Compatibility of (co)actions and localizations
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preliminary version

Earlier, Lunts and Rosenberg studied a notion of compatibility of endofunctors with localization functors, with an application to the study of differential operators on noncommutative rings and schemes. Another compatibility – of Ore localizations of an algebra with a comodule algebra structure over a given bialgebra – introduced in my earlier work – is here described also in categorical language, but the appropriate notion differs from that of Lunts and Rosenberg, and it involves a specific kind of distributive laws. Some basic facts about compatible localization follow from more general functoriality properties of associating comonads or even actions of monoidal categories to comodule algebras. We also introduce localization compatible pairs of entwining structures.

1 Introduction

1.1. (Notation and prerequisites). Throughout the paper $\mathbf{k}$ is a ground field, but for most results it can be taken to be a commutative ring. The unadorned tensor symbol means tensoring over $\mathbf{k}$. We assume that the reader is well familiar with adjoint functors and familiar with (co)monads ([3, 12, 18]) which some call triples ([1]). We will speak (co)modules over (co)monads for what many call (co)algebras over (co)monads.

1.2. (Context and motivation). Apart from general purpose, this article is aimed to create the preliminaries for a natural general theorem on reconstruction of the structure of the *noncommutative scheme* ([13]) on the category of equivariant quasicoherent sheaves over a noncommutative scheme which is locally (in the sense of a cover by biflat affine localizations compatible with (co)actions) of Galois type (noncommutative principal bundle). This theorem involves the construction and exactness properties of various adjoint pairs of functors ([13]), what asked for precise requirements and usage of the correct and natural compatibility properties of localization functors with (co)actions in such geometric setup. This is the main subject of our forthcoming paper [17].

1.3. (Localization.) ([6, 18, 13]) A **localization functor** is a functor which is universal among the functors inverting a given class of morphisms in a domain category. A **continuous localization** functor is a functor $Q^* : \mathcal{A} \to \mathcal{B}$ having a fully faithful right adjoint $Q_* : \mathcal{B} \to \mathcal{A}$ (this implies that Q^* is a localization functor). This is equivalent to having a pair of adjoint functors $Q^* \dashv Q_*$ for which the counit $\epsilon : Q^* Q_* \to \mathrm{id}_{\mathcal{B}}$ is an isomorphism of functors. Consequently the multiplication $Q_* \epsilon Q^* : Q_* Q^* Q_* Q^* \to Q_* Q^*$ of the monad induced by this adjunction is clearly also an isomorphism ("idempotent monad") and the localized category $\mathcal{B}$ is via the comparison functor $N \mapsto (Q_* N, Q_*(\epsilon))$ equivalent to

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the (Eilenberg-Moore) category of modules over that monad. One usually says that $\mathcal{B}$ is a **reflective subcategory** of $\mathcal{A}$ (strictly full subcategory where the inclusion has a left adjoint).

In Abelian categories, one usually considers (additive) flat (= exact and continuous) localization functors: they may be obtained by localization at **thick subcategories** (= full subcategories closed under direct sums, quotients and extensions). The main example in the categories of modules is any *Gabriel localization* (at a Gabriel filter) of the category of left modules over a unital ring R. Even better subclass is the class of **Ore localizations**, which are of the form $Q^*M = S^{-1}R \otimes_R M$ where $S^{-1}R$ is the Ore localized ring, at a (say left) Ore set $S \subset R$. In that case, Q^* and Q_* are exact, $S^{-1}R$ is consequently flat over R and the component of the unit of the adjunction $\eta : \mathrm{id} \to Q_*Q^*$ for the ring R itself, namely $\iota_S : R \to S^{-1}R$, is a morphism of unital rings. The multiplication induces an isomorphism of S-modules $S^{-1}R \otimes_R S^{-1}R \to S^{-1}R$ (because the monad $M \mapsto S^{-1}R \otimes_R M = Q^*Q_*M$ is an idempotent monad).

2 Functoriality of actegories from comodule algebras

2.1. Let B be a **k**-bialgebra. The category $_B\mathcal{M}$ of left B-modules is a monoidal category in standard way: the tensor product is the tensor product of the underlying **k**-modules with the left B-action given by $b(x \otimes_\mathbf{k} y) = \sum b_{(1)}x \otimes_\mathbf{k} b_{(2)}y$.

2.2. Given a **k**-bialgebra B, a left (right) B-**comodule algebra** is a pair (A, ρ) of an algebra A and a left (right) B-coaction $\rho : A \to B \otimes A$ (resp. $\rho : A \to A \otimes B$) which is an algebra map. We use extended Sweedler notation $\rho(e) = \sum e_{(0)} \otimes e_{(1)}$ ([11]).

2.2.1. $_B\mathcal{M}$ acts on $_\mathbf{k}\mathcal{M}$ in a trivial way: on objects just tensor the underlying **k**-modules; bialgebra B lives in the category of **k**-modules, and its meaning is related to the tensor product in $_\mathbf{k}\mathcal{M}$. Thus this distinguished *defining* or **base action** is however important, because in noncommutative geometry it is natural to consider actions of $_B\mathcal{M}$ which are **geometrically admissible**. These are the actions of the type $\lhd : \mathcal{C} \times {_B\mathcal{M}} \to \mathcal{C}$ on an abstract category $\mathcal{C}$ equipped with a direct image functor $U : \mathcal{C} \to {_\mathbf{k}\mathcal{M}}$ such that $U \circ \lhd = \lhd_0 \circ (U \times \mathrm{Id}_{_B\mathcal{M}})$, where $\lhd_0$ is the base action $\lhd_0 : {_\mathbf{k}\mathcal{M}} \times {_B\mathcal{M}} \to {_\mathbf{k}\mathcal{M}}$. Such actions may be called *lifts* of $\lhd_0$ along U. Lifts to $\mathcal{C} = {_E\mathcal{M}}$ where E is a **k**-algebra are in a bijective correspondence with the distributive laws between the base action and monad $E \otimes_\mathbf{k}$ on $_\mathbf{k}\mathcal{M}$)[14, 19]. Such distributive laws are generalizations of Beck' classical distributive laws between two (co)monads.

2.2.2. The distributive law with components $l_{M,Q} : E \otimes (M \otimes Q) \to (E \otimes M) \otimes Q$ given by $e \otimes (m \otimes q) \mapsto \sum e_{(0)} \otimes m \otimes e_{(1)}q$ where $e \in E, m \in M, q \in Q$, where $M \in {_\mathbf{k}\mathcal{M}}$ and $Q \in {_B\mathcal{M}}$ induces thus a right action of $_B\mathcal{M}$ on $_E\mathcal{M}$ lifting the base action. More explicitly, for $N \in {_E\mathcal{M}}$, $N \lhd Q$ is $N \otimes Q$ with left E-module structure $e(n \otimes q) = \sum e_{(0)}n \otimes e_{(1)}q$.

We want to describe in this section the functoriality of this and various dual constructions. We often call monoidal category $\mathcal{D}$ together with a (left or right) $\mathcal{D}$-action on some category $\mathcal{C}$ a (left or right) $\mathcal{D}$-actegory.

2.3. (Comonad for the relative Hopf modules). B is a comonoid in the monoidal category $_B\mathcal{M}$. Therefore the strong monoidal action of $_B\mathcal{M}$ on any category sends it to a comonoid in the category of endofunctors (in our case also additive). The underlying endofunctor $G : {_E}\mathcal{M} \to {_E}\mathcal{M}$ in the category $_E\mathcal{M}$ of left E-modules on objects M in $\mathcal{M}$ is given by the formula $G : M \mapsto M \otimes B$, where the left E-module structure on $M \otimes B$ is given by $e(m \otimes b) := \rho_E(e)(m \otimes b) = \sum e_{(0)} m \otimes e_{(1)} b$ where $e \in E, m \in M, b \in B$. The comultiplication Δ_B on B induces the comultiplication $\delta = \mathrm{id} \otimes \Delta : G \to GG$ on G with counit $\epsilon^G = \mathrm{id} \otimes \epsilon$ making $\mathbf{G} = (G, \delta, \epsilon^G)$ a comonad (cf. the coring picture in [5]).

It is well-known that the category $(_E\mathcal{M})_\mathbf{G}$ of $\mathbf{G}$-comodules (coalgebras) is equivalent to the category $_E\mathcal{M}^B$ of left-right relative (E, B)-Hopf modules; thus we say that $\mathbf{G}$ is the *comonad for relative Hopf modules*. Left-right **relative (E, B)-Hopf module** is a m left module (N, ν_N) where $\nu_N : E \otimes N \to N$ is a left E-action, equipped with a right B-coaction $\rho_N : N \to N \otimes B$ such that $\rho_N(\nu(e, n)) = (\nu \otimes \mu_B)(\mathrm{id} \otimes \tau_{B,N} \otimes \mathrm{id})(\rho_E(e) \otimes \rho_N(n))$ for all $e \in E$, $n \in N$; here $\tau_{B,N} : B \otimes N \to N \otimes B$ is the flip of tensor factors. Morphisms of relative Hopf modules are morphisms of underlying $\mathbf{k}$-modules, which respect E-actions and B-coactions.

2.4. We will work in part of the article not only with maps of comodule algebras over a fixed bialgebra, but we will also allow variable bialgebras. Thus let $\phi : B \to B'$ be a map of bialgebras, (E, ρ) a B-comodule algebra and (E', ρ') a B'-comodule algebra. Then a map of underlying algebras $f : E \to E'$ is a **map of comodule algebras over** ϕ if $\rho' \circ f = (f \otimes \phi) \circ \rho : E \to E' \otimes B'$. Alternatively, one says that the pair (f, ϕ) is a *map of comodule algebras over varying bialgebras*.

2.5. For any algebra map $f : A \to A'$ we use geometric *inverse image* notation for the extension of scalars $f^* : M \to f^*M = A' \otimes_A M$ for the categories of left modules, though this implies that $f \mapsto f^*$ is a covariant functor.

2.6. Theorem. *There is a canonical 2-cell*

$$
\begin{array}{ccc}
{_E}\mathcal{M} \times {_B}\mathcal{M} & \xrightarrow{\;f^* \times \phi^*\;} & {_{E'}}\mathcal{M} \times {_{B'}}\mathcal{M} \\
\downarrow{\scriptstyle \triangleleft} & \quad {\scriptstyle \alpha}\nearrow\!\!\!\nearrow & \downarrow{\scriptstyle \triangleleft'} \\
{_E}\mathcal{M} & \xrightarrow[\;f^*\;]{} & {_{E'}}\mathcal{M}
\end{array}
$$

that is a natural transformation $\alpha = \alpha^{f,\phi} : f^* \circ \triangleleft \Rightarrow \triangleleft' \circ (f^* \times \phi^*)$.

Proof. The components

$$
\alpha_{M,Q} = \alpha^{f,\phi}_{M,Q} : E' \otimes_E (M \triangleleft Q) \to (E' \otimes_E M) \triangleleft' (B' \otimes_B Q)
$$

of the natural transformation α, where $M \in {}_E\mathcal{M}$ and $Q \in {}_B\mathcal{M}$ are defined as **k**-linear extensions of the formulas

$$\alpha_{M,Q}(e' \otimes_E (m \otimes q)) = \sum (e'_{(0)} \otimes_E m) \otimes (1_{B'} \otimes_B e'_{(1)}q).$$

One checks that $\alpha_{M,Q}$ is well defined (it is well-defined before we quotient to $\otimes_E$; consider values on $e'e \otimes (m \otimes q)$ and $\sum e' \otimes e_{(0)}m \otimes e_{(1)}q$ and calculate that both give the same) and that $\alpha_{M,Q}$ is indeed a morphism in ${}_{E'}\mathcal{M}$.

2.7. Consider now the comonad for Hopf modules $\mathbf{G} = (G, \delta, \epsilon)$ on ${}_E\mathcal{M}$.

    Corollary. *There is a 2-cell*

which is in fact a morphism of comonads.

2.8. Theorem. *2-cells α paste correctly with respect to composition of comodule algebra maps over varying base. In other words, for components at B, the pasting*

equals the two cell

where the symbols for canonical invertible 2-cells $g^* f^* \cong (gf)^*$ are ommitted.

    Similar diagrams hold for other components.

    Proof is an easy direct calculation.

2.9. Now we use the transformation α to induce the map for the categories of Hopf modules ${}_E\mathcal{M}^B \to {}_{E'}\mathcal{M}^{B'}$. It is known that ${}_E\mathcal{M}^B \cong ({}_E\mathcal{M})_{\mathbf{G}_E}$ so this procedure is standard.

    Let $M \in {}_E\mathcal{M}$ and $\rho_M : M \to M \otimes B$ be a coaction making M a relative Hopf module.

    Proposition. *The extension of scalars $f^* : {}_E\mathcal{M} \to {}_{E'}\mathcal{M}'$ lifts to the functor $f^{*B} : {}_E\mathcal{M}^B \to {}_{E'}\mathcal{M}^{B'}$ between the categories of relative Hopf modules, which is at objects given by $f^{*B} : (M, \rho_M) \mapsto (f^*M, \alpha_M \circ f^*(\rho))$.*

3 Compatibility for comodule algebras

3.1. (Compatibility of coactions and localizations).

Let (E, ρ) be a right B-comodule algebra. An Ore localization of rings $\iota_S : E \to S^{-1}E$ is ρ-**compatible** ([16]) if there exist an (automatically unique) coaction $\rho_S : S^{-1}E \to S^{-1}E \otimes B$ making $S^{-1}E$ a B-comodule algebra, such that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ \rho\ } & E \otimes B \\
\downarrow{\scriptstyle \iota_S} & & \downarrow{\scriptstyle \iota_S \otimes B} \\
S^{-1}E & \xrightarrow{\ \rho_S\ } & S^{-1}E \otimes B
\end{array}
$$

commutes. It is easy to check that this is equivalent to an effective criterium that for all $s \in S$, $(\iota_S \otimes \mathrm{id}_B)\rho(s)$ is invertible in $S^{-1}E \otimes B$. If the Ore localization is ρ-compatible, ρ_S is called (the) **localized coaction**. The elements $u \in S^{-1}E$ satifying $\rho_S(u) = u \otimes 1$ i.e. the coinvariants under the localized coaction are called **localized coinvariants**. It is a basic and important observation that the localization and taking coinvariants do not commute: the subalgebra $(S^{-1}E)^{\mathrm{co}B} \subset S^{-1}E$ of localized coinvariants typically contains some extra elements which do not naturally belong to the **k**-submodule $\iota_S(E^{\mathrm{co}B})$; moreover typically ι_S restricted to the subring $E^{\mathrm{co}B} \subset E$ is not underlying a ring localization $U^{-1}E^{\mathrm{co}B}$ with respect to any Ore subset U in $E^{\mathrm{co}B}$.

3.2. Theorem.
*Let B be a **k**-bialgebra, (E, ρ) a B-comodule algebra, **G** a comonad from **2.3**, and $\iota : E \to E_\mu$ a perfect (e.g. Ore) localization of rings, which happens to be ρ-compatible.*

*The **k**-linear map*

$$
l_{M,P} : E_\mu \otimes_E (M \otimes P) \Rightarrow (E_\mu \otimes_E M) \otimes P, \ \ e \otimes (m \otimes b) \mapsto \sum (e_{(0)} \otimes m) \otimes e_{(1)},
$$

for $m \in M, p \in P, e \in E$, where P is a B-module and M a E-module is a well-defined morphism of left E-modules. All $l_{M,P}$ together form a mixed distributive law between the localization monad Q_*Q^* and the categorical action of $_B\mathcal{M}$ on $_E\mathcal{M}$.

Proof. This is a slight generalization of the case $P = B$ which gives the distributive law between the localization monad Q_*Q^* and the comonad **G** which is proved in [18]. The general proof is analogous.

3.3. Proposition.
*Given any continuous localization functor $Q^* : \mathcal{A} \to \mathcal{A}_\mu$ and a comonad **G** together with any mixed distributive law $l : Q_*Q^*G \Rightarrow GQ_*Q^*$,*

1) $G_\mu = Q^*GQ_*$ underlies a comonad $\mathbf{G}_\mu = (G_\mu, \delta^\mu, \epsilon^{G_\mu})$ in $\mathcal{A}_\mu$ with comultiplication δ^μ given by the composition

$$
Q^*GQ_* \xrightarrow{Q^*\delta^G Q_*} Q^*GGQ_* \xrightarrow{Q^*G\eta GQ_*} Q^*GQ_*Q^*GQ_*
$$

and whose counit ϵ^{G_μ} is the composition

$$
Q^*GQ_* \xrightarrow{Q^*\epsilon^G Q_*} Q^*Q_* \xrightarrow{\ \epsilon\ } \mathrm{Id}_{\mathcal{A}_\mu}
$$

(where the right-hand side comultiplication ϵ is the counit of the adjunction $Q_* \dashv Q^*$).

2) the composition

$$Q^*GM \xrightarrow{\quad Q^*(\eta_{GM}) \quad} Q^*Q_*Q^*GM \xrightarrow{\quad Q^*(l_M) \quad} G_\mu Q^*M$$

defines a component of a natural transformation $\alpha = \alpha_l : Q^*G \Rightarrow G_\mu Q^*$ for which the mixed pentagon diagram of transformations

$$
\begin{array}{ccc}
Q^*G & \xrightarrow{\quad\quad \alpha \quad\quad} & G_\mu Q^* \\
{\scriptstyle Q^*\delta^G}\downarrow & & \downarrow{\scriptstyle \delta^\mu Q^*} \\
Q^*GG & \xrightarrow[\alpha G]{} G_\mu Q^*G \xrightarrow[G\alpha]{} & G_\mu G_\mu Q^*
\end{array}
$$

commutes and $(\epsilon^{G_\mu} Q^*) \circ \alpha = Q^* \epsilon^G$. In other words, $(Q^*, \alpha_l) : (\mathcal{A}, \mathbf{G}) \to (\mathcal{A}_\mu, \mathbf{G}_\mu)$ is (up to orientation convention which depends on an author) a map of comonads ([15, 20]).

3.4. Theorem. *Under assumptions in **3.2**, there is a unique induced continuous localization functor $Q^{B*} : {}_E\mathcal{M}^B \to {}_{E_\mu}\mathcal{M}^B$ between the categories of relative Hopf modules such that $U_\mu Q^{B*} = Q^*U$ where U and U_μ are the forgetful functors from the category of relative Hopf modules to the categories of usual modules over E and E_μ respectively.*

Proof. We have stated this theorem and given direct proof in [19]. The more general results from the previous sections make it a special case of **2.5**.

3.5. Corollary. *Let $E = B$ and $\mathbf{k}$ is a field. The only Δ-compatible Ore localization $B \to B_\mu$ is the trivial one.*

This is an analogue of the statement that the only G-invariant Zariski open subset of an algebraic group over a field is the whole group.

Proof. The compatible localization functor induces in this case a localization functor $Q^{B*} : {}_B\mathcal{M}^B \to {}_{B_\mu}\mathcal{M}^B$. By the fundamental theorem on relative Hopf modules the domain of this functor is ${}_B\mathcal{M}^B \cong {}_{\mathbf{k}}\mathcal{M}$. But ${}_{\mathbf{k}}\mathcal{M}$ is just a category of vector spaces over a field which does not have nontrivial continuous localizations with contradiction.

4   Compatibility for entwinings

4.1. (Localization-compatible pairs of entwinings.) Let A be a $\mathbf{k}$-algebra, C a $\mathbf{k}$-coalgebra, $\iota : A \to A_\mu$ a perfect localization of rings, and $\psi : A \otimes C \to C \otimes A$, $\psi_\mu : A_\mu \otimes C \to C \otimes A_\mu$ entwinings. We say that (ψ, ψ_μ) is ι-compatible pair of entwinings if the diagram

$$
\begin{array}{ccc}
A \otimes C & \xrightarrow{\quad \psi \quad} & C \otimes A \\
{\scriptstyle \iota \otimes C}\downarrow & & \downarrow{\scriptstyle C \otimes \iota} \\
A_\mu \otimes C & \xrightarrow{\quad \psi_\mu \quad} & C \otimes A_\mu
\end{array}
$$

commutes.

4.2. Define comonad **G** on $_A\mathcal{M}$ as usual: $G(M, \nu) = (C \otimes M, (C \otimes \nu) \circ \psi_M)$.

Proposition. *Given a ι-compatible pair (ψ, ψ_μ) of entwinings, the $\mathbf{k}$-linear map $\psi_\mu \otimes M : A_\mu \otimes C \otimes M \to C \otimes A_\mu \otimes M$ factors to a well-defined map of A-modules*

$$l_M : A_\mu \otimes_A GM \to G(A_\mu \otimes_A M).$$

Proof. Consider the diagram

$$
\begin{array}{ccccc}
A_\mu \otimes A \otimes C \otimes M & \underset{A_\mu \otimes (C \otimes \nu) \circ (\psi \otimes M)}{\overset{\mu \otimes C \otimes M}{\rightrightarrows}} & A_\mu \otimes C \otimes M & \longrightarrow & A_\mu \otimes_A GM \qquad (1) \\
\downarrow & & \downarrow {\scriptstyle \psi_\mu \otimes M} & & \vdots {\scriptstyle l_M} \\
C \otimes A_\mu \otimes A \otimes M & \underset{A_\mu \otimes (C \otimes \nu) \circ (\psi \otimes M)}{\overset{C \otimes \mu \otimes M}{\rightrightarrows}} & C \otimes A_\mu \otimes M & \longrightarrow & G(A_\mu \otimes_A M)
\end{array}
$$

where the rows are equalizer forks and the left vertical arrow is the composition

$$A_\mu \otimes E \otimes C \otimes M \overset{A_\mu \otimes \psi \otimes M}{\longrightarrow} A_\mu \otimes C \otimes E \otimes M \overset{\psi_\mu \otimes A \otimes M}{\longrightarrow} C \otimes A_\mu \otimes A \otimes M$$

If the left square in (1) sequentially commutes, then clearly the right vertical arrow factors to a well-defined map l.

In the following two diagrams we omit the tensor product sign $\otimes_\mathbf{k}$; by abuse of notation we denote by m both multiplications (in A and A_μ).

$$
\begin{array}{ccccc}
& & \overset{mC}{\underset{}{\longrightarrow}} & & \\
A_\mu AC & \overset{mC}{\underset{A_\mu \iota C}{\rightrightarrows}} A_\mu A_\mu C & \overset{mC}{\longrightarrow} & A_\mu C & \qquad (2) \\
{\scriptstyle A_\mu \psi} \downarrow & {\scriptstyle A_\mu \psi_\mu} \downarrow & & \downarrow {\scriptstyle \psi_\mu} & \\
A_\mu CA & \overset{A_\mu C\iota}{\longrightarrow} A_\mu CA_\mu & & & \\
{\scriptstyle A_\mu} \downarrow & \downarrow {\scriptstyle \psi_\mu A_\mu} & & \downarrow {\scriptstyle \psi_\mu} & \\
CA_\mu A & \overset{CA_\mu \iota}{\longrightarrow} CA_\mu A_\mu & \overset{Cm}{\longrightarrow} & CA_\mu & \\
& & \underset{Cm}{\longrightarrow} & &
\end{array}
$$

This diagram clearly commutes and when we tensor the whole diagram with M from the right we see that the upper left square in (1) commutes.

$$
\begin{array}{ccccc}
A_\mu ACM & \overset{A_\mu \psi M}{\longrightarrow} & A_\mu CAM & \overset{A_\mu C\nu}{\longrightarrow} & A_\mu CM \\
{\scriptstyle A_\mu \psi M} \downarrow & & & & \downarrow {\scriptstyle \psi_\mu M} \\
A_\mu CAM & & & & \\
{\scriptstyle \psi_\mu AM} \downarrow & & & & \downarrow {\scriptstyle \psi_\mu M} \\
CA_\mu AM & & \overset{CA_\mu \nu}{\longrightarrow} & & CA_\mu M
\end{array}
$$

This diagram commutes by naturality, and shows that the lower left square in (1) commutes.

We conclude that the map l is well-defined. We need to check that it is a map of A-modules. But again, the commutativity of the diagram (2) shows that $\psi_\mu \otimes M : A_\mu \otimes GM \to G(A_\mu \otimes M)$ is a map of left A-modules (in fact it is a map of A_μ-modules simply by the pentagon for ψ and m); hence *a fortiori* the induced map on quotients respects A-module structure.

4.3. Proposition. l_M *above form a distributive law.*
    Proof. Direct check.

4.4. Theorem. *Every localization compatible pair of entwinings induces a continuous localization $Q_\psi^* : {}_A^C\mathcal{M}_\psi \to {}_{A_\mu}^C\mathcal{M}_{\psi_\mu}$ between the categories of entwined modules for the two entwinings such that $U_\mu Q_\psi^* = Q^* U$ where U and U_μ are the forgetful functors from the categories of entwined to the categories of usual modules over A and A_μ respectively.*

5   The case of module algebras

5.1. We say that the left action $\triangleright$ of a bialgebra H on an algebra A is Hopf or that $(A, \triangleright)$ is a left H-module algebra if $h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright a)$. Let $\iota : A \to A_\mu$ be an Ore localization and $(A, \triangleright)$ a left H-module algebra. We say that ι is compatible with module algebra structure if there is a Hopf action $\triangleright'$ of H on A_μ such that $\iota \circ \triangleright = \triangleright' \circ (H \otimes \iota)$.

5.2. *The monoidal category of right H-comodules has a canonical action on the category of modules ${}_A\mathcal{M}$ over a left H-module algebra A.* Again, A is a monoid in that category so we get in particular a monad $\mathbf{T}$ on ${}_A\mathcal{M}$.

The action is induced by the distributive law with components $l_{M,P} : A \otimes (M \otimes P) \to (A \otimes M) \otimes P$ given by the formula $a \otimes (m \otimes p) \mapsto \sum (p_{(1)} \triangleright_A \otimes m) \otimes p_{(0)}$.

One should check that one indeeds get an action. For simplicity of notation we do it for $P = H$; that is we check that $\mathbf{T}$ is a monad. General case is almost the same.

Define the endofunctor T on the category of left A-modules

$$T(M, \triangleright_M) := (M \otimes_{\mathbf{k}} H, \triangleright_{TM}), \quad M \in A - \mathrm{Mod};$$
$$Tf := f \otimes 1_H \in \mathrm{Hom}_A(TM, TN), \quad \forall f \in \mathrm{Hom}_A(M, N).$$

where the A-action $\triangleright_{TM}$ on $M \otimes_{\mathbf{k}} H$ is given by $\mathbf{k}$-linear extension of the formula $a \triangleright_{TM} (m \otimes h) := \sum ((h_{(2)} \triangleright_A a) \triangleright_M m)) \otimes h_{(1)}$.

We have to check that $\triangleright_{TM}$ is indeed an A-action:

$$
\begin{aligned}
a \triangleright_{TM} (a' \triangleright_{TM} (m \otimes h)) &= a \triangleright_{TM} ((h_{(2)} \triangleright_A a') \triangleright_M m)) \otimes h_{(1)} \\
&= ((h_{(2)} \triangleright_A a) \triangleright_M ((h_{(3)} \triangleright_A a') \triangleright_M m)) \otimes h_{(1)} \\
(\triangleright_A \text{ is a Hopf action}) \quad &= (h_{(2)} \triangleright_A (aa')) \triangleright_M m) \otimes h_{(1)} \\
&= (aa') \triangleright_{TM} (m \otimes h).
\end{aligned}
$$

$$
\begin{aligned}
1 \triangleright_{TM} (m \otimes h) \quad &= \quad ((h_{(2)} \triangleright_A 1) \triangleright_M m)) \otimes h_{(1)} \\
(\triangleright_A \text{ is a Hopf action}) \quad &= \quad (\epsilon(h_{(2)})1 \triangleright_M m)) \otimes h_{(1)} \\
&= \quad m \otimes h.
\end{aligned}
$$

We also check that Tf is indeed a map of left A-modules:

$$
\begin{aligned}
(Tf)[a \triangleright_{TM} (m \otimes h)] \quad &= \quad (Tf)[(h_{(2)} \triangleright_A a) \triangleright_M m \otimes h_{(1)}] \\
&= \quad f((h_{(2)} \triangleright_A a) \triangleright_M m) \otimes h_{(1)} \\
&= \quad (h_{(2)} \triangleright_A a) \triangleright_M f(m) \otimes h_{(1)} \\
&= \quad a \triangleright_{TM} (f(m) \otimes m) \\
&= \quad a \triangleright_{TM} [(Tf)(m \otimes h)]
\end{aligned}
$$

Define the natural transformations $\mu : TT \Rightarrow T$ and $\eta : \mathrm{Id} \Rightarrow T$ by

$$
\mu_{(M,\triangleright_M)}(\sum_i m_i \otimes h_i \otimes g_i) := \sum_i m_i \otimes h_i g_i,
$$

$$
\eta_{(M,\triangleright_M)}(m) := m \otimes 1.
$$

Here we have to check that $\mu_M := \mu_{(M,\triangleright_M)}$ and $\eta_M := \eta_{(M,\triangleright_M)}$ are indeed maps of left A-modules.

$$
\begin{aligned}
a \triangleright_{TTM} [(m \otimes h) \otimes g] \quad &= \quad [(g_{(2)} \triangleright_A a) \triangleright_{TM} (m \otimes h)] \otimes g_{(1)} \\
&= \quad [h_{(2)} \triangleright_A (g_{(2)} \triangleright_A a)] \triangleright_m m \otimes h_{(1)} \otimes g_{(1)} \\
&= \quad (h_{(2)} g_{(2)} \triangleright_A a) \triangleright_M m \otimes h_{(1)} \otimes g_{(1)} \\
&\overset{\mu_M}{\mapsto} \quad (h_{(2)} g_{(2)} \triangleright_A a) \triangleright_M m \otimes h_{(1)} g_{(1)} \\
&= \quad a \triangleright_{TM} (m \otimes hg) \\
&= \quad a \triangleright_{TM} [\mu_M((m \otimes h) \otimes g)].
\end{aligned}
$$

Now we have a straightforward

5.3. Proposition. *Compatibility of Hopf action with localization induces a distributive law between the induced monad* **T** *defined above and the localization monad.*

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