Contramodules

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Corings

First: fix an algebra A over a commutative ring k.

Definition

A *coring* (*co-ring*) is an A-bimodule C with A-bimodule maps

- $\Delta : C \to C \otimes_A C$ (coproduct)
- $\varepsilon : C \to A$ (counit)

such that

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes_A C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle C\otimes_A \Delta} \\ C\otimes_A C & \xrightarrow{\Delta\otimes_A C} & C\otimes_A C\otimes_A C \end{array},$$

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes_A C \\ {\scriptstyle \Delta}\downarrow & \searrow^{C} & \downarrow{\scriptstyle \varepsilon\otimes_A C} \\ C\otimes_A C & \xrightarrow{C\otimes_A \varepsilon} & C \end{array}.$$

Comodules

Definition

A *right A-comodule* is a pair (M, ϱ), where M is a right A-module, and $\varrho : M \to M \otimes_A C$ is a right A-module map such that

$$\begin{array}{ccc} M & \xrightarrow{\quad\varrho\quad} & M\otimes_A C \\ {\scriptstyle\varrho}\downarrow & & \downarrow{\scriptstyle M\otimes_A\Delta} \\ M\otimes_A C & \xrightarrow{\varrho\otimes_A C} & M\otimes_A C\otimes_A C \end{array}, \qquad \begin{array}{ccc} M & \xrightarrow{\varrho} & M\otimes_A C \\ & {\scriptstyle\simeq}\searrow & \downarrow{\scriptstyle M\otimes_A\varepsilon} \\ & & M\otimes_A A. \end{array}$$

A morphism of C-comodules $(M, \varrho) \to (N, \varrho^N)$ is a right A-module map $f : M \to N$ such that

$$(f \otimes_A C) \circ \varrho = \varrho^N \circ f.$$

The category is denoted by $\mathbf{M}^C$.

Definition of contramodules

Definition

A *C-contramodule* is a pair (M, α):

- M is a right A-module;
- $\alpha : \mathrm{Hom}_A(C, M) \to M$, is a right A-module map;

$$\begin{array}{ccc}
\mathrm{Hom}_A(C, \mathrm{Hom}_A(C, M)) & \xrightarrow{\mathrm{Hom}_A(C,\alpha)} & \mathrm{Hom}_A(C, M) \\
\simeq \downarrow & & \downarrow \alpha \\
\mathrm{Hom}_A(C \otimes_A C, M) \xrightarrow{\mathrm{Hom}_A(\Delta, M)} \mathrm{Hom}_A(C, M) & \xrightarrow{\alpha} & M,
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_A(A, M) & \xrightarrow{\mathrm{Hom}_A(\varepsilon, M)} & \mathrm{Hom}_A(C, M) \\
& {\scriptstyle \simeq} \searrow \quad \swarrow {\scriptstyle \alpha} & \\
& M. &
\end{array}$$

Definition

A morphism of right C-contramodules (M, α_M), (N, α_N) is a right A-module map $f : M \to N$ rendering commutative the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(C, M) & \xrightarrow{\mathrm{Hom}_A(C,f)} & \mathrm{Hom}_A(C, N) \\ \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_N} \\ M & \xrightarrow{f} & N. \end{array}$$

The category of right C-contramodules is denoted by $\mathbf{M}_C$. Morphism sets (k-modules) are denoted by $\mathrm{Hom}_C(M, N)$.

A bit of history

- 1965–70: mentioned in relative homological algebra (Eilenberg-Moore), and in category theory (Vazquez Garcia, Barr);
- 2007: Positselski (arxiv:0708.3398) uses contramodules in an algebraic approach to semi-infinite cohomology (Voronov, Arkhipov).
- MathSciNet hits:
    - comodules = 797;
    - contramodules = 3.

Questions

- Is there a natural explanation for the existence of contramodules and are they natural objects to consider?
- In addition to modules of a ring, are there also ‘contramodules’ for rings?
- What do we know about contramodules?
- Why were contramodules ‘forgotten’?

Monads and comonads

Definition

A functor $G : \mathbf{X} \to \mathbf{X}$ is a *comonad* if there are natural transformations $\delta : G \to GG$, $\sigma : G \to \mathrm{id}_{\mathbf{X}}$ such that, for all objects $X \in \mathbf{X}$,

$$\begin{array}{ccc} G(X) & \xrightarrow{\ \delta_X\ } & GG(X) \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle G(\delta_X)} \\ GG(X) & \xrightarrow{\ \delta_{G(X)}\ } & GGG(X) \end{array},$$

$$\begin{array}{ccc} G(X) & \xrightarrow{\ \delta_X\ } & GG(X) \\ {\scriptstyle \delta_X}\downarrow & \searrow^{G(X)} & \downarrow{\scriptstyle G(\sigma_X)} \\ GG(X) & \xrightarrow{\ \sigma_{G(X)}\ } & G(X) \end{array}.$$

Monads defined dually: a functor $F : \mathbf{X} \to \mathbf{X}$ with natural transformations $\mu : FF \to F$, $\eta : \mathrm{id}_{\mathbf{X}} \to F$.

Algebras and coalgebras

Definition

A *coalgebra* or *comodule* of (G, δ, σ) is a pair (X, ϱ^X), where X is an object in $\mathbf{X}$ and $\varrho^X : X \to G(X)$ is a morphism,

$$\begin{array}{ccc} X & \xrightarrow{\varrho^X} & G(X) \\ {\scriptstyle \varrho^X}\downarrow & & \downarrow{\scriptstyle \delta_X} \\ G(X) & \xrightarrow{G(\varrho^X)} & GG(X)\,, \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varrho^X} & G(X) \\ & {\scriptstyle X}\searrow & \downarrow{\scriptstyle \sigma_X} \\ & & X\,. \end{array}$$

A morphism of G-coalgebras (X, ϱ^X), (Y, ϱ^Y) is $f \in \mathbf{X}(X, Y)$ such that

$$\varrho^Y \circ f = G(f) \circ \varrho^X.$$

Algebras of (F, μ, η) defined as pairs (X, ϱ_X), where $\varrho_X : F(X) \to X$.

The categories of algebras and coalgebras

G-coalgebras with their morphisms form a category $\mathbf{X}_G$.

- If $\mathbf{X}$ has colimits, coproducts, cokernels, so does $\mathbf{X}_G$.
- The forgetful functor $\mathbf{X}_G \to \mathbf{X}$ has a right adjoint, the *free coalgebra* functor:

$$X \mapsto (G(X), \delta_X).$$

- Free coalgebras form a full subcategory of $\mathbf{X}_G$ – the Kleisli category $\mathbf{K}_G$.

F-algebras with their morphisms form a category $\mathbf{X}^F$.

- If $\mathbf{X}$ has limits, products, kernels, so does $\mathbf{X}^F$.
- The forgetful functor $\mathbf{X}^F \to \mathbf{X}$ has a left adjoint, the *free algebra* functor:

$$X \mapsto (F(X), \mu_X).$$

- Free algebras form a full subcategory of $\mathbf{X}^F$ – the Kleisli category $\mathbf{K}^F$.

(Co)monads in adjoint pairs

Theorem (Eilenberg-Moore)

Consider an adjoint pair (L, R) of endofunctors on $\mathbf{X}$.

1. *L is a comonad if and only if R is a monad.*
2. *L is a monad if and only if R is a comonad. Furthermore:*

$$\mathbf{X}^L \equiv \mathbf{X}_R.$$

In the case (1), $\mathbf{K}_L \equiv \mathbf{K}^R$.

Corings as (co)monads

Theorem

Let C be an A-bimodule. The following statements are equivalent:

1. *C is an A-coring.*
2. *The functor*
$$-\otimes_A C : \mathbf{M}_A \to \mathbf{M}_A$$
is a comonad.
3. *The functor*
$$\mathrm{Hom}_A(C, -) : \mathbf{M}_A \to \mathbf{M}_A$$
is a monad.

Comodules vs contramodules

- $\mathbf{M}^C \equiv$ coalgebras of comonad $(- \otimes_A C, - \otimes_A \Delta, - \otimes_A \varepsilon)$.
- $\mathbf{M}_C \equiv$ algebras of monad $(\mathrm{Hom}_A(C, -), \mathrm{Hom}_A(\Delta, -), \mathrm{Hom}_A(\varepsilon, -))$.
- **Contramodules seem to be as natural as comodules.**

The case of algebras

Theorem

Let B be a k-module. The following statements are equivalent:

1. *B is a k-algebra.*
2. *The functor*
$$- \otimes_k B : \mathbf{M}_k \to \mathbf{M}_k$$
is a monad.
3. *The functor*
$$\mathrm{Hom}_k(B, -) : \mathbf{M}_k \to \mathbf{M}_k$$
is a comonad.

But $- \otimes_k B$ is the **left** adjoint of $\mathrm{Hom}_k(B, -)$, hence algebras of $- \otimes_k B$ = coalgebras of $\mathrm{Hom}_k(B, -)$ = $\mathbf{M}_B$

There are no 'contramodules' for rings.

'Free' knowledge about $\mathbf{M}_C$

- $\mathbf{M}_C$ has limits, products and kernels.
- $\mathbf{M}_C$ is abelian provided C_A is projective.
- The forgetful functor $\mathbf{M}_C \to \mathbf{M}_A$ has a left adjoint, the *free contramodule* functor:

$$M \mapsto (\mathrm{Hom}_A(C, M), \mathrm{Hom}_A(\Delta, M)).$$

- $C^* = \mathrm{Hom}_A(C, A)$ is a (free) C-contramodule by $\mathrm{Hom}_A(\Delta, A)$.
- For any contramodule (M, α),

$$\mathrm{Hom}_C(C^*, M) \simeq M,$$

- C^* is a generator in $\mathbf{M}_C$.

Contramodules vs modules

C^* is an algebra with the unit ε and product

$$(\xi * \xi')(c) = \sum \xi(\xi'(c_{(1)})c_{(2)}).$$

Theorem

1. *There is a faithful functor $F : \mathbf{M}_C \to \mathbf{M}_{C^*}$ defined as follows. $F(M, \alpha) = M$ is a right C^*-module with the action*

   $$\varrho_M : m \otimes \xi \mapsto \alpha(m\xi(-)).$$

   For morphisms, $F(f) = f$.
2. *The following statements are equivalent:*
   - *F is a full functor.*
   - *C is a finitely generated and projective right A-module.*
   - *F is an isomorphism.*

Projective contramodules

Definition

A C-contramodule P is (C, A)*-projective* if any diagram

$$\begin{array}{ccccc} M & \underset{\iota}{\overset{f}{\rightleftarrows}} & N & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle g} & & \\ & P & & & \end{array},$$

where f, g are C-contramodule maps and ι is an A-module map, can be completed by a C-contramodule map $h : P \to M$.

Theorem

A C-contramodule (P, α) is (C, A)-projective if and only if α has a C-contramodule section. In particular, every free contramodule is (C, A)-projective.

Definition (F Guzman)

C is *coseparable* if there exists an A-bimodule map $\delta : C \otimes_A C \to A$ such that $\delta \circ \Delta = \varepsilon$ and

$$(C \otimes_A \delta) \circ (\Delta \otimes_A C) = (\delta \otimes_A C) \circ (C \otimes_A \Delta).$$

Theorem

The following statements are equivalent:

1. *C is a coseparable coring.*
2. *The forgetful functor $\mathbf{M}^C \to \mathbf{M}_A$ is separable (the unit of adjunction $(\mathrm{Forget}, -\otimes_A C)$ has a natural retraction).*
3. *The forgetful functor $\mathbf{M}_C \to \mathbf{M}_A$ is separable (the counit of adjunction $(\mathrm{Hom}_A(C, -), \mathrm{Forget})$ has a natural section).*

Every contramodule of a coseparable coring is projective.

From comodules to contramodules

Fix an A-coring C and a B-coring D. Aim: describe (reasonable) functors $\mathbf{M}^D \to \mathbf{M}_C$.

Theorem

1. *Given a (C, D)-bicomodule N, and a right D-comodule M, there is a C-contramodule*

$$(\mathrm{Hom}^D(N, M), \mathrm{Hom}^D({}^N\varrho, M)).$$

2. *The functor $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_C$ has a left adjoint.*
3. *Any right adjoint functor $\mathbf{M}^D \to \mathbf{M}_C$ is naturally isomorphic to $\mathrm{Hom}^D(N, -)$ for some (C, D)-bicomodule N.*

The contratensor product

Definition (Positselski)

For all (C, D)-bicomodules N and right C-contramodules (M, α) the *(contra)tensor product* $M \otimes_C N$ is defined as a coequaliser

$$\mathrm{Hom}_A(C, M) \otimes_A N \rightrightarrows M \otimes_A N \longrightarrow M \otimes_C N,$$

where the coequalised maps are $f \otimes_A n \mapsto (f \otimes_A N) \circ {}^N\!\varrho(n)$ and $\alpha \otimes_A N$. Here ${}^N\!\varrho$ is the left C-coaction on N.

$M \otimes_C N$ is a coequaliser of right D-comodule maps, hence $M \otimes_C N$ is a D-comodule. There is a functor $- \otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$.

Theorem (Positselski)

The functor $- \otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$ *is the left adjoint of* $\mathrm{Hom}^D(N, -)$.

A descent triangle

A (C, D)-bicomodule N determines a commutative diagram of right adjoint functors

$$\begin{array}{ccc} & & \mathbf{M}_C \\ & \overset{\mathrm{Hom}^D(N,-)}{\nearrow} & \big\downarrow (-)_A \\ \mathbf{M}^D & \xrightarrow[\mathrm{Hom}^D(N,-)]{} & \mathbf{M}_A \,. \end{array}$$

There is a corresponding monad morphism

$$\mathrm{can}^N : \mathrm{Hom}_A(C, -) \to \mathrm{Hom}^D(N, -\otimes_A N), \quad \mathrm{can}^N_Q(f) = (f \otimes_A N) \circ {}^N\varrho.$$

Decent type

Theorem

The functor $-\otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$ *is fully faithful if and only if the following assertions hold.*

(i) *The natural transformation* can^N *is an isomorphism.*

(ii) *For all contramodules* (M, α)*, the functor* $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$ *preserves the coequaliser defining* $M \otimes_C N$*.*

Corollary

If can^N *is a natural isomorphism and* $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$ *is a right exact functor, then* $-\otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$ *is fully faithful and* C *is a projective right* A*-module.*

Effective descent

Theorem (Beck's theorem)

The categories $\mathbf{M}_C$ *and* $\mathbf{M}^D$ *are equivalent if and only if there exists a* (C, D)*-bicomodule* N *such that*

(i) *The natural transformation* can^N *is an isomorphism.*

(ii) *The functor* $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$ *reflects isos.*

(iii) *The functor* $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$ *preserves reflexive* $\mathrm{Hom}^D(N, -)$*-contractible coequalisers.*

Corollary

For a (C, D)*-bicomodule* N*, TFAE:*

(i) $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_C$ *is an equivalence and* C *is a projective right* A*-module.*

(ii) can^N *is a natural iso,* N *is a generator in* $\mathbf{M}^D$ *and* $\mathrm{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$ *is right exact.*