Non-Orthomodular Models for Both Standard Quantum Logic and Standard Classical Logic: Repercussions for Quantum Computers

By Mladen Pavičić[1]

University of Zagreb, GF, Kačićeva 26, HR-10000 Zagreb, Croatia.

and Norman D. Megill[2]

Boston Information Group, 30 Church St., Belmont MA 02478, U. S. A.

Abstract. It is shown that propositional calculuses of both quantum and classical logics are non-categorical. We find that quantum logic is in addition to an orthomodular lattice also modeled by a weakly orthomodular lattice and that classical logic is in addition to a Boolean algebra also modeled by a weakly distributive lattice. Both new models turn out to be non-orthomodular. We prove the soundness and completeness of the calculuses for the models. We also prove that all the operations in an orthomodular lattice are five-fold defined. In the end we discuss possible repercussions of our results to quantum computations and quantum computers.

PACS numbers: 03.65.Bz, 02.10.By, 02.10.Gd

Keywords: quantum logic, logic of quantum mechanics, quantum computation, orthomodular lattices, weakly orthomodular lattices, classical logic, Boolean algebra, weakly distributive lattices, model theory, categoricity, non-categorical models

[1]E-mail: mpavicic@faust.irb.hr; Web page: http://m3k.grad.hr/pavicic
[2]E-mail: nm@alum.mit.edu

1 Introduction

For more than a century it has been taken for granted that the propositional calculus of classical logic has a Boolean algebra (complemented distributive lattice) as its only lattice model for which completeness of the logic can be proved and for more than half a century it has been taken for granted that an orthomodular lattice is the only such model of the propositional calculus of quantum logic—the logic of quantum mechanics [1]. In this paper we prove that both assumptions are incorrect by finding a new lattice model for classical logic and another for quantum logic neither of which is orthomodular (any distributive lattice is orthomodular). We also show that the reason why distributive and orthomodular lattices also model classical and quantum logics, respectively, lies in the way their completeness proofs have been carried out in the past. We show that the proofs contained a hidden statement which introduced the property of orthomodularity into not necessarily orthomodular Lindenbaum algebras of the logics. This is because mappings of the logic to an ortholattice does not turn the lattice into an orthomodular one as usually assumed. In particular, the orthomodularity law and the distributivity law do not map into the corresponding lattice expressions at all: the orthomodularity in quantum logic and the distributivity in classical logic when mapped into a lattice are valid in a non-orthomodular ortholattice and do not have anything to do with making the lattices orthomodular [2] and distributive.

In terms of computability our results mean that, structurally, a computation and inference of formulas neither in classical nor in quantum logic correspond to a computation and inference of formulas in their models. This discrepancy has not been noticed so far because classical calculations in classical computers and classical physics in phase space are based not on classical logic proper but on its model, i.e., on its distributive model, a Boolean algebra. Also an algebra of two valued (yes and no, 1 and 0) propositions of classical logic must be a Boolean one and any Boolean algebra can be shown equivalent to a Kolmogorovian probability theory (which is therefore another possible model for classical logic) [3]. As opposed to this, quantum algebra which would give a Hilbertian probability theory as a proper universal language for quantum computers is still not known. Therefore the first idea is to rely on quantum logic of elementary input propositions themselves. However, ascribing yes-no values to all quantum propositions is precluded by the Kochen-Specker theorem. [4] Hence, if one wanted to build a quantum simulator (a general purpose quantum computer which would not be limited to particular algorithms such as Shor's or Grover's [5]) one should first develop a proper quantum computer language, i.e., an algebra which would enable typing in any many-system Schrödinger equation and then solve it in a polynomial time by simulating the systems the equation describes. The need for such an algebra also stems from the fact that no operation in quantum logic is unique: as we show in Sec. 2 all the operations, including the identity, are fivefold defined. And with five identity operations and no definite values ascribable to measurement propositions we obviously must seek a new algebraic way of valuating propositions in order to find, for example, which of them give the same measurement results.

In terms of the model theory, our result means that neither classical nor quantum logic are categorical. A formal system is called categorical (monomorphic) if all its models are isomorphic with each other. In 1934 Tarski was—in spite of the Gödel's results—of the opinion that "a non-categorical set of sentences (especially if it is used as an axiomatic system of a deductive theory) does not give the impression of a closed and organic unity and does not seem to determine precisely the meaning of the concepts contained in it." [6] For, the usual set theories are non-categorical simply because they are incomplete as a consequence of Gödel's theorem. The first-order predicate calculus with Peano's natural number sequence axioms is non-categorical and complete. In general, it has been "proved that no consistent first-order theory which possesses an infinite model is categorical" simply because "each such theory possesses models of arbitrary power." ([7], p. 298) Still, simple propositional calculuses not endowed with quantifiers and numbers which were complete were apparently expected to be categorical. Now we prove that surprisingly even such calculuses can be non-categorical.

The paper is organized as follows. In Sec. 2 we show that there are four quantum identities ($a \equiv_i b$, $i = 1, ..., 4$) in an orthomodular lattice which are not symmetric and one which is ($a \equiv_5 b$). They all boil down to the classical identity ($a \equiv_0 b$) in a Boolean algebra. Nevertheless the following implication $a \equiv_i b = 1 \;\Rightarrow\; a = b,\; i = 1, ..., 5$ makes an ortholattice orthomodular. Also $a \equiv_0 b = 1 \;\Rightarrow\; a = b$ makes an ortholattice distributive. These results we use in Sec. 3 where we show that a logic which does have an orthomodular lattice for its model is not necessarily orthomodular—it also has a weakly orthomodular model—and in Sec. 4 that classical logic which does have a distributive lattice for its model is not necessarily distributive: it also has a weakly distributive model. We give soundness and completeness proofs for all the models.

2 Asymmetrical Quantum Identities

An ortholattice (OL) is an algebra $\mathcal{L}_{\rm O} =< \mathcal{L}_{\rm O}^\circ, ', \cap, \cup >$ such that the following conditions are satisfied for any $a, b, c \in \mathcal{L}_{\rm O}{}^\circ$:

L1. $\quad a \cup b \;=\; b \cup a$

L2. $\quad (a \cup b) \cup c \;=\; a \cup (b \cup a)$

L3. $\quad a'' \;=\; a$

L4. $\quad a \cup (b \cup b') \;=\; b \cup b'$

L5. $\quad a \cup (a \cap b) \;=\; a$

L6. $\quad a \cap b \;=\; (a' \cup b')'$

An orthomodular lattice OML is an OL in which the following additional condition is satisfied:

L7. $a \cup b = ((a \cup b) \cap b') \cup b$

A weakly orthomodular lattice WOML is an OL in which the following additional condition is satisfied:

L8. $(a' \cap (a \cup b)) \cup b' \cup (a \cap b) = 1$

A distributive lattice (Boolean algebra) DL is an OL in which the following additional condition is satisfied:

L9. $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

It is well-known that in every orthomodular lattice five polynomial implications satisfy the Birkhoff-von Neumann requirement [8]:

$$a \to_i b = 1 \qquad \Rightarrow \qquad a \le b, \qquad\qquad i = 1, \dots, 5, \tag{2.1}$$

where $a \to_1 b \overset{\text{def}}{=} a' \cup (a \cap b)$, $a \to_2 b \overset{\text{def}}{=} b' \to_1 a'$, $a \to_3 b \overset{\text{def}}{=} (a' \cap b) \cup (a' \cap b') \cup (a \to_1 b)$, $a \to_4 b \overset{\text{def}}{=} b' \to_3 a'$, and $a \to_5 b \overset{\text{def}}{=} (a \cap b) \cup (a' \cap b) \cup (a' \cap b')$.

Even more, it can be proved [9] that the rule (2.1) makes an ortholattice orthomodular, i.e., that (2.1) can be substituted for L7. Since it can also be proved [10] that the following rule

$$a \to_0 b = 1 \qquad \Rightarrow \qquad a \le b, \tag{2.2}$$

where $a \to_0 b \overset{\text{def}}{=} a' \cup b$, makes an ortholattice distributive (Boolean algebra), i.e., that (2.2) can be substituted for L9, it is clear that $a \to_i b$, $i = 1, \dots, 5$ all merge to $a \to_0 b$ in a classical theory. In addition, one can write any a' as $a \to_i 0$ and one can prove[11]:

$$a \cup b = (a \to_i b) \to_i (((a \to_i b) \to_i (b \to_i a)) \to_i a) \tag{2.3}$$

for $i = 1, \dots, 5$. Thus one can form a quantum implication algebra with the operation of implication as a single primitive and to prove that an orthomodular (distributive) lattice can model quantum (classical) logic seems to be obvious since it is easy to prove that in any orthomodular lattice we have: $a \leftrightarrow_i b = a \equiv_5 b$, $i = 1, \dots, 5$, where $a \leftrightarrow_i b \overset{\text{def}}{=} (a \to_i b) \cap (b \to_i a)$ and $a \equiv_5 b \overset{\text{def}}{=} (a \cap b) \cup (a' \cap b')$ and the identity operation $a \equiv_5 b$ reduces to $a \equiv_0 b \overset{\text{def}}{=} (a' \cup b) \cap (b' \cup a)$ in a classical theory. For, $a \equiv_i b = 1$, $i = 0, 5$ is reflexive, symmetric, and transitive and therefore is a relation of equivalence and seems applicable for completeness proofs of our logics.

However, the first doubts are raised by the results that

$$a \equiv_5 b = 1 \qquad \Rightarrow \qquad a = b, \tag{2.4}$$

makes an ortholattice orthomodular [12] and that

$$a \equiv_0 b = 1 \qquad \Rightarrow \qquad a = b, \tag{2.5}$$

makes an ortholattice distributive [10].

A real confirmation of our doubts comes from considering mixed biimplications. All implications reduce to the classical one in a classical theory, so, not only $a \leftrightarrow_i b$ but also $(a \to_i b) \cap (b \to_j a)$, $i \neq j$ must reduce to $a \equiv_0 b$ in a classical theory. Let us have a look at what we get in an orthomodular lattice in Table 1, where $a \equiv_1 b \stackrel{\text{def}}{=} (a \cup b') \cap (a' \cup (a \cap b))$, $a \equiv_2 b \stackrel{\text{def}}{=} (a \cup b') \cap (b \cup (a' \cap b'))$, $a \equiv_3 b \stackrel{\text{def}}{=} (a' \cup b) \cap (a \cup (a' \cap b'))$ and $a \equiv_4 b \stackrel{\text{def}}{=} (a' \cup b) \cap (b' \cup (a \cap b))$. We omit the easy proof. We can also send the reader a computer program which reduces any two-variable orthomodular lattice expression to one of the 96 simplest possible ones as given in [13].

$i \downarrow \quad j \rightarrow$	$b \to_0 a$	$b \to_1 a$	$b \to_2 a$	$b \to_3 a$	$b \to_4 a$	$b \to_5 a$
$a \to_0 b$	$a \equiv_0 b$	$a \equiv_4 b$	$a \equiv_3 b$	$a \equiv_2 b$	$a \equiv_1 b$	$a \equiv_5 b$
$a \to_1 b$	$a \equiv_1 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_1 b$	$a \equiv_5 b$
$a \to_2 b$	$a \equiv_2 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_2 b$	$a \equiv_5 b$	$a \equiv_5 b$
$a \to_3 b$	$a \equiv_3 b$	$a \equiv_5 b$	$a \equiv_3 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$
$a \to_4 b$	$a \equiv_4 b$	$a \equiv_4 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$
$a \to_5 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$	$a \equiv_5 b$

Table 1: Products $(a \to_i b) \cap (b \to_j a)$, $i = 0, \dots, 5$ (rows), $j = 0, \dots, 5$ (columns). "Identities" $a \equiv_i b$, $i = 1, \dots, 4$ are asymmetrical.

Also, we are able to prove:

Lemma 2.1. *In any* OML *we have:*

$$a \equiv_i b = (a \to_i b) \cap (b \to_0 a) \qquad\qquad i = 0, \dots, 5. \tag{2.6}$$

This also holds in any OL *for* $i = 0, 1, 2$ *and in some* OL*s weaker than* OML *for* $i = 3, 4, 5$.

Proof. We omit the easy proof that Eq. (2.6) holds in any OML. For $i = 0, 1, 2$ that it holds in any OL is apparent from the definitions. For $i = 3, 4, 5$ it fails in the non-orthomodular ortholattice from Fig. 1[3] but does not fail either in O6 (Fig. 2) or in WOML, non-OML lattices from [13], Figs. 7b, 9f, 9h, and 11. ☐

[3]The authors would like to thank to William McCune, Argonne National Lab, Argonne IL, U. S. A. (`http://www.mcs.anl.gov/home/mccune/ar/ortholattice/`), for finding this lattice, using the matrix-finding program MACE.

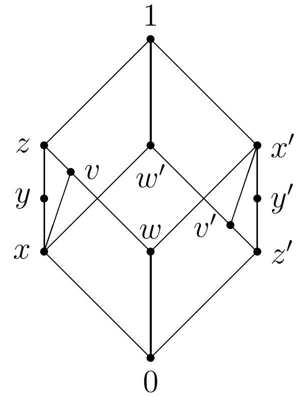

Figure 1: Ortholattice M12.

The expressions $a \equiv_i b, = 1, \ldots, 4$ are all asymmetrical and at first we would think it would be inappropriate to name them identities. But we are able to prove the following theorem.

Theorem 2.2. *An ortholattice in which*

$$a \equiv_i b = 1 \qquad \Rightarrow \qquad a = b, \qquad\qquad i = 1, \ldots, 4 \tag{2.7}$$

holds is an orthomodular lattice and vice versa.

Proof. We give here the proof only for $i = 1$. Others are completely analogous. Let us write the premise $a \equiv_1 b = 1$ as $(a \cup b') \cap (a \to_1 b) = 1$. Hence, $(a \to_1 b) = 1$ and according to [14] $a \le b$. This, together with the other consequence of the premise: $(a \cup b') = 1$, yields $b \le a$ [14], what proves the statement.

As for the vice versa part, all four implications fail in O6 which means that they must be orthomodular. □

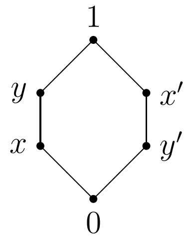

Figure 2: Ortholattice O6

Hence, putting together Eq. (2.4) and Eq. (2.7) we have an indication that the relation of equivalence which establishes a connection between quantum logic and its models might turn out to be based on several different operations of identity at the same time thus making a direct evaluation of elementary logical propositions impossible. In Sec. 3 we prove the conjecture. In Sec. 4 we analyze classical logic and show that although its syntactical structure can map into a weakly distributive lattice as a model, properties of the Boolean algebra as another model enable a consistent direct evaluation of elementary logical propositions.

3 Non-Orthomodular Model for Quantum Logic

A reader which is not at home with methods and parlance of mathematical logic can follow this section by reading logical expressions of the form $\vdash A$ as $a = 1$ in the lattice language. In doing so he or she will miss some features of a proper logic as, for example, that in a logic $a \wedge b$ and $b \wedge a$ are distinct formulas (they coincide in a lattice) but these features do not play an important role in our proofs. We are only interested in connecting the equivalence relations in our logic—which coincide with those in a lattice—with equations in lattices.

A crucial difference we find between quantum logic and orthomodular lattice as its standard model is that properties that play a decisive role in the lattice do not play such a role in the logics. This is in contrast with the properties of our new model, weakly orthomodular lattice, whose properties do correspond to those of the logic. To explain these differences let us consider the orthomodularity property. When we add the orthomodularity property to an ortholattice it becomes an orthomodular lattice. We can compare what happens in a logic by looking at a lattice we obtain by mapping logical axioms $\vdash A$ to an ortholattice where they take over the form $a = 1$; here $a = f(A)$ and f is a morphism from the logic to the lattice. As we have shown in [2] the property $(a \cup (a' \cap (a \cup b))) \equiv_5 (a \cup b) = 1$, we obtain by mapping the logical formula for "orthomodularity" $\vdash (A \vee (\neg A \wedge (A \vee B)) \equiv_5 (A \vee B)$ into an ortholattice, is true in all ortholattices. The reason for such different structures of logic as opposed to its standard model lies in the way we prove the completeness of the standard modeling. To understand this better we give both completeness proofs: in Subsection 3.3 for the standard model and in 3.4 for the new one.

We first consider a quantum logic ($\mathcal{QL}$) derived directly from the properties of a weakly orthomodular lattice WOML without taking the orthomodularity property into account. We do so in order to show that orthomodularity appears only at the stage of proving the completeness and as a property of equivalence classes we can define on a logic. $\mathcal{QL}$ is equivalent to the logics of other authors, e. g., Kalmbach's [8], Dishkant's [15] , Dalla Chiara's [16] , Mittelstaedt's [17], Stachow's [18], Hardegree's [19], Rüttimann's [20], etc. We proved explicitly the equivalence to Kalmbach's and Dishkant's systems in [2] but a general equivalence to all systems follows from our completeness proof given below.

3.1 Quantum Logic

Quantum logic $\mathcal{QL}$ contains the connectives $\rightarrow$, $\leftrightarrow$, $\equiv$, $\vee$, $\wedge$, and $\neg$ which we represent with their lattice counterparts: $\rightarrow$, $\leftrightarrow$, $\equiv$, $\cup$, $\cap$, and $'$. Let $\mathcal{F}^\circ$ be the set of all logical expressions, i.e., well formed formulas (wff). Of these $\vee$, $\neg$ and $\cup$, $'$ are primitive ones. The latter constitutes an algebra $\mathcal{F} = \langle \mathcal{F}^\circ, \neg, \vee, \wedge \rangle$. $\mathcal{QL}$ is given by the following axioms and rules of inference, representing five distinct but equivalent systems.

Axioms

$\mathcal{QL}$**1.** $\vdash A \vee B \equiv_i B \vee A$

$\mathcal{QL}$**2.** $\vdash A \vee (B \vee C) \equiv_i (A \vee B) \vee C$

$\mathcal{QL}$**3.** $\vdash A \equiv_i \neg\neg A$

$\mathcal{QL}$**4.** $\vdash \neg A \vee A \equiv_i (\neg A \vee A) \vee B$

$\mathcal{QL}$**5.** $\vdash A \vee (A \wedge B) \equiv_i A$

$\mathcal{QL}$**6.** $\vdash (A \wedge B) \equiv_i \neg(\neg A \vee \neg B)$

where $i = 1, \ldots, 5$ and will be considered to take over a specific value throughout.

Rules of Inference

$\mathcal{QL}$**R1.** $\vdash A \equiv_i B \quad \Rightarrow \quad \vdash A \vee C \equiv_i B \vee C$

$\mathcal{QL}$**R2.** $\vdash A \equiv_i B \quad \& \quad \vdash B \equiv_i C \quad \Rightarrow \quad \vdash A \equiv_i C$

$\mathcal{QL}$**R3.** $\vdash A \equiv_i B \quad \Leftrightarrow \quad \vdash \neg A \equiv_i \neg B$

$\mathcal{QL}$**R4.** $\vdash A \equiv_i B \quad \Rightarrow \quad \vdash B \equiv_i A$

$\mathcal{QL}$**R5.** $\vdash \neg A \vee A \equiv_i B \quad \Leftrightarrow \quad \vdash B$

Axioms $\mathcal{QL}$1-6 coincide with L1-6 of OL, and $\mathcal{QL}$R1 with L8 (in the form of L8.1 shown in Theorem 3.8 below).

Definition 3.1. *For $\Gamma \subseteq \mathcal{F}^\circ$ we say A is derivable from Γ and write $\Gamma \vdash A$ if there is a sequence of formulas ended by A each of which is either one of the axioms of $\mathcal{QL}$ or is a member of Γ or is obtained from its precursors with the help of a rule of inference of the logic.*

Definition 3.2. *We call $\mathcal{M} = \langle \mathcal{L}, f \rangle$ a model of a set of formulas Γ, if $\mathcal{L}$ is a lattice* (WOML *or* OML)*, $f : \mathcal{F}^\circ \longrightarrow \mathcal{L}$ is a morphism of algebra of wff's which satisfies $f(A) = 1$ for any $A \in \Gamma$; we call the latter A true in the model $\mathcal{M}$.*

3.2 Soundness Proof for Quantum Logic

Lemmas 3.3–3.7 provide some technical results for use in subsequent proofs.

Lemma 3.3. *In any* OL *we have:*

$$(a \cap b) \cup (a \cap c) \le a \cap (b \cup c) \tag{3.1}$$

$$a = 1 \qquad \& \qquad a \to_0 b = 1 \qquad \Rightarrow \qquad b = 1 \tag{3.2}$$

$$(a \equiv_5 b) \to_0 (a \leftrightarrow_1 b) = 1 \tag{3.3}$$

$$b \to_2 a = 1 \qquad \Rightarrow \qquad a \to_2 (a \equiv_5 b) = a \equiv_5 b \tag{3.4}$$

$$a \equiv_5 b = 1 \qquad \Rightarrow \qquad a \to_1 (b \cup c) = 1 \tag{3.5}$$

$$a \to_2 (b \cup c) = (a \cup c) \to_2 (b \cup c) \tag{3.6}$$

$$a \to_i (a \cap b) \;=\; a \equiv_i (a \cap b) \;=\; (a \cap b) \equiv_i a \;=\; a \to_1 b \qquad i = 0, \ldots, 5 \tag{3.7}$$

$$(a \cup b) \to_i b \;=\; (a \cup b) \equiv_i b \;=\; b \equiv_i (a \cup b) \;=\; a \to_2 b \qquad i = 0, \ldots, 5 \tag{3.8}$$

Proof. For (3.1): This is well known and we omit the proof. For (3.2): See [8, p. 237]. For (3.3): $a \equiv_5 b \le a \to_1 b$ and $\le b \to_1 a$, so $a \equiv_5 b \le (a \to_1 b) \cap (b \to_1 a)$; $1 = (a \equiv_5 b)' \cup (a \equiv_5 b) \le (a \equiv_5 b)' \cup ((a \to_1 b) \cap (b \to_1 a))$. For (3.4): From L5 and DeMorgan's law we have $a' \cap (a' \cup b') = a'$, so $a' \cap (a' \cup b') \cap (a \cup b) = a' \cap (a \cup b)$; from hypothesis and DeMorgan's we have $a' \cap (a \cup b) = 0$, so $a' \cap (a' \cup b') \cap (a \cup b) = 0$; from DeMorgan's we have $(a \equiv_5 b)' = (a' \cup b') \cap (a \cup b)$, so $a' \cap (a \equiv_5 b)' = 0$, so $(a \equiv_5 b) \cup (a' \cap (a \equiv_5 b)') = a \equiv_5 b$. For (3.5): From hypothesis and (3.3) and (3.2) we have $a \leftrightarrow_1 b = 1$, so $1 = a \leftrightarrow_1 b \le a \to_1 b = a' \cup (a \cap b) \le a' \cup (a \cap (b \cup c)) = a \to_1 (b \cup c)$. For (3.6): $a' \cap (b \cup c)' = (a' \cap b') \cap c' = (a' \cap c') \cap (b' \cap c') = (a \cup c)' \cap (b \cup c)'$, so $(b \cup c) \cup (a' \cap (b \cup c)') = (b \cup c) \cup ((a \cup c)' \cap (b \cup c)')$. For (3.7) and (3.8): We omit the easy verifications. □

Lemma 3.4. *In any* WOML *we have:*

$$(a \to_1 b) \to_0 (a \to_2 b) = 1 \tag{3.9}$$

$$a \to_1 b = 1 \qquad \Leftrightarrow \qquad a \to_2 b = 1 \tag{3.10}$$

$$a \to_2 b = 1 \qquad \Rightarrow \qquad a \to_2 (a \equiv_5 b) = 1 \tag{3.11}$$

$$a \to_2 b = 1 \qquad \& \qquad b \to_2 a = 1 \qquad \Rightarrow \qquad a \equiv_5 b = 1 \tag{3.12}$$

$$a \equiv_5 b = 1 \qquad \Rightarrow \qquad a \to_2 (b \cup c) = 1 \tag{3.13}$$

$$a \equiv_5 b = 1 \qquad \Rightarrow \qquad (a \cup c) \equiv_5 (b \cup c) = 1 \tag{3.14}$$

Proof. For (3.9): Immediate from L8, L1, L3 and definitions. For (3.10): Immediate from L8 and (3.9), using (3.2). For (3.11): Using (3.1), $a \cap b = a \cap (a \cap b) \le (a \cap (a \cap b)) \cup (a \cap (a' \cap b')) \le a \cap ((a \cap b) \cup (a' \cap b')) = a \cap (a \equiv_5 b)$; so $a \to_1 b = a' \cup (a \cap b) \le a' \cup (a \cap (a \equiv_5 b)) = a \to_1 (a \equiv_5 b)$; so from hypothesis and (3.10) we have $1 = a \to_1 b \le a \to_1 (a \equiv_5 b)$; so from (3.10) we conclude $1 = a \to_2 (a \equiv_5 b)$. For (3.12): Immediate from (3.4) and (3.11). For (3.13): Immediate from (3.5) and (3.10). For (3.14): From (3.13) we have $a \to_2 (b \cup c) = 1$ and $b \to_2 (a \cup c) = 1$; so from (3.6) we have $(a \cup c) \to_2 (b \cup c) = 1$ and $(b \cup c) \to_2 (a \cup c) = 1$; so from (3.12) we have $(a \cup c) \equiv_5 (b \cup c) = 1$. □

Lemma 3.5. *Let t be any term (such as $a \cup a'$). If the equation $t = 1$ holds in all* OML*s, then $t = 1$ holds in* OL + (3.14)*.*

Proof. Theorem 2.15 in [2] and the remark after Theorem 2.12 in [2], which applies to any OL in which (3.14) holds. □

Lemma 3.6. *(a) An ortholattice in which* (3.14) *holds is a* WOML *and vice versa. (b) An ortholattice in which either direction of* (3.10) *holds is a* WOML *and vice versa.*

Proof. (a) We have shown that (3.14) holds in a WOML. On the other hand, it is easy to prove (using the Foulis-Holland theorem for example) that $(a' \cap (a \cup b)) \cup b' \cup (a \cap b) = 1$ holds in an OML; thus by Lemma 3.5 it also holds in OL + (3.14). In other words, the WOML we have defined here is equivalent to the WOML of [2], and L8 and (3.14) are interchangeable as the WOM law added to an OL. (b) It is easy to prove either direction of (3.10) from the other using only L1–L6. In the proof of (3.14), we used only (3.10) along with L1–L6. Thus in an OL, (3.10) follows from L8, and (3.14) follows from (3.10). □

Lemma 3.7. *Let $t_1, ..., t_n, t$ be any terms ($n \geq 0$). If the inference $t_1 = 1$ & ... & $t_n = 1 \;\Rightarrow\; t = 1$ holds in all* OML*s, then it holds in any* WOML.

Proof. We extend the proof of Theorem 2.12 of [2] using the completeness proof for unary quantum logic (e.g. [8, p. 238]) where $t_1 = 1, \ \ldots, \ t_n = 1$ are the ortholattice mappings for the hypotheses of a deduction. □

Theorem 3.8. WOML *given as* L1–L6 + L8 [*which can also be written as* $(a \to_1 b) \to_0 (a \to_2 b) = 1$] *is an* OL *to which the following mapping of* $\mathcal{QL}$R1

$$\text{L8.1.} \qquad a \equiv_i b = 1 \qquad \Rightarrow \qquad a \cup c \equiv_i b \cup c = 1 \qquad i = 1, \ldots, 5$$

is added and vice versa.

Proof. Since $a = b$ implies $a \cup c = b \cup c$, by Theorem 2.2 and (2.4) we have that $a \equiv_i b = 1$ implies $(a \cup c) \equiv_i (b \cup c) = 1$ in any OML. By Lemma 3.7 this also holds in any WOML. On the other hand, assume L8.1 holds. If $a \to_1 b = 1$ then $a \equiv_i (a \cap b) = 1$ by (3.7), so $(a \cup b) \equiv_i ((a \cap b) \cup b) = 1$ by L8.1, so $(a \cup b) \equiv_i b = 1$, so $a \to_2 b = 1$ by (3.8), so L8 holds by Lemma 3.6b. □

Let us also prove the following theorem which we shall use later on.

Theorem 3.9. WOML *is an* OL *to which either of the following properties is added:*

$$\text{L8.2.} \qquad a \to_1 b = 1 \qquad \Rightarrow \qquad b' \to_1 a' = 1$$

$$\text{L8.3.} \qquad ((a \to_1 b) \to_0 b) \;\equiv_5\; (a \cup b) = 1$$

and vice versa.

Proof. For L8.2: Immediate from Lemma 3.6b. For L8.3: We have, using L1–L6, $(a \to_1 b) \to_0 b = (a \cap (a' \cup b')) \cup b \le a \cup b$, so $((a \to_1 b) \to_0 b) \cap (a \cup b) = (a \to_1 b) \to_0 b$ and $((a \to_1 b) \to_0 b)' \cap (a \cup b)' = (a \cup b)'$. Hence $((a \to_1 b) \to_0 b) \equiv_5 (a \cup b) = ((a \to_1 b) \to_0 b) \cup (a \cup b)' = (a \cap (a' \cup b')) \cup b \cup (a' \cap b')$, which becomes the left-hand side of L8 after substituting a' for a and b' for b then applying L3. □

We now prove the soundness of modeling quantum logic by a weakly orthomodular lattice.

Theorem 3.10. *(Soundness) If* $\Gamma \vdash A$*, then* A *is true in any* WOML *model.*

Proof. Any axiom $\mathcal{QL}1$—$\mathcal{QL}6$ is true in any model WOML. Let us put $a = f(A)$ and $b = f(B)$ and let us verify for example $\mathcal{QL}1$ for $i = 5$. It maps to $((a \cup b) \cap (b \cup a)) \cup ((a' \cap b') \cap (b' \cap a')) = 1$. By L1, L2, L4, and L8 we get $(a \cup b) \cup (a \cup b)' = 1$ which is true by definition. $\mathcal{QL}2$—$\mathcal{QL}6$ we prove analogously. We also have to verify that the set of formulas true in a model $\mathcal{M}$ is closed under the rules of inference: $\mathcal{QL}$R1-5. $\mathcal{QL}$R1 maps to L8.1. $\mathcal{QL}$R2 maps to $a \equiv_i b = 1 \ \&\ b \equiv_i c = 1 \ \Rightarrow\ a \equiv_i c = 1$ which according to Lemma 3.7, Theorem 2.2 and (2.4) holds in any WOML since $a = b \ \&\ b = c \ \Rightarrow a = c$ holds in any OML. $\mathcal{QL}$R3-5 mappings we verify analogously. □

3.3 Standard Completeness Proof for Quantum Logic

Let us now see how a standard completeness of modeling $\mathcal{QL}$ can be proved to see where the orthomodularity in such a proof emerges from. First we have to check whether $\equiv_i$ defines a relation of equivalence. That the relation is symmetric, for $\equiv_5$ it follows from $\mathcal{QL}$R4. The other four identities are themselves asymmetric but the symmetry nevertheless holds for the equivalence relation since that symmetry is metaimplicational—not equational. For example, for $\equiv_1$ we prove it as follows. From $\vdash A \equiv_1 B$ by $\mathcal{QL}$R3 we get $\vdash (\neg A \vee B) \wedge (A \vee (\neg A \wedge \neg B))$ wherefrom by a $\mathcal{QL}$ equivalent of L8.2 we get the required result. Note that the transitivity of $\mathcal{QL}$R2 when mapped to a lattice: $a \equiv_i b = 1 \ \&\ b \equiv_i c = 1 \ \Rightarrow\ a \equiv_i c = 1$ fail in ortholattices shown in Fig. 3. It therefore does not hold in all ortholattices as does the relational transitivity $a = b \ \&\ b = c \ \Rightarrow\ a = c$, but requires L8.

Now we can prove the following lemmas and introduce a definition.

Lemma 3.11. *Relation* $\approx$ *defined as*

$$A \approx B \ \stackrel{\text{def}}{=}\ \Gamma \vdash A \equiv_i B \tag{3.15}$$

is a relation of congruence in the algebra $\mathcal{F}$*.*

Proof. As we have shown above, $\approx$ is an equivalence relation. In order to be a relation of congruence, the relation of equivalence must be compatible with the operations $\neg$ and $\vee$. $\vdash A \equiv_i B \ \Rightarrow \vdash \neg A \equiv_i \neg B$ is nothing but $\mathcal{QL}$R3 and $\vdash A \equiv_i B \ \Rightarrow \vdash (A \vee C) \equiv_i (B \vee C)$ is $\mathcal{QL}$R1. □

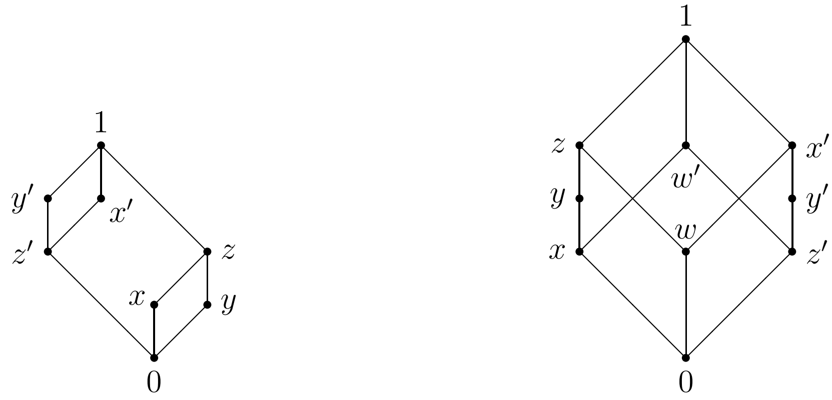

Figure 3: (a) Ortholattice from [13], Fig. 9g; (b) Ortholattice from [2], Fig. 3.

Definition 3.12. *The equivalence class under the relation of equivalence is defined as* $|A| = \{B \in \mathcal{F}^\circ : A \approx B\}$ *and we denote* $\mathcal{F}^\circ/\approx\, = \{|A| \in \mathcal{F}^\circ\}$ *The equivalence classes define the natural morphism* $f : \mathcal{F}^\circ \longrightarrow \mathcal{F}^\circ/\approx$ *which gives* $f(A) \stackrel{\mathrm{def}}{=} |A|$. *We write* $a = f(A)$, $b = f(A)$, *etc.*

Corrolary 3.13. *The relation* $a = b$ *on* $\mathcal{F}^\circ/\approx$ *is given as:*

$$|A| = |B| \qquad \Leftrightarrow \qquad \Gamma \vdash A \equiv_i B. \tag{3.16}$$

Lemma 3.14. *The Lindenbaum algebra* $\mathcal{A} = \langle \mathcal{F}^\circ/\approx, \neg/\approx, \vee/\approx\rangle, \wedge/\approx\rangle$ *is a* WOML, *i.e.,* L1–L6 *and* L8 *hold for* $\neg/\approx$, $\vee/\approx$, *and* $\wedge/\approx$ *as* $'$, $\cup$, *and* $\cap$, *respectively.*

Proof. That L1–L6 hold in $\mathcal{A}$ is obvious. $\mathcal{QL}$R1 gives L8.1 which together with L1–L6 gives L8 according to Theorem 3.8. ☐

We see that, as we already stressed above, to prove the orthomodularity from $\mathcal{QL}$ in the Lindenbaum algebra the latter need not be an orthomodular lattice. WOML suffices. Previously we proved that the "orthomodularity" in $\mathcal{QL}$ is given as: $\vdash A\vee(\neg A\wedge(A\vee B)) \equiv_5 A\vee B$ is no more than an "ortho-property" (i.e., its lattice mapping holds in any ortholattice). Another way of expressing orthomodularity is $\vdash A \vee (B \wedge (\neg A \vee \neg B)) \equiv_5 A \vee B$ whose lattice mapping is nothing but L8.3, i.e., it is a WOM property itself. This means that the "orthomodularity" from $\mathcal{QL}$ sometimes maps to an ortho-property and sometimes to a weakly orthomodular property but never to a proper orthomodular lattice property. The following theorem explains the peculiarity.

Theorem 3.15. *The orthomodularity lattice property L7 holds in* $\mathcal{A}$ *as a consequence of the way we define the equivalence relation in Lemma 3.11. Hence,* $\mathcal{A}$ *is also an* OML.

Proof. The theorem is a direct consequence of Theorem 2.2 and rule $\mathcal{QL}$R5. ☐

As we see the orthomodularity follows from Lemma 3.11 for five different operations of identity $\equiv_i$, $i = 1, \ldots, 5$ for which in an ortholattice Theorem 2.2 holds. The previous theorem is a consequence of the very definition of the relation = between the equivalence classes given by the Definition 3.12 as the following lemma shows.

Lemma 3.16. *There is no operation of identity* $\equiv_6$ *for which* $\mathcal{QL}$1-6 *and* $\mathcal{QL}$R1-5 *would hold, whose lattice mapping would satisfy Eq.* (2.7) *in* WOML *and for which the orthomodularity property would not be satisfied in* $\mathcal{A}$.

Proof. Let $\mathcal{QL}$*1-6* and $\mathcal{QL}$*R1-5* hold for $\equiv_6$. We form the Lindenbaum algebra $\mathcal{A}$ for this logic using $A \approx B \stackrel{\text{def}}{=} \Gamma \vdash A \equiv_6 B$ and Lemma 3.14 formulated for this $A \approx B$. Let us further assume that the so obtained $\mathcal{A}$ is not orthomodular. But by Lemma 3.13 from $\vdash A \vee (\neg A \wedge (A \vee B)) \equiv_6 A \vee B$ which must hold in such $\mathcal{QL}$ we obtain the orthomodularity, i.e., the contradiction. □

The remaining lemmas and theorems we adopt from [15]. The first two lemmas are obvious and we omit the proofs.

Lemma 3.17. $$\Gamma \vdash A \equiv_i B \qquad \Leftrightarrow \qquad f(A) = f(B)$$

Lemma 3.18. $\mathcal{M} \stackrel{\text{def}}{=} \langle \mathcal{F}^\circ/\approx, f\rangle$ *is a model of* Γ.

Lemma 3.19. $$f(A) = 1 \qquad \Rightarrow \qquad \Gamma \vdash A$$

Proof. Since $f(\neg B \vee B) = 1$, the premise $f(A) = 1$ yields $A \equiv_i \neg B \vee B$ by Lemma 3.17, wherefrom by $\mathcal{QL}$R5 we get $\Gamma \vdash A$. □

Thus for the five above defined operations of identities—and through them above defined equality between equivalence classes—we obtain:

Theorem 3.20. *(Completeness) If a formula* A *is true in all OML models of a set of wff's* Γ*, i.e., if* $f(A) = 1$*, then* $\Gamma \vdash A$.

Proof. Since $\mathcal{M}$ is a model of Γ, to be true for A in $\mathcal{M}$ means: $f(A) = 1$. Hence, by the previous lemma, we get: $\Gamma \vdash A$. □

The soundness of quantum logic given by Theorem 3.10 must be valid for OML as soon as it is valid for WOML. Thus we obtain:

Theorem 3.21. $\Gamma \vdash A$ *iff* A *is true in all* OML *models.*

These theorems show that in the syntactical structure of quantum logic there is nothing orthomodular. The orthomodularity appears through the definition of the equivalence relation. By defining it in the standard way as above, we, in effect, introduce an additional axiom in the lattice structure of the equivalence classes as the Theorem 2.2 shows. In Sec. 4 we show that one obtains an analogous result for classical logic. The only difference will be a possibility of {0,1} evaluation of every proposition which is not possible in quantum logic.

3.4 Non-Orthomodular Completeness Proof for Quantum Logic

As we have seen in the previous subsection the orthomodularity in the standard completeness proof of quantum logic emerges from nothing else but the very definition of the relation of equivalence defined on $\mathcal{QL}$. Therefore, if we were to stay with WOML (which served us to prove soundness) in a completeness proof for $\mathcal{QL}$ as well, we should change the definition of $A \approx B$. What we do not want in the same equivalence classes are those A and B whose lattice equality $f(A) = f(B)$ would make an ortholattice orthomodular when added to it. And this is exactly what O6 lattice offers us. Any such equality fails in it and any WOML expression holds in it.

Definition 3.22. *Letting* O6 *represent the lattice shown in Fig. 2, we define* $\mathcal{O}6$ *as the set of all mappings* $o : \mathcal{F}^\circ \longrightarrow \mathrm{O6}$ *such that for* $A, B \in \mathcal{F}^\circ$, $o(\neg A) = o(A)'$ *and* $o(A \vee B) = o(A) \cup o(B)$.

Lemma 3.23. *Relation* $\approx$ *defined as*

$$A \approx B \overset{\text{def}}{=} \Gamma \vdash A \equiv_i B \,\&\, (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \Rightarrow o(A) = o(B)], \tag{3.17}$$

where $i = 1, \ldots, 5$, *is a relation of congruence in the algebra* $\mathcal{F}$.

Proof. Let us first prove that $\approx$ is an equivalence relation. $A \approx A$ and $A \approx B \ \Rightarrow\ B \approx A$ are obvious. The proof of the transitivity runs as follows.

$$A \approx B \quad \& \quad B \approx C \tag{3.18}$$

$$\begin{aligned} \Rightarrow\ \Gamma \vdash A \equiv_i B \quad & \& \quad \Gamma \vdash B \equiv_i C \\ & \& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) = o(B)] \\ & \& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(B) = o(C)] \end{aligned} \tag{3.19}$$

$$\begin{aligned} \Rightarrow\ \Gamma \vdash A \equiv_i C \quad & \& \\ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ & \Rightarrow\ o(A) = o(B) \,\&\, o(B) = o(C)]. \end{aligned} \tag{3.20}$$

Since all the WOML axioms and rules hold in O6, the last metaconjunction in line 3.20 reduces to $o(A) = o(C)$ by transitivity. Hence the conclusion $A \approx C$ by definition.

In order to be a relation of congruence, the relation of equivalence must be compatible with the operations $\neg$ and $\vee$. The proofs of the compatibilities run as follows.

$$A \approx B \tag{3.21}$$

$$\Rightarrow\ \Gamma \vdash A \equiv_i B \quad \& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) = o(B)] \tag{3.22}$$

$$\Rightarrow\ \Gamma \vdash \neg A \equiv_i \neg B \,\&\, (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A)' = o(B)'] \tag{3.23}$$

$$\Rightarrow\ \Gamma \vdash \neg A \equiv_i \neg B \,\&\, (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(\neg A) = o(\neg B)] \tag{3.24}$$

$$\Rightarrow\ \neg A \approx \neg B \tag{3.25}$$

$$A \approx B \tag{3.26}$$

$$\Rightarrow \quad \Gamma \vdash A \equiv_i B \quad \& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow \ o(A) = o(B)] \tag{3.27}$$

$$\Rightarrow \quad \Gamma \vdash (A \vee C) \equiv_i (B \vee C)$$

$$\& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow \ o(A) \cup o(C) = o(B) \cup o(C)] \tag{3.28}$$

$$\Rightarrow \quad (A \vee C) \approx (B \vee C) \tag{3.29}$$

In these proofs we used $\mathcal{QL}$R3 and $\mathcal{QL}$R1 and the corresponding lattice mappings in O6. □

Definition 3.24. *The equivalence class under the relation of equivalence is defined as* $|A| = \{B \in \mathcal{F}^\circ : A \approx B\}$ *and we denote* $\mathcal{F}^\circ/\approx\ = \{|A| \in \mathcal{F}^\circ\}$ *The equivalence classes define the natural morphism* $f : \mathcal{F}^\circ \longrightarrow \mathcal{F}^\circ/\approx$ *which gives* $f(A) \stackrel{\rm def}{=} |A|$. *We write* $a = f(A)$, $b = f(A)$, *etc.*

Corrolary 3.25. *The relation* $a = b$ *on* $\mathcal{F}^\circ/\approx$ *is given as:*

$$|A| = |B| \qquad \Leftrightarrow \qquad A \approx B \tag{3.30}$$

Lemma 3.26. *The Lindenbaum algebra* $\mathcal{A} = \langle \mathcal{F}^\circ/\approx, \neg/\approx, \vee/\approx \rangle, \wedge/\approx\rangle$ *is a* WOML, *i.e.,* L1–L6 *and* L8.2 *hold for* $\neg/\approx$, $\vee/\approx$, *and* $\wedge/\approx$ *as* $'$, $\cup$, *and* $\cap$, *respectively* [where—for simplicity—we use the same symbols ($'$ and $\cup$) as for O6 since in the paper there are no ambiguous expressions in which the origin of the operations would not be clear from the context].

Proof. Since all the WOML axioms and rules hold in O6 the proof follows from the proof of Lemma 3.14. □

Theorem 3.27. *The orthomodularity lattice property L7 does not hold in* $\mathcal{A}$.

Proof. We assume $\mathcal{F}^\circ$ contains at least two propositional variables (or "primitive" or "starting" wffs). We pick an evaluation o that maps two of them, A and B, to distinct nodes $o(A)$ and $o(B)$ of O6 that are neither 0 nor 1 such that $o(A) \le o(B)$ [i.e. $o(A)$ and $o(B)$ are on the same side of hexagon O6 in Fig. 2]. From the structure of O6 we obtain $o(A) \cup o(B) = o(B)$ and $o(A) \cup (o(A)' \cap (o(A) \cup o(B))) = o(A) \cup (o(A)' \cap o(B)) = o(A) \cup 0 = o(A)$. Therefore $o(A) \cup o(B) \ne o(A) \cup (o(A)' \cap (o(A) \cup o(B))$, i.e., $o(A \vee B) \ne o(A \vee (\neg A \wedge (A \vee B)))$. This falsifies $(A \vee B) \approx (A \vee (\neg A \wedge (A \vee B))$. Therefore $a \cup b \ne a \cup (a' \cap (a \cup b))$, providing a counterexample to the OM law for $\mathcal{F}^\circ/\approx$. □

Let us now reformulate the remaining lemmas and theorems from the previous subsection.

Lemma 3.28. $\langle \mathcal{F}^\circ/\approx, f\rangle$ *is a* WOML *model of* Γ.

Theorem 3.29. *(Completeness) If a formula* A *is true in all* WOML *models of a set of wff's* Γ, *i.e., if* $f(A) = 1$, *then* $\Gamma \vdash A$.

Proof. $f(A) = 1$ is equivalent to $|A| = |B \vee \neg B|$ and therefore to

$$
\begin{aligned}
& \Gamma \vdash A \equiv_i B \vee \neg B \quad \& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \;\Rightarrow\; o(A) = 1] && (3.31)\\
\Leftrightarrow \;& \Gamma \vdash A \quad \& \quad (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \;\Rightarrow\; o(A) = 1] && (3.32)\\
\Rightarrow \;& \Gamma \vdash A && (3.33)
\end{aligned}
$$

□

Theorem 3.30. *$\Gamma \vdash A$ iff A is true in all* WOML *models.*

Proof. Right to left metaimplication in the line 3.33 of Theorem 3.29 holds because all deductions of QL are sound in WOML, and O6 is a WOML. □

4 Non-Distributive Model for Classical Logic

As in the previous section, a reader which is not at home with methods and parlance of mathematical logic can follow this section by reading logical expressions of the form $\vdash A$ as $a = 1$ in the lattice language.

A difference we find between classical logic and the Boolean algebra (distributive lattice) as its standard model is that properties that play a decisive role in the lattice do not play such a role in the logic. And again, this is in contrast to the new model, weakly distributive lattice which is even non-orthomodular. To explain the difference let us consider the distributivity property. When we add the distributivity property to an ortholattice it becomes distributive. As in the previous section to see what then happens in a logic we mimic logical axioms $\vdash A$ by their lattice form $a = 1$; here $a = g(A)$ and g is a morphism from the logic to the lattice. Thus $(a \wedge (b \vee c)) \equiv_0 ((a \wedge b) \vee (a \wedge c)) = 1$ which we obtain by a mapping of the distributivity is true in all weakly distributive lattices which are not even orthomodular. However, the lattice distributivity $(a \wedge (b \vee c)) = ((a \wedge b) \vee (a \wedge c))$ is true only in a distributive lattice, not in a weakly distributive one. To understand this difference better we briefly review a completeness proof for the standard model in Subsection 4.2 and subsequently for the new one in Subsection 4.

4.1 Classical Logic

Classical logic $\mathcal{CL}$ contains the connectives $\rightarrow$, $\leftrightarrow$, $\equiv$, $\vee$, $\wedge$, and $\neg$ which we represent with their lattice counterparts: $\rightarrow$, $\leftrightarrow$, $\equiv$, $\cup$, $\cap$, and $'$. When we omit parentheses, we assume these connectives bind from weakest to strongest in this order. We also represent logical formulas, wff's, A by means of a lattice expression $a = 1$ where necessary. Let $\mathcal{G}^\circ$ be the set of all logical expressions, i.e., well formed formulas (wff). The latter constitutes an algebra $\mathcal{G} = \langle \mathcal{G}^\circ, \neg, \vee \rangle$.

We make use of the PM classical logical system $\mathcal{CL}$ [Whitehead and Russell's *Principia Mathematica* axiomatization in the Hilbert and Ackermann's presentation [21] (without the associativity axiom which P. Bernays proved redundant) but in the schemata form so that we dispense with their rule of substitution] where $A \to_0 B \overset{\text{def}}{=} \neg A \vee B$.

Axioms

$\mathcal{CL}$**1.** $\vdash A \vee A \to_0 A$

$\mathcal{CL}$**2.** $\vdash A \to_0 A \vee B$

$\mathcal{CL}$**3.** $\vdash A \vee B \to_0 B \vee A$

$\mathcal{CL}$**4.** $\vdash (A \to_0 B) \to_0 (C \vee A \to_0 C \vee B)$

Rule of Inference—*Modus ponens*

$\mathcal{CL}$**R1.** $\vdash A \quad \& \quad \vdash A \to_0 B \quad \Rightarrow \quad \vdash B$

Definition 4.1. *For $\Delta \subseteq \mathcal{G}^\circ$ we say A is derivable from Δ and write $\Delta \vdash A$ if there is a sequence of formulas ended by A each of which is either one of the axioms of $\mathcal{CL}$ or is a member of Δ or is obtained from its precursors with the help of a rule of inference of the logic.*

Definition 4.2. *We call $\mathcal{N} = \langle \mathcal{L}, h \rangle$ a model of a set of formulas Δ, if $\mathcal{L}$ is a lattice* (WOML *or* OML) *$g : \mathcal{G}^\circ \longrightarrow \mathcal{L}$ is a morphism of algebra of wff's which satisfies $g(A) = 1$ for any $A \in \Delta$; we call the latter A true in the model $\mathcal{N}$.*

4.2 Standard Soundness and Completeness Proof for Classical Logic

The following theorem holds in $\mathcal{CL}$:

$$\mathcal{CL}\mathbf{5}. \qquad \vdash A \vee (B \wedge C) \equiv_i (A \vee B) \wedge (A \vee C) \tag{4.1}$$

where $i = 0, \ldots, 5$.

The theorem is usually called a distributivity law. However, when its lattice mapping

$$\mathbf{L10}. \qquad a \cup (b \cap c) \equiv_i (a \cup b) \cap (a \cup c) = 1 \tag{4.2}$$

is added to an ortholattice it does not make the ortholattice even orthomodular: it does not fail in O6. We call this property a weakly distributive one and a weakly orthomodular lattice to which the property is added a weakly distributive lattice, WDL.

We see that, as with the orthomodularity in quantum logic, in the syntactical structure of classical logic there is nothing distributive. The distributivity will appear as a result of the way the relation of equivalence is usually defined in a proof of the completeness of classical logic. To better see this we shell first try to make $\mathcal{CL}$ complete by using the equivalence relation given in Lemma 3.11 instead of the usually used one. (Note that former reduces to the latter in a distributive algebra.) In particular, we are going to check whether a conjecture we disproved in Theorem 3.16 for WOML, would perhaps work for WDL.

It is easy to verify that in $\mathcal{CL}$1-4 and $\mathcal{CL}$R1 all expressions of the form $\vdash A$ can be written as $\vdash A \equiv_i B \vee \neg B$. So, we can repeat the procedure from the previous section and obtain the following theorem.

Theorem 4.3. *(Soundness) If* $\Delta \vdash A$*, then* A *is true in any* WDL *model.*

The critical point is definition of the equivalence relation for the completeness proof. Standard completeness procedure introduces it as follows.

Definition 4.4. *The equivalence relation on* $\mathcal{G}$ *is defined as:*

$$A \approx B \qquad \Leftrightarrow \qquad \vdash A \equiv_i B, \qquad\qquad i = 1, \ldots, 5. \tag{4.3}$$

The equivalence class under the relation of equivalence is defined as $|A| = \{B \in \mathcal{G} : A \approx B\}$ *and we denote* $\mathcal{G}/\approx\, = \{|A| \in \mathcal{G}\}$.

Only from $\mathcal{CL}$1–4 and $\mathcal{CL}$R1 we obtain:

Lemma 4.5. *The Lindenbaum algebra* $\mathcal{B} = \langle \mathcal{G}/\approx, \neg/\approx, \vee/\approx\rangle, \wedge/\approx\rangle$ *is* at least *a* WDL*, i.e.,* L1–L6 *and* L10 *hold for* $\neg/\approx$*,* $\vee/\approx$*, and* $\wedge/\approx$ *as* $'$*,* $\cup$*,and* $\cap$*, and respectively.*

However, as we have shown in Subsection 3.3, the very definition of the equivalence relation makes the Lindenbaum algebra orthomodular so that we are able to prove the following theorem.

Theorem 4.6. *An* OML *to which* L10 *is added is a distributive lattice.*

Proof. In $\mathcal{CL}$ $\vdash A \equiv_0 B$ is equivalent to $\vdash A \equiv_i B$. Therefore in WDL $a \equiv_0 b = 1$ is equivalent to $a \equiv_i b = 1$. Therefore, since Eq. (2.4) holds, Eq. (2.5) gives the required result. □

Thus we end up with:

Theorem 4.7. $\Delta \vdash A$ *iff* A *is true in all DL models.*

Had we used the following usual definition,

Definition 4.8. *The equivalence relation on* $\mathcal{G}$ *is defined as:*

$$A \approx B \qquad \Leftrightarrow \qquad \vdash A \equiv_0 B, \tag{4.4}$$

it would make the Lindenbaum algebra $\mathcal{B}$ distributive directly by Eq. (2.5).

4.3 Non-Distributive Completeness Proof for Classical Logic

Thus, we need an equivalence relation which does not introduce orthomodularity to WDL. The following one serves the purpose.

Lemma 4.9. *Relation $\approx$ defined as*

$$A \approx B \stackrel{\text{def}}{=} \Delta \vdash A \equiv_0 B \ \& \ (\forall o \in \mathcal{O}6)[(\forall X \in \Delta)(o(X) = 1) \Rightarrow o(A) = o(B)], \tag{4.5}$$

is a relation of congruence in the algebra $\mathcal{G}$.

Proof. The proof actually follows from the proof of Lemma 3.23. We only have to prove that the rules $\vdash A \equiv_0 B \ \Rightarrow \vdash \neg A \equiv_0 \neg B$ and $\vdash A \equiv_0 B \ \Rightarrow \vdash A \vee C \equiv_0 B \vee C$ do hold in $\mathcal{CL}$. But this is well known. (E.g., rules *29 and *30 on p. 116, §26 [22].) ☐

As a direct consequence of the Theorem 3.27 we obtain

Theorem 4.10. *The Lindenbaum algebra $\mathcal{B}$ is not orthomodular and therefore not distributive.*

Hence we obtain:

Theorem 4.11. $\Delta \vdash A$ *iff* A *is true in all WDL models.*

5 Conclusion

In Sec. 3 we show that there are two non-isomorphic models of the propositional calculus of quantum logic: an orthomodular lattice and a weakly orthomodular lattice. In Sec. 4 we show that there are two non-isomorphic models of the propositional calculus of classical logic: a distributive lattice (Boolean algebra) and a weakly distributive lattice. Hence, both calculuses are non-categorical and neither of them maps its syntactical structure to both its models. They do to one of the models and do not to the other. Surprisingly the models which do preserve the syntactical structure of the logics are not the standard ones—Boolean algebra and the orthomodular lattice—but the other ones—weakly distributive and weakly orthomodular lattice. This immediately raises fundamental questions: How come no one realized syntactical discrepancy between the logics and their standard models so far? Why has the usage of classical logic in mathematical and scientific applications not shown contradictions? What are the repercussions for computations and computers? ...

As for classical logic one can answer these questions as follows: First, very many applications have not used the logic itself but its model instead—Boolean algebra. Secondly, the usual two-valued logic does have only one model: the two-element Boolean algebra—and the usual many-valued classical logic also admits only Boolean algebra as its model. The former claim one can easily check by means of the truth tables: both Eq. (2.2) and Eq. (2.5) hold. The latter claim can be checked in the same way, using., e.g., Łukasiewicz's three- and many-valued logic [23] or Post's m-valued logic [24]. It is therefore possible that a numerical valuation of classical logic always implies that Boolean algebra can be the only model. In that case Eq. (2.2) would just reflect the ordering of valuation. O6 which is a weakly distributive model for classical logic cannot be numerically valuated: its left and its right nodes are not comparable, they are non-archimedean. Hence, the main aspect of our result is that the syntactical structure of classical logic corresponds to (maps to) the structure of the weakly distributive lattice not the one of the Boolean algebras. The result does not affect our usage of the models based on numerical valuation of classical logic but opens a possibility of using non-ordered lattice models which would in turn faithfully reflect the syntax of the logic.

With quantum logic it is just the opposite—yes-no values cannot be ascribed to all quantum propositions due to the Kochen-Specker theorem. [4] It is true, most applications of quantum logic also have not used the logic itself but its orthomodular model instead. Actually, what is usually called *quantum logic* in the mathematical physics literature is not the very logic but its orthomodular model: an orthomodular lattice itself or together with states defined on it. [25] This is because one straightforwardly arrives at a Hilbert space representation of quantum logic propositions by using its orthomodular model. [26] On the other hand, what is called quantum logic in the quantum computation literature is an algebra of qubits (quantum bits, two dimensional Hilbert space pure quantum systems) determined by quantum logic gates and particular algorithms (e.g., Shor's or Grover's). [27] However, a possible quantum logic of instructions for manipulating arbitrary qubits in general quantum computers (quantum simulators) can—in the absence of numerical valuation of elementary propositions—rely only on a syntactical structure of the qubits. "Quantum computers require quantum logic, something fundamentally different to classical Boolean logic." [5]

Whether the "required" logic is the quantum logic proper (we considered above) or one of its two models, requires further investigation but certainly none of them suffices for a complete logic of qubits or for modeling the Hilbert space. For, a necessary ingredient of the latter logic is the superposition principle which is a property of the second order. Also one should define a probability measure on an orthomodular lattice as well as a unitary map and assume infinite-dimensionality if one wanted a Hilbert space description of qubits. It is a question whether one can simulate infinite-dimensionality by means of quantum logic gates of a quantum computer. Therefore in our newest work [28] we investigate further stronger than weakly orthomodular (WOM) conditions, on the one hand, and stronger than orthomodular (OM) ones, on the other, which Hilbertian lattices should satisfy. In particular, we consider generalizations of the so-called orthoarguesian (OA) property which when added to WOM lattices (WOMLs) and OM lattices OMLs) make them rich enough for definitions of a superposition property. A WOM OA lattice is then still not orthomodular (does not fail in O6) although its OA condition fails in all other non-OA Greechie lattices. (We use computer programs which do automated testing of most Greechie lattices with up to 14 blocks and beyond.)

On the other hand, we will investigate whether one could use a finite-dimensional Hilbert space based on our OALs for qubits. A finite-dimensional Hilbert space allows nonstandard non-archimedean Keller fields in addition to the standard (real, complex, and quaternionic) ones. This could open a possibility for a direct usage of WOMOALs in the qubit logic.

Acknowledgment

M. P. acknowledges a support of the Ministry of Science of Croatia.

References

[1] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).

[2] M. Pavičić and N. D. Megill, Helv. Phys. Acta **71**, 610 (1998).

[3] H. Leblanc, Alternatives to standard first-order semantics, in *Handbook of Philosophical Logic*, edited by D. Gabbay and F. Guenthner, volume I: Elements of Classical Logic, pages 189–274, D. Reidel, Dordrecht, 1983.

[4] A. Peres, Found. Phys **26**, 807 (1996).

[5] V. Vedral and M. B. Plenio, Prog. Quant. Electron. **22**, 1 (1998), Retrievable at http://xxx.lanl.gov/abs/quant-ph/9802065.

[6] A. Tarski, Some methodological investigations on the definiability of concepts, in *Logic, Semantics, Metamathematics (Papers from 1923 to 1938) by Alfred Tarski*, edited by J. H. Woodger, pages 296–319, Oxford University Press, Oxford, 1956.

[7] A. A. Fraenkel, Y. Bar-Hillel, and A. Levy, *Foundations of Set Theory*, North-Holland, Amsterdam, 1973.

[8] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London, 1983.

[9] M. Pavičić, Int. J. Theor. Phys. **26**, 845 (1987).

[10] M. Pavičić, Int. J. Theor. Phys. **37**, 2099 (1998).

[11] M. Pavičić and N. D. Megill, Int. J. Theor. Phys. **37**, 2091 (1998).

[12] M. Pavičić, Int. J. Theor. Phys. **32**, 1481 (1993).

[13] L. Beran, *Orthomodular Lattices; Algebraic Approach*, D. Reidel, Dordrecht, 1985.

[14] M. Pavičić, Found. Phys. **19**, 999 (1989).

[15] H. Dishkant, Rep. Math. Logic **3**, 9 (1974).

[16] M. L. Dalla Chiara, Quantum logic, in *Handbook of Philosophical Logic*, edited by D. Gabbay and F. Guenthner, volume III., pages 427–469, D. Reidel, Dordrecht, 1986.

[17] P. Mittelstaedt, *Quantum Logic*, L. M. S. Monographs; v. 18, Academic Press, London, 1983.

[18] E.-W. Stachow, J. Phil. Logic **5**, 237 (1976).

[19] G. M. Hardegree, The conditional in abstract and concrete quantum logic, in *The Logico-Algebraic Approach to Quantum Mechanics*, edited by C. A. Hooker, volume II. Contemporary Consolidation, pages 49–108, D. Reidel, Dordrecht, 1979.

[20] G. T. Rüttimann, *Logikkalküle der Quantenphysik. Eine Abhandlung zur Ermittlung der formal-logischen Systeme, die der nicht-relativistischen Quantentheorie zugrundeliegen*, Duncker & Humblot, Berlin, 1977.

[21] D. Hilbert and W. Ackermann, *Principles of Mathematical Logic*, Chelsea, New York, 1950.

[22] S. C. Kleene, *Introduction to Metamathematics*, North-Holland, Amsterdam, 1974.

[23] J. Łukasiewicz, *Selected Works*, North-Holland, Amsterdam, 1970.

[24] E. L. Post, Am. J. Math. **43**, 163 (1921).

[25] P. Pták and S. Pulmannová, *Quantum Structures as Quantum Logics*, Kluwer, Dordrecht, 1991.

[26] S. S. Holland, JR., Bull. Am. Math. Soc. **32**, 205 (1995).

[27] A. Barenco et al., Phys. Rev. A **52**, 3457 (1995).

[28] N. D. Megill and M. Pavičić, [submitted] (1999).