x,y)$, hence i_u induces a $\boldsymbol{k}$-algebra homomorphism $i'_u: SW(M)\to SW(M)[\varepsilon]$. Now for any $x\in SW(M)$ we have $i'_u(x)=x+D_u(x)\varepsilon$ for some $D_u(x)$, and this determines the convolution D_u.

8.7 *(Deformation).* Suppose $M=M_1\oplus M_2$, but M_1 and M_2 are not necessarily orthogonal. Inclusions $M_i\to M$ induce maps $SW(M_i)\to SW(M)$, hence (together with multiplication in $SW(M)$) a map $\rho: SW(M_1)\otimes SW(M_2)\to SW(M)$. In general, this is not a homomorphism of $\boldsymbol{k}$-algebras, but still a $\boldsymbol{k}$-linear map.

Claim. *If one of M_1 or M_2 is flat, then ρ is an isomorphism of $\boldsymbol{k}$-module.*

Proof. (a) Suppose M_2 is flat; write it in form $M_2=\varinjlim N_\alpha$ where all N_α are free of finite rank and consider on $M\oplus N_\alpha$ the symplectic form induced from M by $M_1\oplus N_\alpha\to M_1\oplus M_2=M$. Since everything commutes with inductive limits, it is sufficient to prove that $SW(M_1)\otimes SW(N_\alpha)\to SW(M_1\oplus N_\alpha)$ are isomorphisms, i.e. we can assume M_2 to be free of finite rank. An easy induction argument shows that we can assume M_2 free of rank one.

(b) Assume $M_2=\boldsymbol{k}\theta$ is free of rank 1, $M=M_1\oplus\boldsymbol{k}\theta$. Put $u(x):=\Phi(\theta,x)$ for any $x\in M_1$, so we get $u\in M_1^*$. Consider the action $\star$ of M on $SW(M_1)\otimes_{\boldsymbol{k}} S(\boldsymbol{k}\theta)=SW(M_1)\otimes_{\boldsymbol{k}}\boldsymbol{k}[\theta]=SW(M_1)[\theta]$: M_1 acts on $SW(M_1)[\theta]$ by left multiplication $x\star\sum_{j\geqslant 0}\alpha_j\theta^j=\sum_{j\geqslant 0}(x\alpha_j)\theta^j$ for any $x\in M_1$, $\alpha_j\in SW(M_1)$; and $\theta\star\sum_{j\geqslant 0}\alpha_j\theta^j=\sum_{j\geqslant 0}\alpha_j\theta^{j+1}+\sum_{j\geqslant 0}D_u(\alpha_j)\theta^j=\sum_{j\geqslant 0}(\alpha_{j-1}+D_u(\alpha_j))\theta^j$. So we get a $\boldsymbol{k}$-linear map $\gamma: M=M_1\oplus\boldsymbol{k}\theta\to\mathrm{End}_{\boldsymbol{k}}(SW(M_1)[\theta])$. It is easy to see that $[\gamma(x),\gamma(y)]=\Phi(x,y)$; if $x,y\in M_1$ or if $x=y=\theta$ this is evident; and in the mixed case $x=\theta$, $y\in M_1$ we calculate

$$\gamma(\theta)\gamma(y)\Bigl(\sum_{j\geqslant 0}\alpha_j\theta^j\Bigr)=\theta\star\Bigl(y\star\sum_{j\geqslant 0}\alpha_j\theta^j\Bigr)=\theta\star\sum_{j\geqslant 0}(y\alpha_j)\theta^j$$
$$=\sum_{j\geqslant 0}y\alpha_j\theta^{j+1}+\sum_{j\geqslant 0}D_u(y\alpha_j)\theta^j,$$
$$\gamma(y)\gamma(\theta)\Bigl(\sum_{j\geqslant 0}\alpha_j\theta^j\Bigr)=\sum_{j\geqslant 0}y\alpha_j\theta^{j+1}+\sum_{j\geqslant 0}yD_u(\alpha_j)\theta^j,$$
$$\bigl[\gamma(\theta),\gamma(y)\bigr]\Bigl(\sum_{j\geqslant 0}\alpha_j\theta^j\Bigr)=\sum_{j\geqslant 0}D_u(y)\alpha_j\theta^j=u(y)\cdot\sum_{j\geqslant 0}\alpha_j\theta^j,$$
$$\bigl[\gamma(\theta),\gamma(y)\bigr]=u(y)=\Phi(\theta,y).$$

This means that γ induces a map $\tilde{\gamma}: SW(M)\to\mathrm{End}_{\boldsymbol{k}}(SW(M_1)[\theta])$; now it is immediate that $\alpha\mapsto\tilde{\gamma}(\alpha)(1)$ gives a map $SW(M)\to SW(M_1)\otimes S(\boldsymbol{k}\theta)$ inverse to ρ, hence ρ is an isomorphism. $\square$

8.8. Note that if $M = \boldsymbol{k}e_1 \oplus \cdots \oplus \boldsymbol{k}e_n$, 8.7 implies that $S(M) = S(\boldsymbol{k}e_1) \otimes \cdots \otimes S(\boldsymbol{k}e_n) \to SW(M)$ is an isomorphism; this proves 8.4 in this case; the general case of 8.4 can be deduced from this by taking inductive limits.

8.9. Now suppose $M = Q \oplus P$ with flat and Φ-isotropic P and Q (i.e. $\Phi|_{P\times P} = 0$ and $\Phi|_{Q\times Q} = 0$). Then Φ is uniquely determined by the bilinear form $\varphi := \Phi|_{Q\times P} : Q \times P \to K$ since $\Phi(q + p, q' + p') = \varphi(q, p') - \varphi(q', p)$, and any bilinear form φ defines such a Φ. Algebra $\mathscr{D}_{Q,P,\varphi} = \mathscr{D}_\varphi := SW(Q \oplus P, \Phi)$ is called the *Weyl algebra defined by* φ. Since Q and P are isotropic, $SW(Q) \cong S(Q)$ and $SW(P) \cong S(P)$; since they are flat, the map $S(Q) \otimes_{\boldsymbol{k}} S(P) = SW(Q) \otimes_{\boldsymbol{k}} SW(P) \to SW(Q \oplus P, \varphi) = \mathscr{D}_\varphi$ is an isomorphism of $\boldsymbol{k}$-modules by 8.7.

Any $p \in P$ defines a form $d_\varphi(p) \in Q^*$ by the rule $d_\varphi(p) : q \mapsto \varphi(q, p)$, hence a derivation (= convolution) $D_p := D_{-d_\varphi(p)}$ on $S(Q) = SW(Q)$ (note the minus sign!). On the other hand, any $q \in Q$ acts on $S(Q)$ by multiplication $L_q : \alpha \mapsto q\alpha$. Since $[D_p, D_{p'}] = [L_q, L_{q'}] = 0$ and $[D_p, L_q] = -\varphi(q, p)$, we see that the map $q + p \mapsto L_q + D_p$ defines a $\boldsymbol{k}$-linear map $\mathscr{D}_\varphi = SW(Q \oplus P) \to \operatorname{End}_{\boldsymbol{k}}(S(Q))$, i.e. a *$\mathscr{D}_\varphi$-module structure on* $S(Q)$. Note that $S(Q) \subset \mathscr{D}_\varphi$ acts on $S(Q)$ with respect to this structure in the natural way, and $S(P) \subset \mathscr{D}_\varphi$ acts by convolutions, and, in particular, $P \subset S(P) \subset \mathscr{D}_\varphi$ by derivations, hence $S(Q) \otimes P \subset \mathscr{D}_\varphi$ also acts on $S(Q)$ by $\boldsymbol{k}$-derivations.

Notice that $S^r(Q) \cdot S^n(Q) \subset S^{r+n}(Q)$, $S^r(P) \cdot S^n(Q) \subset S^{n-r}(Q)$. The construction is preserved up to signs when we interchange Q and P, hence $\mathscr{D}_\varphi$ acts on $S(P)$ as well.

8.10 *(Completed Weyl algebra and its action on the completed symmetric algebra).* Consider the completed symmetric algebra

$$\hat{S}(Q) = \prod_{n \geqslant 0} S^n(Q) = \varprojlim \frac{S(Q)}{S^{\geqslant n}(Q)} = \varprojlim \frac{S(Q)}{(S^+(Q))^n}.$$

Clearly, $Q \subset S(Q) \subset \hat{S}(Q)$; if Q is a free $\boldsymbol{k}$-module of rank n, then $S(Q)$ is the algebra of polynomials in n variables, and $\hat{S}(Q)$ the algebra of formal power series.

Recall that $S(Q) \otimes_{\boldsymbol{k}} S(P) \xrightarrow{\rho} \mathscr{D}_\varphi$ is an isomorphism of $\boldsymbol{k}$-modules; we want to construct an algebra $\hat{\mathscr{D}}_\varphi \supset \mathscr{D}_\varphi$ and an isomorphism $\hat{\rho} : \hat{S}(Q) \otimes_{\boldsymbol{k}} S(P) \to \hat{\mathscr{D}}_\varphi$, compatible with ρ on $S(Q) \otimes_{\boldsymbol{k}} S(P) \subset \hat{S}(Q) \otimes_{\boldsymbol{k}} S(P)$ (recall that $S(P)$ is flat!). Take $\hat{\mathscr{D}}_\varphi := \hat{S}(Q) \otimes_{\boldsymbol{k}} S(P)$, $\hat{\rho} := \mathrm{id}$. For any $p \in P$, the convolution $D_p : S(Q) \to S(Q)$ maps $S^n(Q)$ into $S^{n-1}(Q)$, hence it is continuous and lifts to $\hat{D}_p : \hat{S}(Q) \to \hat{S}(Q)$. Since maps $\hat{L}_p : \hat{\mathscr{D}}_\varphi \to \hat{\mathscr{D}}_\varphi$ defined by $\hat{L}_p : \alpha \otimes \delta \mapsto \alpha \otimes p\delta + \hat{D}_p(\alpha) \otimes \delta$, considered for different $p \in P$, mutually commute, they define some $\hat{L}_\delta$ for all $\delta \in S(P)$.

We *define* the multiplication $\star$ on $\hat{\mathscr{D}}_\varphi$ by $(\alpha \otimes \delta) \star (\alpha' \otimes \delta') := ((L_\alpha \otimes 1) \circ \hat{L}_\delta)(\alpha' \otimes \delta')$, where $L_\alpha : \hat{S}(Q) \to \hat{S}(Q)$, $\beta \mapsto \alpha\beta$ is the usual multiplication map. It is straightforward to check that $\hat{\mathscr{D}}_\varphi$ is an associative $\boldsymbol{k}$-algebra and that $\mathscr{D}_\varphi \to \hat{\mathscr{D}}_\varphi$ is compatible with multiplication. It is enough to check associativity for $u, v, w \in \hat{S}(Q) \otimes S_{\leqslant n}(P) \subset \hat{\mathscr{D}}_\varphi$, for arbitrary $n \geqslant 0$. For any $k > 0$, one can find some $u', v', w' \in S_{<k+3n}(Q) \otimes S_{\leqslant n}(P)$ such that $u \equiv u'$ (mod $\hat{S}_{\geqslant k+3n}(Q) \otimes S(P)$) and so on. Notice that $(\hat{S}_{\geqslant k}(Q) \otimes S_{\leqslant n}(P)) \star (\hat{S}_{\geqslant l}(Q) \otimes S_{\leqslant n}(P)) \subset \hat{S}_{\geqslant k+l-n}(Q) \otimes S_{\leqslant 2n}(P)$ for any $k, l \geqslant 0$, hence $(u \star v) \star w \equiv u'v'w' \equiv u \star (v \star w)$ (mod $\hat{S}_{\geqslant k}(Q) \otimes S(P)$) for all $k > 0$. In a similar way, we construct an action of $\hat{\mathscr{D}}_\varphi \cong \hat{S}(Q) \otimes S(P)$ on $\hat{S}(Q)$ (elements of P act on $\hat{S}(Q)$ by means of the derivation maps $\hat{D}_p$ constructed above), and an action of $\hat{\mathscr{D}}_\varphi$ on $S(P)$ as well.

8.11 *(Classical Weyl algebras).* Suppose P is projective of finite type, and put $Q := P^*$ and $\varphi : Q \times P \to K$ be *minus* the canonical pairing. We put $\mathscr{D}_P := \mathscr{D}_\varphi$, $\hat{\mathscr{D}}_P := \hat{\mathscr{D}}_\varphi$. Then $\mathscr{D}_P = \mathscr{D}_\varphi = SW(P^* \oplus P)$ is a filtered associative algebra, and $\mathrm{gr}(\mathscr{D}_P) \cong S(Q \oplus P) \cong S(Q) \otimes S(P)$; $\mathscr{D}_P$ acts on $S(Q)$ and $S(P)$, and $\hat{\mathscr{D}}_P$ acts on $\hat{S}(Q)$ and $S(P)$.

8.12. If in addition P is free with base $\{e_j\}_{j=1}^n$ and $\{e^k\}_{k=1}^n$ is the dual base of $Q = P^*$, then $\hat{\mathscr{D}}_P = SW(Q \oplus P)$ is a free associative $\boldsymbol{k}$-algebra in $x_k := i(e^k)$ and $\partial^j := i(e_j)$ subject to the relations $[\partial^k, \partial^l] = 0 = [x_i, x_j]$, $[\partial^k, x_j] = \delta_j^k$. In this way, we obtain the classical Weyl algebra written in coordinates. $\hat{\mathscr{D}}_P$ in this situation corresponds to differential operators of the form $\sum f_{i_1 \dots i_n}(x_1, \dots, x_n)(\partial^1)^{i_1} \cdots (\partial^n)^{i_n}$, where $f_{i_1 \dots i_n}$ are formal power series, all but finitely many equal to zero.

8.13. In the situation of 8.9, the $\boldsymbol{k}$-submodule $\mathscr{L}_P := S(Q) \cdot P \subset \mathscr{D}_P$ is a Lie subalgebra. Indeed, $\forall \alpha, \alpha' \in S(Q)$ $\forall p, p' \in P$, $(\alpha p) \cdot (\alpha' p') = \alpha\alpha' pp' + \alpha D_p(\alpha')p'$, hence $[\alpha p, \alpha' p'] = \alpha D_p(\alpha') \cdot p' - \alpha' D_{p'}(\alpha) \cdot p \in \mathscr{L}_P$. Recall that there is a $\mathscr{D}_\varphi$-module structure on $S(Q)$, for which $S(Q) \subset \mathscr{D}_\varphi$ acts by multiplication and $P \subset \mathscr{D}_\varphi$ by derivations (namely, convolutions), hence $\mathscr{L}_P = S(Q) \cdot P \subset \mathscr{D}_\varphi$ acts on $S(Q)$ by derivations. This way we obtain a Lie algebra homomorphism $\tau : \mathscr{L}_P \cong S(Q) \otimes P \to \mathrm{Der}_{\boldsymbol{k}}(S(Q))$.

Proposition. *τ is an isomorphism under assumptions of* 8.11.

Proof. Any derivation $D \in \mathrm{Der}_{\boldsymbol{k}}(S(Q))$ corresponds to an algebra homomorphism $\sigma := 1_{S(Q)} + D\varepsilon : S(Q) \to S(Q)[\varepsilon]$, $\alpha \mapsto \alpha + D(\alpha) \cdot \varepsilon$, such that $\pi \circ \sigma = 1_{S(Q)}$ for $\pi : S(Q)[\varepsilon] \to S(Q)$, $\varepsilon \mapsto 0$. By the universal property of $S(Q)$, the map σ is defined by its restriction $\sigma|_Q : Q \to S(Q)[\varepsilon] = S(Q) \oplus S(Q) \cdot \varepsilon$. Clearly, $\sigma|_Q(x) = x + \varphi(x)\varepsilon$ for some map $\varphi : Q \to S(Q)$. Since P and Q are projective of finite rank, $\mathrm{Hom}_{\boldsymbol{k}}(Q, S(Q)) \cong S(Q) \otimes P$, so φ gives us an element $\tilde{\varphi} \in S(Q) \otimes P$. One checks that $\tau(\tilde{\varphi}) = D$ (it is enough to check this on $Q \subset S(Q)$ since a derivation of $S(Q)$ is completely determined by its restriction on Q). This way we obtain a map $\mathrm{Der}_{\boldsymbol{k}}(S(Q)) \to \mathscr{L}_P \cong S(Q) \otimes P$ inverse to τ. $\square$

8.14. Similarly, $\hat{\mathscr{L}}_P := \hat{S}(Q) \cdot P \subset \hat{\mathscr{D}}_P$ is closed under Lie bracket, and it acts by derivations on $\hat{S}(Q)$. All *continuous* derivations of $\hat{S}(Q)$ arise in this way.

9. Vector fields on formal affine spaces and end of the proof

9.0. Fix a projective $\boldsymbol{k}$-module P of finite type, put $Q = P^*$. We are going to compute the $\boldsymbol{k}$-algebras of vector fields on $\mathbf{W}(P)$ and $\mathbf{W}^\omega(P)$. More precisely, we will identify these vector fields with derivations of $S(Q)$ (respectively $\hat{S}(Q)$), hence with elements of $\mathscr{L}_P = S(Q) \cdot P \subset \mathscr{D}_P$ (respectively of $\hat{\mathscr{L}}_P \subset \hat{\mathscr{D}}_P$); we will show that this identification respects Lie bracket. Then we are going to use this to compute some vector fields defined in Section 7.

9.1 *(Representable functors).* Suppose $F \in \mathcal{E}_{\boldsymbol{k}} = \mathbf{Funct}({}_{\boldsymbol{k}\backslash}\mathcal{P}, \mathrm{Sets})$ is representable by some $A = (A, J) \in \mathrm{Ob}_{\boldsymbol{k}\backslash}\mathcal{P}$. This means that we have an element $X \in F(A)$, such that for any

$R = (R, I) \in {}_{k\backslash}\mathcal{P}$ and any $\xi \in F(R)$ there is a unique morphism $\varphi : A \to R$ in ${}_{k\backslash}\mathcal{P}$, such that $(F(\varphi))(X) = \xi$. One can also write $F(R) \cong \mathrm{Hom}_{{}_{k\backslash}\mathcal{P}}(A, R)$ or $F = \mathrm{Hom}(A, -)$.

Now consider $TF := \prod_{\boldsymbol{k}[\varepsilon]/\boldsymbol{k}} F : (R, I) \mapsto F(R[\varepsilon], I \oplus R\varepsilon)$ together with the projection $\pi : TF \to F$ induced by $R[\varepsilon] \xrightarrow{p} R$, $\varepsilon \mapsto 0$. By definition, $\mathrm{Vect}(F) = \Gamma(TF/F) = \mathrm{Hom}_F(F, TF)$ is the set of sections of TF/F. By Yoneda lemma, any $\sigma \in \mathrm{Vect}(F)$, i.e. a section $\sigma : F \to TF$, is determined by $\sigma_0 := \sigma_A(X) \in TF(A) = F(A[\varepsilon], J \oplus R\varepsilon) \cong \mathrm{Hom}_{{}_{k\backslash}\mathcal{P}}((A, J), (A[\varepsilon], J \oplus R\varepsilon))$. Denote by $\tilde{\sigma}_0 : (A, J) \to (A[\varepsilon], J \oplus R\varepsilon)$ the corresponding morphism in ${}_{k\backslash}\mathcal{P}$. Since σ is a section of π iff $p_A \circ \tilde{\sigma}_0 = \mathrm{id}_A$, then $\tilde{\sigma}_0 = \mathrm{id}_A + \varepsilon \cdot D$ for a uniquely determined $D : A \to A$, and $\tilde{\sigma}_0$ is $\boldsymbol{k}$-algebra homomorphism iff D is a $\boldsymbol{k}$-derivation of A: $\tilde{\sigma}_0(ab) = (a + \varepsilon D(a))(b + \varepsilon D(b)) = ab + \varepsilon(D(a)b + aD(b))$. We have constructed a bijection $\mathrm{Vect}(F) \xrightarrow{\lambda} \mathrm{Der}_{\boldsymbol{k}}(A)$. One sees immediately that λ is an isomorphism of $\boldsymbol{k}$-modules, where the $\boldsymbol{k}$-structure on $\mathrm{Vect}(F)$ comes from the $\boldsymbol{k}$-action $[c]_R : R[\varepsilon] \to R[\varepsilon]$, $x + y\varepsilon \mapsto x + cy\varepsilon$ for any $c \in \boldsymbol{k}$.

9.2. Proposition. $\lambda : \mathrm{Vect}(F) \to \mathrm{Der}_{\boldsymbol{k}}(A)$ *is an isomorphism of Lie algebras.*

Proof. Recall that the Lie bracket on $\mathrm{Vect}(F) = \Gamma(TF/F)$ is defined as follows. Consider three copies of the dual number algebra, $\boldsymbol{k}[\varepsilon]$, $\boldsymbol{k}[\eta]$, $\boldsymbol{k}[\zeta]$, the tensor product $\boldsymbol{k}[\varepsilon, \eta] = \boldsymbol{k}[\varepsilon] \otimes_{\boldsymbol{k}} \boldsymbol{k}[\eta]$ and the embeddings of $\boldsymbol{k}[\varepsilon]$, $\boldsymbol{k}[\eta]$, $\boldsymbol{k}[\zeta]$ into $\boldsymbol{k}[\varepsilon, \eta]$, denoted by $\varphi_\varepsilon, \varphi_\eta, \varphi_\zeta$, where the last map is determined by $\varphi_\zeta : \zeta \mapsto \varepsilon\eta$. Given a pair of sections $\sigma, \tau : F \to TF$, we consider σ as a section of $T_\varepsilon F$ and τ as a section of $T_\eta F$. Here $T_\varepsilon, T_\eta, T_\zeta$ are the corresponding "tangent bundles"; of course $T_\varepsilon F \cong T_\eta F \cong T_\zeta F \cong TF$; besides, $T_\varepsilon T_\eta F = T_\eta T_\varepsilon F = T_{\varepsilon,\eta} F := \prod_{\boldsymbol{k}[\varepsilon,\eta]/\boldsymbol{k}} (F|_{\boldsymbol{k}[\varepsilon,\eta]})$. We have two maps $F \xrightarrow{\sigma} T_\varepsilon F \xrightarrow{T_\varepsilon(\tau)} T_\varepsilon T_\eta F = T_{\varepsilon,\eta} F$ and $F \xrightarrow{\tau} T_\eta F \xrightarrow{T_\eta(\sigma)} T_\eta T_\varepsilon F = T_{\varepsilon,\eta} F$. The section $[\sigma, \tau] : F \to T_\zeta F$ is defined by $(\varphi_\zeta)_*([\sigma, \tau]) = T_\eta(\sigma) \circ \tau - T_\varepsilon(\tau) \circ \sigma$. Now $\forall x \in A$, $\sigma_A(x) = x + \varepsilon\lambda(\sigma)x \in A[\varepsilon]$, $T_\varepsilon(\tau)_A \circ \sigma_A : x \mapsto x + \varepsilon\lambda(\sigma)x + \eta\lambda(\tau)x + \varepsilon\eta\lambda(\tau)\lambda(\sigma)x$, and, similarly, $T_\eta(\sigma)_A \circ \tau_A : x \mapsto x + \varepsilon\lambda(\sigma)x + \eta\lambda(\tau)x + \varepsilon\eta\lambda(\sigma)\lambda(\tau)x$. Therefore $T_\varepsilon(\tau)_A \circ \sigma_A - T_\eta(\sigma)_A \circ \tau_A = [\lambda(\sigma), \lambda(\tau)] \cdot \varepsilon\eta$. $\square$

9.3. All this can be applied to $T^\omega F$ instead of TF: in this case we will have $\sigma_0 \in \mathrm{Hom}_{{}_{k\backslash}\mathcal{P}}((A, J), (A[\varepsilon], J \oplus J\varepsilon))$, so $\sigma_0 = \mathrm{id}_A + \varepsilon D$, where D is a derivation of A such that $D(J) \subset J$. This gives an isomorphism $\lambda^\omega : \mathrm{Vect}^\omega F \to \{D \in \mathrm{Der}_{\boldsymbol{k}}(A) \mid D(J) \subset J\}$.

9.4. Now consider $F := \mathbf{W}(P)$ for a projective $\boldsymbol{k}$-module P of finite type. Functor F is representable by $A := (S(Q), 0)$, where $Q := P^*$. Indeed, $\mathrm{Hom}_{{}_{k\backslash}\mathcal{P}}(A, (R, I)) = \mathrm{Hom}_{\boldsymbol{k}\text{-alg}}(S(Q), R) = \mathrm{Hom}_{\boldsymbol{k}}(Q, R) \cong Q^* \otimes_{\boldsymbol{k}} R \cong R \otimes_{\boldsymbol{k}} P = \mathbf{W}(P)(R)$. In this way we see that $\mathrm{Vect}(\mathbf{W}(P)) \overset{\lambda}{\cong} \mathrm{Der}_{\boldsymbol{k}}(S(Q)) \cong S(Q) \otimes_{\boldsymbol{k}} P \cong S(Q) \cdot P = \mathscr{L}_P \subset \mathscr{D}_P$ (cf. 8.13). Given a $\sigma \in \Gamma(TF/F)$, one obtains the corresponding element $\bar{D}$ in $S(Q) \otimes P$ as follows: $\sigma_A : F(A) \to TF(A) = F(A[\varepsilon]) = \mathbf{W}(P)(A[\varepsilon]) = P \otimes_{\boldsymbol{k}} A[\varepsilon]$, so $\sigma_A(X) = X + \varepsilon \cdot \bar{D}'$ for some $\bar{D}' \in P \otimes_{\boldsymbol{k}} A \cong S(Q) \otimes_{\boldsymbol{k}} P$ (recall $X \in F(A) = P \otimes_{\boldsymbol{k}} A = P \otimes_{\boldsymbol{k}} S(Q)$). One can check that $\bar{D} = \bar{D}'$ (this also provides an alternative proof of 8.13).

9.5 *(Pro-representable functors).* Consider the full subcategories $\mathcal{P}_n \subset \mathcal{P}$ given by $\mathrm{Ob}\,\mathcal{P}_n = \{(R, I) \in \mathrm{Ob}\,\mathcal{P} \mid I^n = 0\}$. These subcategories give an exhaustive filtration $\mathcal{P} = \bigcup_{n \geqslant 1} \mathcal{P}_n$; the corresponding slice categories are ${}_{k\backslash}\mathcal{P}_n \subset {}_{k\backslash}\mathcal{P}$. Fix a functor $F \in \mathrm{Ob}\,\mathcal{E}$. It can happen that F is not representable, but its restrictions $F^{(n)} := F|_{{}_{k\backslash}\mathcal{P}_n}$ are. This means that for each $n \geqslant 0$ we have some $A_n = (A_n, J_n) \in \mathrm{Ob}\,{}_{k\backslash}\mathcal{P}_n$ (in particular, $J_n^n = 0$) and an element $X_n \in F(A_n)$ such that for any $R=(R,I)$ in ${}_{\boldsymbol{k}\backslash}\mathcal{P}_n$ and any $\xi \in F(R)$ there is a unique morphism $\varphi: A_n \to R$ such that $\xi = (F(\varphi))(X_n)$. Since $A_n \in \mathrm{Ob}\,{}_{\boldsymbol{k}\backslash}\mathcal{P}_n \subset \mathrm{Ob}\,{}_{\boldsymbol{k}\backslash}\mathcal{P}_{n+1}$, the universal property of A_{n+1}, $X_{n+1} \in F(A_{n+1})$ gives us a map $\varphi_n : A_{n+1} \to A_n$ such that $(F(\varphi_n))(X_{n+1}) = X_n$. In this way we obtain a projective system $\underline{A} = (\cdots \to A_3 \xrightarrow{\varphi_2} A_2 \xrightarrow{\varphi_1} A_1)$, or even a pro-object $\boldsymbol{A} :=$ "$\varprojlim$" A_n over ${}_{\boldsymbol{k}\backslash}\mathcal{P}$. For any $R=(R,I)$ in ${}_{\boldsymbol{k}\backslash}\mathcal{P}$ we get $\mathrm{Hom}_{\mathrm{Pro}\,{}_{\boldsymbol{k}\backslash}\mathcal{P}}(\boldsymbol{A}, R) = \varinjlim_n \mathrm{Hom}_{{}_{\boldsymbol{k}\backslash}\mathcal{P}}(A_n, R) = F(R)$, since if $R \in \mathrm{Ob}\,{}_{\boldsymbol{k}\backslash}\mathcal{P}_m$, this inductive system stabilizes for $n \geqslant m$.

9.6. A vector field $\sigma \in \mathrm{Vect}(F): F \to TF$ is completely determined by its values $\sigma_n := \sigma_{A_n}(X_n)$ lying in $TF(A_n) = F(A_n[\varepsilon], J_n + A_n\varepsilon) \cong \mathrm{Hom}_{{}_{\boldsymbol{k}\backslash}\mathcal{P}}((A_{n+1}, J_{n+1}), (A_n[\varepsilon], J_n + A_n\varepsilon))$; let $\tilde\sigma_n : A_{n+1} \to A_n[\varepsilon]$ be the corresponding morphism in ${}_{\boldsymbol{k}\backslash}\mathcal{P}$ (here we used that $R \in \mathrm{Ob}\,\mathcal{P}_n$ implies $R[\varepsilon] \in \mathrm{Ob}\,\mathcal{P}_{n+1}$ for $R = A_n$). These $\tilde\sigma_n$ satisfy the obvious compatibility relations

$$\begin{tikzcd} & A_{n+1} \arrow[dl, "\tilde\sigma_n"'] \arrow[d, "\varphi_n"] \\ A_n[\varepsilon] \arrow[r, "p_{A_n}"] & A_n, \end{tikzcd} \qquad \begin{tikzcd} A_{n+2} \arrow[r, "\tilde\sigma_{n+1}"] \arrow[d, "\varphi_{n+1}"] & A_{n+1}[\varepsilon] \arrow[d, "\varphi_n[\varepsilon]"] \\ A_{n+1} \arrow[r, "\tilde\sigma_n"] & A_n[\varepsilon]. \end{tikzcd}$$

We see that $\tilde\sigma_n = \varphi_n + \varepsilon D_n$ for some $D_n : A_{n+1} \to A_n$ satisfying $\varphi_n \circ D_{n+1} = D_n \circ \varphi_{n+1}$ and $D_n(xy) = D_n(x)\cdot\varphi_n(y) + \varphi_n(x)\cdot D_n(y)$; in this way, one can think of $\boldsymbol{D} = (D_n)$ as a derivation of the pro-$\boldsymbol{k}$-algebra $\boldsymbol{A} =$ "$\varprojlim$" A_n. Sections $\mathrm{Vect}^\omega(F)$ of $T^\omega F$ are treated similarly, but we get additional conditions $D_n(J_{n+1}) \subset J_n$. Actually $A_n[\varepsilon]^\omega \in \mathrm{Ob}\,{}_{\boldsymbol{k}\backslash}\mathcal{P}_n$, so in this case we get a compatible family of derivations $D'_n : A_n \to A_n$ such that $D_n = D'_n \circ \varphi_n$ and $D'_n(J_n) \subset J_n$, and the overall description of $\mathrm{Vect}^\omega(F)$ is even simpler than that of $\mathrm{Vect}(F)$.

9.7. Let us apply this to $F = \mathbf{W}^\omega(P)$. First of all, $F|_{{}_{\boldsymbol{k}\backslash}\mathcal{P}_n}$ is representable by $(A_n, J_n) = (S(Q)/S^{\geqslant n}(Q), S^+(Q)/S^{\geqslant n}(Q))$. Namely, for any $(R,I) \in \mathrm{Ob}\,{}_{\boldsymbol{k}\backslash}\mathcal{P}_n$ we have

$$\begin{aligned}&\mathrm{Hom}_{{}_{\boldsymbol{k}\backslash}\mathcal{P}}\big((A_n,J_n),(R,I)\big)\\ &\quad = \big\{\varphi \in \mathrm{Hom}_{\boldsymbol{k}\text{-alg}}\big(S(Q),R\big) \text{ such that } \varphi\big(S^+(Q)^n\big)=0,\ \varphi\big(S^+(Q)\big)\subset I\big\},\end{aligned}$$

what, since $I^n = 0$, equals

$$\big\{\varphi \mid \varphi\big(S^+(Q)\big) \subset I\big\} \cong \big\{\tilde\varphi \in \mathrm{Hom}_{\boldsymbol{k}}(Q,R) \bigm| \tilde\varphi(Q) \subset I\big\} = \mathrm{Hom}_{\boldsymbol{k}}(Q,I) \cong I \otimes_{\boldsymbol{k}} P.$$

We see that the vector fields $\sigma \in \mathrm{Vect}\,\mathbf{W}^\omega(P)$ correspond to compatible families $\boldsymbol{D} = (D_n)$ of "derivations" $D_n : S(Q)/S^{\geqslant n+1}(Q) \to S(Q)/S^{\geqslant n}(Q)$. One can take the "true" projective limit and obtain a continuous derivation $D: \hat S(Q) \to \hat S(Q)$, that corresponds by 8.14 to some element of $\hat{\mathscr{L}}_P \cong \hat S(Q) \otimes P \cong \hat S(Q)\cdot P \subset \hat{\mathscr{D}}_P$. This is a Lie algebra isomorphism by the same reasoning as in 9.2. Again, given a $\sigma \in \mathrm{Vect}\,\mathbf{W}^\omega(P)$, we can construct the corresponding element $\bar D \in \mathscr{L}_P \cong \hat S(Q) \otimes P$ as follows: apply $\sigma_{A_n} : \mathbf{W}^\omega(P)(A_n) = J_n \otimes_{\boldsymbol{k}} P \to T\mathbf{W}^\omega(P)(A_n) = \mathbf{W}^\omega(P)(A_n[\varepsilon]) = J_n \otimes_{\boldsymbol{k}} P \oplus A_n\varepsilon \otimes_{\boldsymbol{k}} P$ to $X_n \in J_n \otimes_{\boldsymbol{k}} P = \mathbf{W}^\omega(P)(A_n)$ and get some element $\sigma_{A_n}(X_n) = X_n + \varepsilon\cdot\bar D_n$, $\bar D_n \in A_n \otimes_{\boldsymbol{k}} P = S(Q)/S^{\geqslant n}(Q) \otimes_{\boldsymbol{k}} P$. These $\bar D_n$ form a compatible family that defines an element of $\hat S(Q) \otimes_{\boldsymbol{k}} P$; this element is exactly $\bar D$ (proof is similar to 9.4).

9.8. Suppose G is a group in $\mathcal{E}$, F is an object of $\mathcal{E}$ and G acts on F from the left: we are given some $\alpha : G \times F \to F$ satisfying usual properties. Since T is left exact, $T(G \times F) = TG \times TF$, and we get a left action $T\alpha : TG \times TF \to TF$ compatible with α:

$$\begin{tikzcd} TG \times TF \arrow[r, "T\alpha"] \arrow[d, "\pi_G \times \pi_F"] & TF \arrow[d, "\pi_F"] \\ G \times F \arrow[r, "\alpha"] & F. \end{tikzcd}$$

Hence $\mathbf{Lie}(G) \subset TG$ also acts on TF, so we get a map $\mathbf{Lie}(G) \times TF \xrightarrow{\beta} TF$. Since π_G maps $\mathbf{Lie}(G)$ into the identity of G, the following diagram is commutative:

$$\begin{tikzcd} \mathbf{Lie}(G) \times TF \arrow[r, "\beta"] \arrow[d, "\mathrm{pr}_2"] & TF \arrow[d, "\pi_F"] \\ TF \arrow[r, "\pi_F"] & F. \end{tikzcd}$$

After composing β with $\mathrm{id}_{\mathbf{Lie}(G)} \times s$ where $s : F \to TF$ is the zero section of TF, we get a map $\gamma : \mathbf{Lie}(G) \times F \to TF$ over F, hence a map $\gamma^\flat : \mathbf{Lie}(G) \to \mathbf{Hom}_F(F, TF) = \mathbf{Vect}(F)$, and by taking the global sections (= evaluating at $\boldsymbol{k}$) we obtain a $\boldsymbol{k}$-linear map $\Gamma(\gamma^\flat) : \Gamma(\mathbf{Lie}(G)) = \mathrm{Lie}(G) \to \Gamma(\mathbf{Vect}(F)) = \mathrm{Vect}(F)$. One checks, by means of the description of Lie bracket on $\mathbf{Vect}(F)$ given in 9.2 and a similar description of the Lie bracket on $\mathbf{Lie}(G)$ recalled in 7.10.4, that $\gamma^\flat$ and $d_e\alpha := \Gamma(\gamma^\flat)$ are Lie algebra homomorphisms.

9.9. Fix a Lie algebra $\mathfrak{g}$ over $\boldsymbol{k}$, finitely generated projective as a $\boldsymbol{k}$-module, and construct the formal group $G := \mathbf{Exp}_\times(\mathfrak{g})$ with $\mathbf{Lie}(G) = \mathbf{W}(\mathfrak{g})$, hence $\mathrm{Lie}(G) = \Gamma(\mathbf{W}(\mathfrak{g})) = \mathfrak{g}$, as in 7.7. Consider first the left action of G on itself given by the multiplication map $\mu : G \times G \to G$. According to 9.8, we get a map $d_e\mu : \mathfrak{g} \to \mathrm{Vect}(G) = \Gamma(TG/G)$. It is clear from the description given in 9.8, that $Y \in \mathfrak{g}$ maps to the *right-invariant vector field* $\sigma_Y : G \to TG$ given by $(\sigma_Y)_R : g \mapsto (1 + Y\varepsilon) \cdot s_R(g)$ [here $1 + Y\varepsilon$ denotes the image of $Y \in \mathbf{Lie}(G)(R)$ in $TG(R)$, $s : T \to TG$ is the zero section of TG and g is an element of $G(R)$, $R \in \mathrm{Ob}_{\boldsymbol{k}\backslash}\mathcal{P}$]. This means that, if we identify TG with $G \times \mathbf{Lie}(G) \cong G \times \mathbf{W}(\mathfrak{g})$ by means of the map $(g, Y) \mapsto (1 + Y\varepsilon) \cdot s(g)$, then $\sigma_Y \in \mathrm{Hom}_G(G, TG) \cong \mathrm{Hom}_G(G, G \times \mathbf{Lie}(G)) \cong \mathrm{Hom}(G, \mathbf{Lie}(G))$ is identified with the constant map $g \mapsto Y$ in $\mathrm{Hom}(G, \mathbf{Lie}(G)) = \mathrm{Hom}(G, \mathbf{W}(\mathfrak{g}))$.

9.10. Recall that the exponential map $\exp' : \mathbf{W}^\omega(\mathfrak{g}) \to G$ gives an isomorphism of formal schemes, hence we can deduce from μ a left action $\alpha : G \times \mathbf{W}^\omega(\mathfrak{g}) \to \mathbf{W}^\omega(\mathfrak{g})$:

$$\begin{tikzcd} G \times \mathbf{W}^\omega(\mathfrak{g}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_G \times \exp'"] & \mathbf{W}^\omega(\mathfrak{g}) \arrow[d, "\exp'", "\sim"'] \\ G \times G \arrow[r, "\mu"] & G. \end{tikzcd}$$

This would give us a Lie algebra homomorphism $d_e\alpha : \mathfrak{g} \to \mathrm{Vect}(\mathbf{W}^\omega(\mathfrak{g})) = \mathrm{Hom}_{\mathbf{W}^\omega(\mathfrak{g})}(\mathbf{W}^\omega(\mathfrak{g}), T\mathbf{W}^\omega(\mathfrak{g})) \cong \mathrm{Hom}(\mathbf{W}^\omega(\mathfrak{g}), \mathbf{W}(\mathfrak{g}))$. We want to compute explicitly the vector fields $\tilde{\sigma}_Y = (d_e\alpha)(Y)$ in terms of this isomorphism.

We have the following diagram (cf. 7.10.6):

$$\begin{tikzcd}
\mathbf{W}^{\omega}(\mathfrak{g}) \arrow[r, "\tilde{\sigma}_Y"] \arrow[d, "\exp'"', "\sim"] & T\big(\mathbf{W}^{\omega}(\mathfrak{g})\big) \arrow[r, "\sim"] \arrow[d, "\sim"', "T(\exp')"] & \mathbf{W}^{\omega}(\mathfrak{g}) \times \mathbf{W}(\mathfrak{g}) \arrow[r] \arrow[d, dashed, "\tau"] & \mathbf{W}^{\omega}(\mathfrak{g}) \arrow[d, "\exp'"] \\
G \arrow[r, "\sigma_Y"] & TG \arrow[r, "\sim"] & G \times \mathbf{W}(\mathfrak{g}) \arrow[r, "\pi_G"'] & G. \arrow[l, dotted, bend right, "s"']
\end{tikzcd}$$

Here τ is given by $(X, Z) \mapsto (\exp(X), \sum_{n\geqslant 0} \frac{(\operatorname{ad} X)^{n+1}}{(n+1)!}(Z))$ for any X lying in $\mathbf{W}^{\omega}(\mathfrak{g})(R) = I_R \cdot \mathfrak{g}_{(R)}$ and any Z from $\mathbf{W}(\mathfrak{g})(R) = \mathfrak{g}_{(R)}$ [I_R is nilpotent, so is $(\operatorname{ad} X)$, hence the sum is finite; note the absence of the factor $(-1)^n$ factor in comparison to 7.10.6; this is due to the fact that we have chosen here another splitting $TG \cong G \times \mathbf{Lie}(G)$, given by *right-invariant* vector fields]. On the other hand, by 9.9, σ_Y is given by $g \mapsto Y$. This means that $\tilde{\sigma}_Y$ maps X into (X, Z) such that $\sum_{n\geqslant 0} \frac{(\operatorname{ad} X)^{n+1}}{(n+1)!}(Z) = Y$, i.e. $P(\operatorname{ad} X)(Z) = Y$, where $P(T) \in \mathbb{Q}[[T]]$ is the series $P(T) = (e^T - 1)/T$. Therefore, $Z = P(\operatorname{ad} X)^{-1}(Y)$, and classically $P(T)^{-1} = T/(e^T - 1) = \sum_{n\geqslant 0} \frac{B_n}{n!} T^n$ (this is actually the definition of Bernoulli numbers B_n).

9.11 *(Definition of embedding θ).* We have just seen that for $Y \in \mathfrak{g}$ the vector field $\tilde{\sigma}_Y$ is given by $X \mapsto (X, \sum_{n\geqslant 0} \frac{B_n}{n!} (\operatorname{ad} X)^n(Y))$. On the other hand, by 9.7, we know that vector fields on $\mathbf{W}(\mathfrak{g})$ correspond to continuous derivations of $\hat{S}(\mathfrak{g}^*)$, or to the elements of $\hat{S}(\mathfrak{g}^*) \otimes \mathfrak{g} \cong \hat{S}(\mathfrak{g}^*) \cdot \mathfrak{g} = \hat{\mathscr{L}}_{\mathfrak{g}} \subset \hat{\mathscr{D}}_{\mathfrak{g}}$, where $\hat{\mathscr{D}}_{\mathfrak{g}}$ is the completed Weyl algebra of $\mathfrak{g}$, cf. 8.10, 8.14. We want to compute the elements of $D_Y \in \hat{\mathscr{L}}_{\mathfrak{g}}$ that correspond to $\tilde{\sigma}_Y \in \operatorname{Vect}(\mathbf{W}^{\omega}(\mathfrak{g}))$; this would give us a Lie algebra homomorphism $\mathfrak{g} \xrightarrow{\theta} \hat{\mathscr{L}}_{\mathfrak{g}} \subset \hat{\mathscr{D}}_{\mathfrak{g}}$, $Y \mapsto D_Y$, hence also a homomorphism $\mathcal{U}(\mathfrak{g}) \xrightarrow{\tilde{\theta}} \hat{\mathscr{D}}_{\mathfrak{g}}$. We will see in 9.14 that both θ and $\tilde{\theta}$ are injective.

9.12. Let us apply 9.7 for $P = \mathfrak{g}$, $Q = \mathfrak{g}^*$, $F = \mathbf{W}^{\omega}(\mathfrak{g})$, to compute the element $D_Y \in \hat{S}(\mathfrak{g}^*) \otimes \mathfrak{g}$ corresponding to $\tilde{\sigma}_Y$ defined by some $Y \in \mathfrak{g}$. We know that F is pro-representable by $\boldsymbol{A} =$ "$\varprojlim$"(A_n, φ_n), where $A_n = (A_n, J_n) = (S(\mathfrak{g}^*)/S^{\geqslant n}(\mathfrak{g}^*), S^+(\mathfrak{g}^*)/S^{\geqslant n}(\mathfrak{g}^*))$, $\varphi_n : A_{n+1} \to A_n$ is the projection. We have also the universal elements $X_n \in A_n$; such an element is equal to the image of the canonical element $c_{\mathfrak{g}} \in \mathfrak{g}^* \otimes \mathfrak{g}$ in $J_n \otimes_{\boldsymbol{k}} \mathfrak{g} = \mathbf{W}^{\omega}(\mathfrak{g})(A_n) \subset \mathfrak{g}_{(A_n)}$. According to 9.7, we have to apply $(\tilde{\sigma}_Y)_{A_n} : \mathbf{W}^{\omega}(\mathfrak{g})(A_n) \to T\mathbf{W}^{\omega}(\mathfrak{g})(A_n) = J_n \otimes_{\boldsymbol{k}} \mathfrak{g} \oplus \varepsilon A_n \otimes_{\boldsymbol{k}} \mathfrak{g}$ to X_n and to take the second component $D_{Y,n}$. According to 9.11, $(\tilde{\sigma}_Y)_{A_n}(X_n) = X_n + \varepsilon \cdot \sum_{k=0}^{n} \frac{B_k}{k!} (\operatorname{ad} X_n)^k(Y)$; this gives us the value of $D_{Y,n}$. Elements $X_n \in A_n$ define an universal element $X \in \varprojlim(J_n \otimes_{\boldsymbol{k}} \mathfrak{g}) \cong \hat{S}^+(\mathfrak{g}^*) \otimes_{\boldsymbol{k}} \mathfrak{g} \subset \mathfrak{g}_{(\hat{A})}$, where $(\hat{A}, \hat{J}) = (\hat{S}(\mathfrak{g}^*), \hat{S}^+(\mathfrak{g}^*))$ is the "true" (topological) projective limit of $\boldsymbol{A}$. Of course, X is still the image of $c_{\mathfrak{g}} \in \mathfrak{g}^* \otimes \mathfrak{g}$ in $\hat{J} \otimes \mathfrak{g} \subset \mathfrak{g}_{(\hat{A})}$. We see that $D_Y = \sum_{k\geqslant 0} \frac{B_k}{k!} (\operatorname{ad} X)^k(Y) \in \mathfrak{g}_{(\hat{A})} \cong \hat{S}(\mathfrak{g}^*) \otimes_{\boldsymbol{k}} \mathfrak{g}$. This power series converges since $X \in \hat{J} \cdot \mathfrak{g}_{(\hat{A})}$ is topologically nilpotent. Here $Y \in \mathfrak{g}$ is considered as an element of $\mathfrak{g}_{(\hat{A})} \supset \mathfrak{g}$, and $(\operatorname{ad} X)^k(Y)$ is computed with respect to the $\hat{A}$-Lie algebra structure on $\mathfrak{g}_{(\hat{A})}$.

9.13 *(Main formula).* Suppose $\mathfrak{g}$ is a free $\boldsymbol{k}$-module of rank n. Fix a base $(e_i)_{1\leqslant i\leqslant n}$ of $\mathfrak{g}$, and consider the structural constants $C_{ij}^k \in \boldsymbol{k}$ defined by $[e_i, e_j] = \sum_k C_{ij}^k e_k$. (Here the completed Weyl algebra is used, hence, unlike in Sections 1–6, there is no need to use a formal variable t, hence to distinguish C_{jk}^i and $(C^0)_{jk}^i$.) Denote by (e^i) the dual base of $\mathfrak{g}^*$, denote by ∂^i the images of e_i in $S(\mathfrak{g}) \subset \hat{S}(\mathfrak{g}^*) \otimes_{\boldsymbol{k}} S(\mathfrak{g}) \cong \hat{\mathscr{D}}_{\mathfrak{g}}$ and by x_i—the images of e^i in $\hat{S}(\mathfrak{g}^*) \subset \hat{\mathscr{D}}_{\mathfrak{g}}$. (The apparent loss of covariance/contravariance here is due to the fact we will need to apply the Weyl algebra automorphism $x_i \mapsto -\partial^i$, $\partial^i \mapsto x_i$ to recover the main formula in form (1); there does not seem to be a completely satisfactory way of fixing this.)

Clearly $c_{\mathfrak{g}} = \sum_i e^i \otimes e_i$, hence $X = \sum_i x_i e_i \in \mathfrak{g}_{(\hat{A})} = \mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$ is the universal element, and for any $Y \in \mathfrak{g}$, D_Y is given by $\sum_{s \geqslant 0} \frac{B_s}{s!} (\operatorname{ad} X)^s(Y)$. In coordinates, $\operatorname{ad} X \in \operatorname{End}_{\hat{A}}(\mathfrak{g}_{(\hat{A})})$ is given by $\operatorname{ad} X : e_j \mapsto \sum_i x_i [e_i, e_j] = \sum_{i,k} C_{ij}^k x_i e_k$, hence the matrix $M = (M_j^i)$ of $\operatorname{ad} X$ is given by $M_j^i = \sum_k C_{kj}^i x_k$ (cf. with $\mathbf{C}_j^i$ from Sections 1–5, which involve $-\partial^k$s in the place of x_ks).

For $Y = e_j$ we obtain

$$\begin{aligned} D_{e_j} = D_Y &= \sum_{s \geqslant 0} \frac{B_s}{s!} (\operatorname{ad} X)^s(e_j) = \sum_{s \geqslant 0} \sum_{i=1}^n \frac{B_s}{s!} \left(M^s\right)_j^i e_i \\ &= \sum_{i=1}^n \left(\sum_{s=0}^\infty \frac{B_s}{s!} \left(M^s\right)_j^i \right) \partial^i \in \mathscr{L}_{\mathfrak{g}} \subset \hat{\mathscr{D}}_{\mathfrak{g}}. \end{aligned} \tag{36}$$

Thus we have constructed an explicit embedding $e_j \mapsto D_{e_j}$ of $\mathfrak{g}$ into the completed Weyl algebra $\hat{\mathscr{D}}_{\mathfrak{g}}$. Recall that $\hat{\mathscr{D}}_{\mathfrak{g}}$ is some completion of the Weyl algebra $\mathscr{D}_{\mathfrak{g}}$, and that $\mathscr{D}_{\mathfrak{g}}$ in this situation is the free algebra over $\boldsymbol{k}$ generated by $x_1, \dots, x_n, \partial^1, \dots, \partial^n$ subject to the relations $[x_i, x_j] = [\partial^i, \partial^j] = 0$, $[\partial^k, x_i] = \delta_i^k$, i.e. is the classical Weyl algebra over $\boldsymbol{k}$ with $2n = 2 \dim \mathfrak{g}$ generators.

9.14 *(Injectivity of θ and $\tilde{\theta}$).* Now it remains to show that our homomorphisms $\theta : \mathfrak{g} \to \hat{\mathscr{L}}_{\mathfrak{g}}$ and $\tilde{\theta} : \mathcal{U}(\mathfrak{g}) \to \hat{\mathscr{D}}_{\mathfrak{g}}$ are injective. To achieve this we consider the "evaluation at origin map" $\beta : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*)/\hat{S}^+(\mathfrak{g}^*) = \boldsymbol{k}$ and the induced maps $\beta \otimes 1_{\mathfrak{g}} : \hat{\mathscr{L}}_{\mathfrak{g}} \to \mathfrak{g}$ and $\beta \otimes 1_{S(\mathfrak{g})} : \hat{\mathscr{D}}_{\mathfrak{g}} \to S(\mathfrak{g})$. One checks immediately that $(\beta \otimes 1_{\mathfrak{g}}) \circ \theta = 1_{\mathfrak{g}}$, hence θ is injective; for $\tilde{\theta}$ observe that $(\beta \otimes 1_{S(\mathfrak{g})}) \circ \tilde{\theta}$ maps $\mathcal{U}_n(\mathfrak{g})$ into $S_n(\mathfrak{g})$ and induces the identity map between the associated graded $\operatorname{gr}(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{g})$ and $\operatorname{gr}(S(\mathfrak{g})) = S(\mathfrak{g})$, hence is injective, hence $\tilde{\theta}$ is also injective.

10. Another proof in the language of coderivations

10.1. In this section, $\boldsymbol{k} \supset \mathbb{Q}$. If H is any $\boldsymbol{k}$-coalgebra with comultiplication Δ_H, then a $\boldsymbol{k}$-linear map $D \in \operatorname{End}_{\boldsymbol{k}}(H)$ is a *coderivation* if $(D \otimes 1 + 1 \otimes D) \circ \Delta_H = \Delta_H \circ D$. We denote by $\operatorname{Coder}_{\boldsymbol{k}}(H)$ the set of all $\boldsymbol{k}$-linear coderivations of H.

10.2. Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{k}$ projective and finitely generated as a $\boldsymbol{k}$-module. Put $H := \mathcal{U}(\mathfrak{g})$ and $C := S(\mathfrak{g})$. Both C and H are Hopf algebras, cocommutative as coalgebras. Moreover, there is a unique canonical (functorial in $\mathfrak{g}$) isomorphism of *coalgebras* $\xi : C \xrightarrow{\sim} H$ that is identity on $\mathfrak{g}$ [Bourbaki, Chapter II]. This *coexponential map* may be described as follows: it maps $x_1 \cdots x_n \in C^n = S^n(\mathfrak{g})$ into $\frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \in \mathcal{U}(\mathfrak{g})_n \subset H$.

10.3. Consider the left action of H on itself defined by its algebra structure: $h \mapsto L_h \in \operatorname{End}_{\boldsymbol{k}}(H)$, $L_h : h' \mapsto hh'$. Since $H = \mathcal{U}(\mathfrak{g})$, this restricts to a Lie algebra action of $\mathfrak{g}$ on H. *This is an action by coderivations.* Indeed, for any $h \in \mathfrak{g}$, we have $\Delta_H(h) = 1 \otimes h + h \otimes 1$, hence $(L_h \otimes 1 + 1 \otimes L_h)(\Delta_H(h')) = \Delta_H(h) \cdot \Delta_H(h') = \Delta_H(hh') = (\Delta_H \circ L_h)(h')$.

10.4. The coexponential isomorphism $\xi : C \xrightarrow{\sim} H$ allows us to define a left action $h \mapsto D_h := \xi^{-1} \circ L_h \circ \xi$ of H on C, such that $\mathfrak{g} \subset H$ acts on C by *coderivations*, and $D_h(1) = h$ for any $h \in \mathfrak{g}$. Thus we get $\theta : h \mapsto D_h$, $\theta : \mathfrak{g} \to \mathrm{Coder}_{\boldsymbol{k}}(C) = \mathrm{Coder}_{\boldsymbol{k}}(S(\mathfrak{g}))$. By "duality" $S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*)$ (we use here that $\mathfrak{g}$ is projective f.g., $\boldsymbol{k} \supset \mathbb{Q}$) since $S(\mathfrak{g})^* = (\bigoplus S^n(\mathfrak{g}))^* = \prod S^n(\mathfrak{g})^* \cong \prod S^n(\mathfrak{g}^*) = \hat{S}(\mathfrak{g}^*)$, and D_h give us derivations ${}^t D_h : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*)$. Hence we get a Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathrm{Der}(\hat{S}(\mathfrak{g}^*)) \cong \hat{S}(\mathfrak{g}^*) \otimes \mathfrak{g} \cong \hat{\mathscr{L}}_{\mathfrak{g}} \subset \hat{\mathscr{D}}_{\mathfrak{g}}$, where $\hat{\mathscr{D}}_{\mathfrak{g}}$ is the completed Weyl algebra of $\mathfrak{g}$, $\theta : h \mapsto {}^t D_h$.

10.5. The pairing between $S^n(\mathfrak{g}^*)$ and $S^n(\mathfrak{g})$ that induces the isomorphism $S^n(\mathfrak{g})^* \cong S^n(\mathfrak{g}^*)$ used above, is given by $\langle x_1 \dots x_n, u_1 \dots u_n \rangle = \sum_{\sigma \in S_n} \langle x_i, u_{\sigma(i)} \rangle$, i.e. $\langle x^n, u^n \rangle = n! \langle x, u \rangle^n$. Elements of the form x^n generate $S^n(\mathfrak{g})$ and $\xi(x^n) = x^n \in \mathcal{U}(\mathfrak{g})$ $\forall x \in \mathfrak{g}$. We see that $D_h(x^n) = \xi^{-1}(hx^n)$. Moreover, $\xi(x^k h) = \frac{1}{k+1} \sum_{p+q=k} x^p h x^q$ and $(\mathrm{ad}\, x).\xi(x^k h) = \frac{1}{k+1}(x^{k+1}h - hx^{k+1})$. For the following computations, fix $x \in H$ and denote by $L = L_x$ and $R = R_x$ the left and right multiplication by x. Since L and R commute and $\boldsymbol{k} \supset \mathbb{Q}$, the polynomials from $\mathbb{Q}[L, R]$ act on H. In particular,

$$\xi\left(x^k h\right) = \frac{1}{k+1} \sum_{p+q=k} x^p h x^q = \left(\frac{1}{k+1} \sum_{p+q=k} L^p R^q\right)(h),$$

$$(\mathrm{ad}\, x).h = xh - hx = (L - R)(h),$$

$$\xi\left(x^k (\mathrm{ad}\, x)^l h\right) = \frac{1}{k+1} \left(\sum_{p+q=k} L^p R^q\right)(L - R)^l(h).$$

We would like to find rational coefficients $a_{k,l}^{(n)} \in \mathbb{Q}$, such that $D_h(x^n) = \sum_{k,l} a_{k,l}^{(n)} x^k (\mathrm{ad}\, x)^l.h$. This condition can be rewritten as

$$hx^n = \sum_{k,l} a_{k,l}^{(n)} \xi\left(x^k (\mathrm{ad}\, x)^l h\right), \quad \text{i.e.} \quad R^n(h) = \sum_{k,l} \frac{a_{k,l}^{(n)}}{k+1} \left(\sum_{p+q=k} L^p R^q\right)(L - R)^l(h).$$

For this it is enough to require

$$R^n = \sum_{k,l} \frac{a_{k,l}^{(n)}}{k+1} \left(\sum_{p+q=k} L^p R^q\right)(L - R)^l \quad \text{to hold in } \mathbb{Q}[L, R].$$

Since $L - R$ is not a zero divisor in $\mathbb{Q}[L, R]$, this identity is equivalent to

$$(L - R)R^n = \sum_{k,l} \frac{a_{k,l}^{(n)}}{k+1} \left(L^{k+1} - R^{k+1}\right)(L - R)^l.$$

Now the LHS is a homogeneous polynomial of degree $n + 1$, hence in finding $a_{k,l}^{(n)}$ we may also require that all summands of other degrees on the RHS vanish as well, i.e. assume that $a_{k,l}^{(n)} = 0$ unless $k + l = n$. This also allows to simplify the notation, namely set $a_k^{(n)} := a_{k,n-k}^{(n)}$, $0 \leqslant k \leqslant n$. Next, consider the isomorphism $\mathbb{Q}[L, R] \cong \mathbb{Q}[X, Y]$ given by $R \mapsto X$, $L \mapsto X + Y$; it allows us to rewrite our identity as

$$X^nY=\sum_{k=0}^{n}\frac{a_k^{(n)}}{k+1}\big((X+Y)^{k+1}-X^{k+1}\big)Y^{n-k}.$$

Divide both sides by Y^{n+1} and put $T:=X/Y$; so we get the following identity in $\mathbb{Q}[T]$:

$$T^n=\sum_{k=0}^{n}\frac{a_k^{(n)}}{k+1}\big((T+1)^{k+1}-T^{k+1}\big). \tag{37}$$

For any $P\in\mathbb{Q}[T]$ set $\delta P:=P(T+1)-P(T)$, $DP:=P'(T)$. (We do not use the classical notation ΔP for $P(T+1)-P(T)$ since it might be confused with our notation for the comultiplication.) By Taylor's formula $(e^DP)(T)=\sum_{k\geqslant 0}\frac{P^{(k)}(T)}{k!}=P(T+1)$, hence $\delta P=(e^D-1)P$, i.e. $\delta=e^D-1$. In terms of $P_n(T):=\sum_{k=0}^{n}\frac{a_k^{(n)}}{k+1}T^{k+1}$, Eq. (37) may be rewritten as

$$\delta P_n(T)=T^n,\qquad P_n(0)=0. \tag{38}$$

This determines $P_n(T)$ uniquely, and the coefficients of $P_n(T)$ can be expressed in terms of Bernoulli numbers. To obtain this expression, observe that $\delta=\frac{e^D-1}{D}D$, so $(\frac{e^D-1}{D})(DP_n)=T^n$, therefore $DP_n=(\frac{D}{e^D-1})(T^n)$. By the definition of Bernoulli numbers B_k, we have $\frac{D}{e^D-1}=\sum_{k\geqslant 0}\frac{B_k}{k!}D^k$, hence

$$DP_n(T)=\sum_{k\geqslant 0}\frac{B_k}{k!}D^kT^n=\sum_{k\geqslant 0}B_k\binom{n}{k}T^{n-k}=\sum_{k=0}^{n}B_{n-k}\binom{n}{k}T^k.$$

On the other hand, $DP_n(T)=\sum_{k=0}^{n}a_k^{(n)}T^k$, hence $a_k^{(n)}=\binom{n}{k}B_{n-k}$. We have proved the following formula:

10.6. For any $h\in\mathfrak{g}\subset H$, $x\in\mathfrak{g}\subset C$, we have

$$D_h\big(x^n\big)=\xi^{-1}\big(hx^n\big)=\sum_{k=0}^{n}a_k^{(n)}x^k\cdot\big((\operatorname{ad}x)^{n-k}(h)\big)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k\cdot\big((\operatorname{ad}x)^{n-k}(h)\big), \tag{39}$$

or, shortly,

$$D_h\big(x^n\big)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\cdot\big((\operatorname{ad}x)^k(h)\big). \tag{40}$$

This implies for any $x\in\mathfrak{g}$, $\alpha\in\mathfrak{g}^*$, $n\geqslant 0$,

$$\big\langle x^n,{}^tD_h(\alpha)\big\rangle=\big\langle D_h\big(x^n\big),\alpha\big\rangle=\big\langle B_n\cdot(\operatorname{ad}x)^n(h),\alpha\big\rangle=B_n\big\langle(\operatorname{ad}x)^n(h),\alpha\big\rangle.$$

Both sides are polynomial maps in x of degree n, by taking polarizations we get

$$\langle x_1 \dots x_n, {}^t D_h(\alpha) \rangle = \frac{B_n}{n!} \sum_{\sigma \in S_n} \langle (\operatorname{ad} x_{\sigma(1)}) \dots (\operatorname{ad} x_{\sigma(n)})(h), \alpha \rangle, \quad \forall x_1, \dots, x_n \in \mathfrak{g}.$$

The above formula is by no means new: it can be found for example in [Petracci, Remark 3.4].

10.7. The coderivation $D_h : S(\mathfrak{g}) \to S(\mathfrak{g})$ corresponds by duality to a continuous derivation ${}^t D_h : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*)$, which is uniquely determined by its restriction ${}^t D_h|_{\mathfrak{g}^*} : \mathfrak{g}^* \to \hat{S}(\mathfrak{g}^*)$. Such a map corresponds to an element ${}^t \tilde{D}_h \in \hat{S}(\mathfrak{g}^*) \otimes \mathfrak{g} \cong \hat{\mathscr{L}}_{\mathfrak{g}} \subset \hat{S}(\mathfrak{g}^*) \otimes S(\mathfrak{g}) \cong \hat{\mathscr{D}}_{\mathfrak{g}}$. Now we want to compute this element $\tilde{D}_h := ({}^t D_h)^{\sim} \in \hat{\mathscr{L}}_{\mathfrak{g}} \subset \hat{\mathscr{D}}_{\mathfrak{g}}$ of the Weyl algebra since it defines (by means of the usual action of $\hat{\mathscr{D}}_{\mathfrak{g}}$ on $\hat{S}(\mathfrak{g}^*)$) the derivation ${}^t D_h$ as well as the coderivation D_h.

10.8. Any element $u \in \mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))} = \hat{S}(\mathfrak{g}^*) \otimes \mathfrak{g} = \hat{\mathscr{L}}_{\mathfrak{g}}$ corresponds to a $\boldsymbol{k}$-linear map $u^\sharp : S(\mathfrak{g}) \to \mathfrak{g}$ and conversely, since $\mathfrak{g}$ is a projective $\boldsymbol{k}$-module of finite type. Note that $\mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$ has a $\hat{S}(\mathfrak{g}^*)$-Lie algebra structure obtained by base change from that of $\mathfrak{g}$. Let us compute $[u, v]^\sharp$ in terms of $u^\sharp$ and $v^\sharp : S(\mathfrak{g}) \to \mathfrak{g}$ for any two elements $u, v \in \mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$.

10.8.1. To do this consider the following more general situation: Let M, N, P be projective $\boldsymbol{k}$-modules of finite rank, $B : M \times N \to P$ a $\boldsymbol{k}$-bilinear map, $\tilde{B} : M \otimes N \to P$ the map induced by B, $u \in M_{(\hat{S}(\mathfrak{g}^*))}$, $v \in N_{(\hat{S}(\mathfrak{g}^*))}$. Suppose $u^\sharp : S(\mathfrak{g}) \to M$ and $v^\sharp : S(\mathfrak{g}) \to N$ correspond to u and v, and we want to compute the map $w^\sharp : S(\mathfrak{g}) \to P$ corresponding to $w = B_{(\hat{S}(\mathfrak{g}^*))}(u, v)$ in terms of $u^\sharp$ and $v^\sharp$.

10.8.2. The answer here is the following: $w^\sharp = \tilde{B} \circ (u^\sharp \otimes v^\sharp) \circ \Delta$ where $\Delta : S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g})$ is the comultiplication on $S(\mathfrak{g})$. Indeed, by linearity, it is sufficient to check this for $u = x \otimes \varphi$, $v = y \otimes \psi$, $x \in M$, $y \in N$, $\varphi, \psi \in \hat{S}(\mathfrak{g}^*) = S(\mathfrak{g})^*$. Then $u^\sharp : \lambda \mapsto \varphi(\lambda)x$, $v^\sharp : \lambda \mapsto \psi(\lambda)y$, $w = z \otimes \varphi\psi$ where $z = B(x, y) \in P$ and $\varphi\psi = (\varphi \otimes \psi) \circ \Delta$ by duality between algebra $\hat{S}(\mathfrak{g}^*)$ and coalgebra $S(\mathfrak{g})$. Hence $w^\sharp : \lambda \mapsto (\varphi\psi)(\lambda) \cdot z = (\varphi \otimes \psi)(\Delta(\lambda)) \cdot B(x, y) = (\tilde{B} \circ (u^\sharp \otimes v^\sharp) \circ \Delta)(\lambda)$.

10.8.3. Now we apply this for $M = N = P = \mathfrak{g}$, B—the multiplication map. We see that for any two $u^\sharp, v^\sharp : S(\mathfrak{g}) \to \mathfrak{g}$ we have $[u, v]^\sharp = \mu_{\mathfrak{g}} \circ (u^\sharp \otimes v^\sharp) \circ \Delta$ where $\Delta = \Delta_{S(\mathfrak{g})} : S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g})$ is the comultiplication of $S(\mathfrak{g})$ and $\mu_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $x \otimes y \mapsto [x, y]$ is the bracket multiplication of Lie algebra $\mathfrak{g}$. We introduce a Lie bracket $[\,,\,]_{\mathfrak{g}}$ on $\operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g})$ by this rule; then $[u, v]^\sharp = [u^\sharp, v^\sharp]_{\mathfrak{g}}$, so $\mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))} \to \operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g})$, $u \mapsto u^\sharp$ is an isomorphism of Lie algebras. Observe that there is another Lie bracket $[\,,\,]_{\mathscr{D}}$ on $\operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g}) \cong \operatorname{Der.cont}(\hat{S}(\mathfrak{g})) \cong \operatorname{Coder}(S(\mathfrak{g}))$ given by the usual commutator of coderivations: $[D_1, D_2]_{\mathscr{D}} = D_1 D_2 - D_2 D_1$. (One must be careful when passing from derivations to coderivations since $[{}^t D_1, {}^t D_2] = -{}^t[D_1, D_2]$.)

10.9. We have some specific elements in $\mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$, hence in $\operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g})$. First of all, any $a \in \mathfrak{g}$ lies in $\mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$, inducing therefore a map $a^\sharp : S(\mathfrak{g}) \to \mathfrak{g}$. Clearly, this is the map sending $1 \in S(\mathfrak{g})$ to a, and sending all of $S^+(\mathfrak{g}) = \bigoplus_{n \geqslant 1} S^n(\mathfrak{g})$ to zero.

10.9.1. One checks immediately that $[a^\sharp, b^\sharp]_{\mathfrak{g}} = [a, b]^\sharp$, $\forall a, b \in \mathfrak{g}$, $[a^\sharp, u^\sharp]_{\mathfrak{g}} = (\operatorname{ad} a) \circ u^\sharp$, $[u^\sharp, a^\sharp]_{\mathfrak{g}} = -(\operatorname{ad} a) \circ u^\sharp$, $\forall a \in \mathfrak{g}$, $u^\sharp : S(\mathfrak{g}) \to \mathfrak{g}$.

10.9.2. Besides, the "canonical element" $X \in \mathfrak{g} \otimes \mathfrak{g}^* \subset \mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$ (the image of $\mathrm{id}_{\mathfrak{g}}$ under the identification $\operatorname{Hom}(\mathfrak{g}, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}^*$) provides a map $X^\sharp : S(\mathfrak{g}) \to \mathfrak{g}$, which is clearly the projection of $S(\mathfrak{g}) = \bigoplus_{n \geqslant 0} S^n(\mathfrak{g})$ onto $S^1(\mathfrak{g}) = \mathfrak{g}$. In particular, for all $a \in \mathfrak{g}$, $X^\sharp(a) = a$ if $n = 1$, and $X^\sharp(a^n) = 0$ otherwise.

10.9.3. Let us compute $[X^\sharp, u^\sharp]_\mathfrak{g}$ for any $u \in \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$. Since elements of the form a^n generate $S^n(\mathfrak{g})$, it is sufficient to determine all $[X^\sharp, u^\sharp](a^n)$. Now, $\Delta(a^n) = \Delta(a)^n = (a \otimes 1 + 1 \otimes a)^n = \sum_{k=0}^n \binom{n}{k} a^k \otimes a^{n-k}$, hence $(X^\sharp \otimes u^\sharp)(\Delta(a^n)) = na \otimes u^\sharp(a^{n-1})$, so $[X^\sharp, u^\sharp]_\mathfrak{g}(a^n) = n.[a, u^\sharp(a^{n-1})]$ (for $n = 0$ the RHS is assumed to be zero by convention). In particular, if $u^\sharp$ was zero restricted to $S^n(\mathfrak{g})$ for fixed n, then $[X^\sharp, u^\sharp]_\mathfrak{g}$ is zero on $S^{n+1}(\mathfrak{g})$.

10.9.4. Thus, we have proved $(\mathrm{ad}_\mathfrak{g} X^\sharp)(u^\sharp) : a^n \mapsto n(\mathrm{ad}\, a)(u^\sharp(a^{n-1}))$. By induction, one obtains $(\mathrm{ad}_\mathfrak{g} X^\sharp)^k(u^\sharp) : a^n \mapsto \frac{n!}{(n-k)!}(\mathrm{ad}\, a)^k u^\sharp(a^{n-k})$ for $n \geqslant k$ (for $n < k$ the RHS is assumed to be zero). Using this for $u^\sharp = h^\sharp$ defined by some $h \in \mathfrak{g}$, we obtain $(\mathrm{ad}_\mathfrak{g} X^\sharp)^n(h^\sharp) : x^n \mapsto n!(\mathrm{ad}\, x)^n(h)$, and it is zero outside $S^n(\mathfrak{g})$. By taking polarizations, we get $(\mathrm{ad}_\mathfrak{g} X^\sharp)^n(h^\sharp) : x_1 x_2 \ldots x_n \mapsto \sum_{\sigma \in S_n} (\mathrm{ad}\, x_{\sigma(1)})(\mathrm{ad}\, x_{\sigma(2)}) \ldots (\mathrm{ad}\, x_{\sigma(n)})(h)$.

10.10. Now we can write (39) in another way. Recall that $\forall h \in \mathfrak{g}$ we have constructed a coderivation $D_h : S(\mathfrak{g}) \to S(\mathfrak{g})$ given by (39) such that $h \mapsto D_h$ is a Lie algebra embedding $\mathfrak{g} \to \mathrm{Coder}(S(\mathfrak{g})) \cong \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g}) \cong \hat{\mathscr{L}}_\mathfrak{g}$. Recall that the map $\mathrm{Coder}(S(\mathfrak{g})) \to \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$ maps a coderivation D into $\tilde{D}^\sharp := X^\sharp \circ D : S(\mathfrak{g}) \to \mathfrak{g}$. Hence $\tilde{D}_h^\sharp = X^\sharp \circ D_h$ is given by

$$\tilde{D}_h^\sharp\left(x^n\right) = B_n \cdot (\mathrm{ad}\, x)^n(h).$$

Comparing with 10.9.4 we obtain the following equality in $\mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$:

$$\tilde{D}_h^\sharp = \sum_{n \geqslant 0} \frac{B_n}{n!} \left(\mathrm{ad}_\mathfrak{g} X^\sharp\right)^n \left(h^\sharp\right).$$

By dualizing and taking into account 10.8.3 we obtain an equality in $\hat{\mathscr{L}}_\mathfrak{g} \cong \hat{S}(\mathfrak{g}^*) \otimes \mathfrak{g} \cong \mathfrak{g}_{(\hat{S}(\mathfrak{g}^*))}$:

$${}^t\tilde{D}_h = \sum_{n \geqslant 0} \frac{B_n}{n!} (\mathrm{ad}_\mathfrak{g} X)^n(h).$$

This is exactly the main formula (36) of Section 9 in invariant form. So we have proved it again in a shorter but less geometric way. The above formula has already appeared in a slightly different form in [Petracci], Theorem 5.3 and formulas (20), (13) and (15). We refer to the introduction to the present work for a more detailed comparison of our results with those of *loc.cit.*

10.11. In the proof presented in this section, we started from an invariantly defined isomorphism of coalgebras $\xi : C = S(\mathfrak{g}) \xrightarrow{\sim} H = \mathcal{U}(\mathfrak{g})$ and used it to transport the coderivations $L_h : x \mapsto hx$ from H onto C. However, we could replace in this reasoning ξ by any other isomorphism of coalgebras $\xi' : C \xrightarrow{\sim} H$ and obtain another embedding $\mathfrak{g} \to \hat{\mathscr{L}}_\mathfrak{g}$ in this way, hence another formula. Yet another possibility is to consider on H the coderivations $R_h : x \mapsto -xh$ instead of the L_hs. This gives the same formula but with additional $(-1)^n$ factors in each summand. (One can see this by considering the isomorphism of $\mathfrak{g}$ onto its opposite $\mathfrak{g}^\circ$ given by $x \mapsto -x$.)

10.12. What are the other possible choices of $\xi'' : C \xrightarrow{\sim} H$? Actually the only choice functorial in $\mathfrak{g}$ is the coexponential map. Thus we need some additional data. Suppose for example that $\mathfrak{g}$ is free rank n as a $\boldsymbol{k}$-module, and $e_1, \ldots, e_n$ are a base of $\mathfrak{g}$. Denote by x_i the images of e_i in $C = S(\mathfrak{g})$, and by z_i the images in $H = \mathcal{U}(\mathfrak{g})$. Then, one can take $\xi' : x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mapsto z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, for any $\alpha_i \in \mathbb{N}_0$. This is easily seen to be a coalgebra isomorphism, since $\Delta_C(x_i) = x_i \otimes 1 + 1 \otimes x_i$, $\Delta_H(z_i) = z_i \otimes 1 + 1 \otimes z_i$, and these monomials in z_i form a base of H by PBW theorem. In this way, one obtains another embedding $\mathfrak{g} \hookrightarrow \hat{\mathscr{L}}_{\mathfrak{g}}$, the lower degree terms of which have to be given by the triangular matrices. In the geometric language of previous sections, this corresponds to the map $\mathbf{W}(\mathfrak{g}) \xrightarrow{\exp''} G = \mathbf{Exp}_\times(\mathfrak{g})$ given by $x_1 e_1 + \cdots + x_n e_n \mapsto \exp'(x_1 e_1) \cdot \cdots \cdot \exp'(x_n e_n)$.

11. Conclusion and perspectives

Within the general problematics of finding Weyl-algebra realizations of finitely generated algebras, a remarkable universal formula has been derived in three different approaches, suggesting further generalizations. Our result and proof can apparently also be extended, in a straightforward manner, to Lie superalgebras.

More difficult is to classify all homomorphisms $U(\mathfrak{g}) \to A_n[[t]]$, which are not universal, but rather defined for a given Lie algebra $\mathfrak{g}$. We have computed some examples of such representations (e.g. [MS]), but we do not know any classification results. As usual for the deformation problems, we expect that the homological methods may be useful for the treatment of concrete examples.

In our representation, t is a formal variable. If $\boldsymbol{k}$ is a topological ring, then one can ask if our formal series actually converges for finite t. Let $\rho : A_n \to \mathcal{B}(H)$ be a representation of the Weyl algebra by bounded operators on a Hilbert space H, and t fixed. Then for any $x \in \mathfrak{g}$, $\rho \circ \Phi(x)$ is a power series in bounded operators. Under the conditions when this is a convergent series, for $\boldsymbol{k} = \mathbb{C}$, $\lambda = 1$, our formula is known, see Appendix 1, formula (1.28) of [KarMaslov], with a very different proof.

Similarly to the analysis in [OdesFeigin], it may be useful to compute the commutant of the image of Φ in $A_n[[t]]$. It is an open problem if there are similar homomorphisms for the quantum enveloping algebras. The approach to formal Lie theory taken in [Holtkamp] may be useful in this regards. We expect that our approach may be adapted to the setup of Lie theory over operads [Fresse]. In particular, generalizations to Leibniz algebras (no antisymmetry!) would be very interesting. There is also an integration theory for Lie algebroids (yielding Lie groupoids), with very many applications. This suggests that the vector field computations may be adaptable to that case.

Our main motivation is, however, to explore in future similar representations in study of possible quantum field theories in the backgrounds given by noncommutative spaces, where we may benefit on unifying methods and intuition based on the exploration of the uniform setup of Weyl algebras. Some related physically inspired papers are [AmCam1,AmCam2,Berceanu,Dim, Kathotia,Kontsevich,Lukierski,LukWor,MS].

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