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HOMOTOPSKA SVOJSTVA LOKALNO KOMPAKTNIH PROSTORA U BESKONAČNOSTI – TRIVIJALNOST I POKRETLJIVOST

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Sadržaj

U članku se uvodi nova metoda proučavanja geometrije lokalno kompaktnih prostora u kojoj se u obzir uzimaju samo ponašanja homotopija blizu beskonačnosti. Teorija homotopije u beskonačnosti dobivena na taj način poopćuje Chapman-ovu teoriju slabe prave homotopije. Ona je u uskoj vezi s teorijom oblika preko Chapman-ovog teorema komplemenata. Motivirani tom vezom, u članku se definiraju geomatrijska svojstva lokalno kompaktnih prostora nazvana $\mathscr{C}$-trivijalnost u ∞ i $\mathscr{C}$-pokretljivost u ∞, gdje je $\mathscr{C}$ proizvoljna klasa topoloških prostora, koja odgovara trivijalnosti oblika i pokretljivosti u teoriji oblika. Dokazano je nekoliko teorema o tim svojstvima.

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n-CONNECTEDNESS OF INVERSE SYSTEMS AND APPLICATIONS TO SHAPE THEORY

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Abstract. Let $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ be an n-connected inverse system of CW-pairs such that the restriction $(\mathbf{A}, \mathbf{x})$ is m-connected. We prove that there exists an isomorphic inverse system $(\mathbf{Y}, \mathbf{B}, \mathbf{y})$ having n-connected terms and such that the terms in the restriction $(\mathbf{B}, \mathbf{y})$ are m-connected. This result is then applied in proving analogues of Hurewicz and Blakers-Massey theorems for homotopy pro-groups and shape groups.

1. Introduction

In the present paper we study n-connected inverse systems and applications to homotopy pro-groups and shape groups. We first develop the notion of T-systems, generalizing inverse systems, and prove the reindexing theorem and Morita's lemma for T-systems. Then we apply this results to n-connected inverse systems of CW-complexes and show that every such system can be replaced by an isomorphic system having n-connected terms (the relative case being considered also). This implies that e. g. every shape n-connected Hausdorff continuum can be obtained as the inverse limit of n-connected polyhedra.

Next we use the developed methods to prove a version of Hurewicz theorem for homotopy pro-groups, which is more complete than the one in [10] or [12]. Finally we prove the analogue of the Blakers--Massey excision theorem for homotopy pro-groups and shape groups, and a theorem on the homotopy pro-groups and shape groups of identification spaces.

A few words about the notations. By HCW we shall denote the category of topological spaces having the homotopy type of CW-complexes and homotopy classes of continuous maps. The appropriate pointed category and the category of pointed pairs shall be denoted

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by HCW_0 and HCW_0^2 respectively. An inverse system $\mathbf{X}$ over HCW is said to be *associated* to the topological space X if it satisfies the conditions in Definition 1.2. of Morita [11], and similarly for pairs.

Note that if an inverse system $(\mathbf{X}, \mathbf{A}) = \{(X, A)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is associated to the pair (X, A), then the system $\mathbf{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is associated to X, but the restriction $\mathbf{A} = \{A_\lambda, p_{\lambda\lambda'} \mid A_{\lambda'}, \Lambda\}$ need not be associated to A. The technical condition which assures that $\mathbf{A}$ is associated to A is that of A being *P-embedded* in X [12]. For a Hausdorff compact space X every closed subset is P-embedded. In a way in shape theory P-embeddings play the role of cofibrations in homotopy theory.

The n-th homotopy pro-group of an inverse system $(\mathbf{X}, \mathbf{A}, \mathbf{x}) \in$ pro-HCW_0^2 will be denoted by pro-$\pi_n(\mathbf{X}, \mathbf{A}, \mathbf{x})$. For a pointed pair of topological spaces (X, A, x_0) we define pro-$\pi_n(X, A, x_0) :=$ pro-$\pi_n(\mathbf{X}, \mathbf{A}, \mathbf{x})$ where $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ is any inverse system over HCW^2 associated to (X, A, x_0). The limit $\check{\pi}_n(X, A, x_0) := \varprojlim$ pro-$\pi_n(X, A, x_0)$ is called the n-th shape group of (X, A, x_0). Similarly one has pro-$H_n(\mathbf{X}, \mathbf{A})$ and pro-$H_n(X, A)$ and $H_n(X, A) = \varprojlim$ pro-$H_n(X, A)$ is the n-th Čech homology group of the pair (X, A). Therefore throughout the paper H_n denotes the Čech homology functor. For details see e. g. [9] or [10].

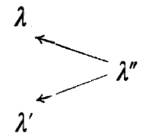
2. T-systems

In shape theory we usually deal with inverse systems over directed sets, but sometimes we do need more general pro-objects. In this paper certain such objects, called T-systems, will be needed. In this section we define T-systems and prove the reindexing theorem (Theorem 1) and Morita's Lemma (Teorem 2).

Following Grothendieck [4] we have the following definition:

A nonvoid category Λ is said to be (left) filtered if the following holds:

(*i*) For any two objects $\lambda, \lambda' \in \Lambda$, there is an object $\lambda'' \in \Lambda$ and a diagram



in Λ.

(*ii*) For any two morphisms $\lambda \leftleftarrows \lambda'$ in Λ, there is a morphism $\lambda' \leftarrow \lambda''$ such that the two compositions $\lambda \leftleftarrows \lambda' \leftarrow \lambda''$ are equal.

Given a category $\mathscr{C}$, a *pro-object* over $\mathscr{C}$ is a covariant functor $\mathbf{X}: \Lambda \to \mathscr{C}$, where Λ is a small filtered category.

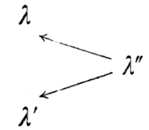
The set of morphisms between two pro-objects X and Y is defined as

$$\text{pro-}\mathscr{C}(\mathbf{X}, \mathbf{Y}) := \varprojlim_{M} \varinjlim_{\Lambda} \mathscr{C}(X_\lambda, Y_\mu),$$

where X_λ denotes $X(\lambda)$. Pro-objects over $\mathscr{C}$ and the morphisms between them form a category, denoted by pro-$\mathscr{C}$ and called the pro-category of $\mathscr{C}$.

DEFINITION: *A nonvoid category Λ is called T-(left) filtered if instead of (i) and (ii) the following stronger conditions hold:*

($T1$) *For any two objects $\lambda, \lambda' \in \Lambda$ there exist a $\lambda'' \in \Lambda$ with $\lambda'' \neq \lambda, \lambda'$ and a diagram*



in Λ.

($T2$) *For any two morphisms $\lambda \leftleftarrows \lambda'$ and any morphism $\lambda' \leftarrow \lambda''$ in Λ with $\lambda'' \neq \lambda'$, the two compositions $\lambda \leftleftarrows \lambda' \leftarrow \lambda''$ are equal.*

A T-system over $\mathscr{C}$ is a covariant functor $\mathbf{X}: \Lambda \to \mathscr{C}$, where V is a small T-filtered category.

We shall use the notation $\mathbf{X} = \{X_\lambda, \{p_{\lambda\lambda'}\}, \Lambda\}$, where $X_\lambda = \mathbf{X}(\lambda)$ and $\{p_{\lambda\lambda'}\} = \mathbf{X}(\{\lambda' \to \lambda\})$, since there might be several Λ-morphisms from λ' to λ, and therefore the set $\{p_{\lambda\lambda'}\}$ of $\mathscr{C}$-morphisms from $X_{\lambda'}$ to X_λ might contain more than one element. Note that because of ($T2$), both compositions in the diagram

$$\lambda \leftarrow \lambda' \leftarrow \lambda'' \leftarrow \lambda^*, \ \lambda^* \neq \lambda''$$

are equal, and thus

$$\{p_{\lambda\lambda'}\, p_{\lambda'\lambda''}\}\, p_{\lambda''\lambda^*} = \{p_{\lambda\lambda''}\}\, p_{\lambda''\lambda^*} \tag{1}$$

for every $p_{\lambda''\lambda^*} \in \{p_{\lambda''\lambda^*}\}$.

A small T-filtered category Λ can always be considered as a directed set by defining $\lambda \leqslant \lambda'$ whenever the set of Λ-morphisms $\{\lambda \leftarrow \lambda'\}$ is nonvoid. The situation in ($T1$) is then shortly denoted by $\lambda, \lambda' < \lambda''$ ($\lambda < \lambda''$ means in addition that $\lambda \neq \lambda''$).

Therefore, T-systems are pro-objects over $\mathscr{C}$ which are more general than inverse systems over $\mathscr{C}$ because there might be more bonding morphisms for the same pair of indices. The uniqueness of bonding morphisms in inverse systems in expressed by

$$p_{\lambda\lambda'}\, p_{\lambda'\lambda''} = p_{\lambda\lambda''}$$

whenever $\lambda \leqslant \lambda' \leqslant \lambda''$, whereas for T-systems only the weaker condition (1) is satisfied.

Now we describe the morphisms between T-systems in terms of objects and morphisms in $\mathscr{C}$ and in the situation in which we are going to use them. Let $\mathbf{X} = \{X_\lambda, \{p_{\lambda\lambda'}\}, \Lambda\}$ and $\mathbf{Y} = \{Y_\mu, \{q_{\mu\mu'}\}, M\}$ be T-systems over $\mathscr{C}$. Let $f: M \to \Lambda$ be a function (not necessarily a functor, i. e. order preserving) and let for each $\mu \in M$ be given a $\mathscr{C}$-morphism $f_\mu: X_{f(\mu)} \to Y_\mu$. $(f, f_\mu): \mathbf{X} \to \mathbf{Y}$ is said to be a *map* of T-systems, provided given any bonding morphism $q_{\mu\mu'}: Y_{\mu'} \to Y_{\mu'}$ there exists an index $\lambda \in \Lambda$ such that $\lambda \geqslant f(\mu), f(\mu')$ and there exist bonding morphisms $p_{f(\mu)\lambda}$ and $p_{f(\mu')\lambda}$ such that

$$f_\mu\, p_{f(\mu)\lambda} = q_{\mu\mu'}\, f_{\mu'}\, p_{f(\mu')\lambda}. \tag{2}$$

Two maps of T-systems (f, f_μ) and (g, g_μ) between X and Y are said to be equivalent, provided given any index $\mu \in M$, there exists an index $\lambda \in \Lambda$ with $\lambda \geqslant f(\mu), g(\mu)$ and there exist bonding morphisms $p_{f(\mu)\lambda}$ and $p_{g(\mu)\lambda}$ such that

$$f_\mu\, p_{f(\mu)\lambda} = g_\mu\, p_{g(\mu)\lambda}. \tag{3}$$

Equivalence classes under this relation are the morphisms of T-systems in pro-$\mathscr{C}$. The morphism represented by the map of T-systems (f, f_μ) will be denoted by $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$.

A particularly simple (and thus important) case is when $\Lambda = M$, $f = 1_\Lambda$ and for every $q_{\lambda\lambda'}$ there is a $p_{\lambda\lambda'}$ such that the relation

$$f_\lambda\, p_{\lambda\lambda'} = q_{\lambda\lambda'}\, f_{\lambda'}$$

holds. Such a map of T-systems $(1, f_\lambda)$ is called *level map*, and the induced morphism $\mathbf{f}$ is refered to as *level* (or *special*) *morphism*.

In dealing with T-systems we usually select one bonding morphism $p_{\lambda\lambda'}$ from each nonempty set $\{p_{\lambda\lambda'}\}$. For $\lambda \leqslant \lambda' \leqslant \lambda'' < \lambda^*$ we have from (1)

$$p_{\lambda\lambda'}\, p_{\lambda'\lambda''}\, p_{\lambda''\lambda^*} = p_{\lambda\lambda''}\, p_{\lambda''\lambda^*}. \tag{4}$$

Given two T-systems $\mathbf{X} = \{X_\lambda, \{p_{\lambda\lambda'}\}, \Lambda\}$ and $\mathbf{Y} = \{Y_\mu, \{q_{\mu\mu'}\}, M\}$ and selected bonding morphisms $p_{\lambda\lambda'} \in \{p_{\lambda\lambda'}\}$ and $q_{\mu\mu'} \in \{q_{\mu\mu'}\}$, a pair $(f, f_\mu): \mathbf{X} \to \mathbf{Y}$ is a map of T-systems iff for every $\mu \leqslant \mu'$ there exist indices $\lambda \geqslant f(\mu), f(\mu')$ and $\lambda' > \lambda$ such that for selected bonding morphisms holds

$$f_\mu\, p_{f(\mu)\lambda}\, p_{\lambda\lambda'} = q_{\mu\mu'}\, f_{\mu'}\, p_{f(\mu')\lambda}\, p_{\lambda\lambda'}. \tag{5}$$

Furthermore, (f, f_μ) and (g, g_μ) are equivalent, iff for every $\mu \in M$ there exist indices $\lambda \geqslant f(\mu), g(\mu)$ and $\lambda' > \lambda$ such that

$$f_\mu\, p_{f(\mu)\lambda}\, p_{\lambda\lambda'} = g_\mu\, p_{g(\mu)\lambda}\, p_{\lambda\lambda'}. \tag{6}$$

We are now ready to prove the reindexing theorem.

THEOREM 1. *Given any T-system $\mathbf{X} = \{X_\lambda, \{p_{\lambda\lambda'}\}, \Lambda\}$ over $\mathscr{C}$, there exists an inverse system $\mathbf{Y} = \{Y_\mu, q_{\mu\mu'}, M\}$ which is isomorphic to $\mathbf{X}$ in pro-$\mathscr{C}$. Moreover, the set M is cofinite, each Y_μ is some X_λ and each $q_{\mu\mu'}$ is the composition of some $p_{\lambda\lambda'}$'s.*

Proof. For every pair $\lambda \leqslant \lambda'$ select a bonding morphism $p_{\lambda\lambda'} \in \{p_{\lambda\lambda'}\}$. Let M be the family of all finite subsets of Λ ordered by inclusion. Define an increasing function $f: M \to \Lambda$ as follows:

For singletons define $f(\{\lambda\}) = \lambda$, and let $\mu \in M$ be of cardinality greater then or equal two. Suppose $f(\mu')$ has already been defined for all μ' with card $\mu' <$ card μ. Let $\lambda(\mu) \in \Lambda$ be such that $\lambda(\mu) \geqslant f(\mu_0)$ for all $\mu_0 < \mu$, and let $f(\mu)$ be any element with $f(\mu) > \lambda(\mu)$.

Let $Y_\mu := X_{f(\mu)}$ and for $\mu < \mu'$ define

$$q_{\mu\mu'} := p_{f(\mu)\lambda(\mu')}\, p_{\lambda(\mu')f(\mu')}: Y_{\mu'} \to Y_\mu.$$

It is easy to check that $\mathbf{Y} := \{Y_\mu, q_{\mu\mu'}, M\}$ is an inverse system.

For $\mu \in M$ let $f_\mu := 1: X_{f(\mu)} \to Y_\mu$. Furthermore, let $g: \Lambda \to M$ be the function defined by $g(\lambda) = \{\lambda\}$, and $g_\lambda := 1: Y_{g(\lambda)} \to X_\lambda$. A straightforward verification shows that $(f, f_\mu): \mathbf{X} \to \mathbf{Y}$ and $(g, g_\lambda): \mathbf{Y} \to \mathbf{X}$ define morphisms in pro-$\mathscr{C}$ which are inverses each other.

Note. In [2] a general reindexing theorem is proved, stating that any pro-object over $\mathscr{C}$ is naturally isomorphic to an inverse system. Both proofs, ours and the one used in [2], are based on an idea in [8].

Morita [12] has proven a very useful criterion for a level map of inverse systems to induce an isomorphism in pro-$\mathscr{C}$, known as Morita's Lemma. This result can be generalised to T-systems. For simplicity we prove only its special version which we shall need later on.

THEOREM 2. *Let Λ be a T-filtered category, $\mathbf{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ an inverse system and $\mathbf{Y} = \{Y_\lambda, \{q_{\lambda\lambda'}\}, \Lambda\}$ a T-system over $\mathscr{C}$, and let $(1, f_\lambda): \mathbf{X} \to \mathbf{Y}$ be a level map of systems. Suppose that for each $\lambda < \lambda'$*

there exist a morphism $g_{\lambda\lambda'}: Y_{\lambda'} \to X_\lambda$ *and a bonding morphism* $q_{\lambda\lambda'} \in \{q_{\lambda\lambda'}\}$ *such that* $g_{\lambda\lambda'} f_{\lambda'} = p_{\lambda\lambda'}$ *and* $f_\lambda g_{\lambda\lambda'} = q_{\lambda\lambda'}$. *Then* $(1, f_\lambda)$ *induces an isomorphism in* pro-$\mathscr{C}$.

Proof. Choose bonding morphisms $q_{\lambda\lambda'} \in \{q_{\lambda\lambda'}\}$ for which the assumptions of the theorem hold.

If $\lambda < \lambda' < \lambda''$, then

$$g_{\lambda\lambda'} q_{\lambda'\lambda''} = p_{\lambda\lambda'} g_{\lambda'\lambda''} \tag{7}$$

since both sides are equal to $g_{\lambda\lambda'} f_{\lambda'} g_{\lambda'\lambda''}$.

Whenever $\lambda < \lambda' < \lambda'' < \lambda^*$ by using (7) one verifies

$$g_{\lambda\lambda''} q_{\lambda''\lambda^*} = p_{\lambda\lambda'} g_{\lambda'\lambda''} q_{\lambda''\lambda^*}. \tag{8}$$

Furthermore, for $\lambda < \lambda' < \lambda'' < \lambda^* < \lambda^{**}$, from (4), (7) and (8) we get

$$g_{\lambda\lambda'} q_{\lambda'\lambda^*} q_{\lambda^*\lambda^{**}} = g_{\lambda\lambda''} q_{\lambda''\lambda^*} q_{\lambda^*\lambda^{**}}. \tag{9}$$

Let $g: \Lambda \to \Lambda$ be any function for which $g(\lambda) > \lambda$, and let $g_\lambda := g_{\lambda g(\lambda)}: Y_{g(\lambda)} \to X_\lambda$. To show that $(g, g_\lambda): \mathbf{Y} \to \mathbf{X}$ is a map of systems, for $\lambda < \lambda'$ and $\lambda'' > g(\lambda), g(\lambda')$ choose λ^* and λ^{**} such that $\lambda'' < \lambda^* < \lambda^{**}$. Then using (9), (8) and (9) again, one verifies

$$g_\lambda q_{g(\lambda)\lambda^*} q_{\lambda^*\lambda^{**}} = p_{\lambda\lambda'} g_{\lambda'} q_{g(\lambda')\lambda^*} q_{\lambda^*\lambda^{**}}$$

which proves the claim. It is easy to see now that the morphisms **f** and **g** represented by $(1, f_\lambda)$ and (g, g_λ) are inverses each other, and hence f is an isomorphism in pro-$\mathscr{C}$.

Further on we shall encounter several times the situation described by the following theorem:

THEOREM 3. ***Let*** $\mathbf{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\} \in$ pro-$\mathscr{C}$ *be an inverse system with* Λ *having no maximal element. Suppose that for each* $\lambda \in \Lambda$ *an object* $Y_\lambda \in \operatorname{Ob}\mathscr{C}$ *and morphism* $i_\lambda: X_\lambda \to Y_\lambda$ *are given and that there exist an index* $\lambda' \geqslant \lambda$ *and a morphism* $g_{\lambda\lambda'}: Y_{\lambda'} \to X_\lambda$ *such that*

$$g_{\lambda\lambda'} i_{\lambda'} = p_{\lambda\lambda'}. \tag{10}$$

Then there is an inverse system $\mathbf{Z} = \{Z_\mu, r_{\mu\mu'}, M\}$ *which is isomorphic to* $\mathbf{X}$ *in* pro-$\mathscr{C}$, *and each* Z_μ *is some* Y_λ.

Proof. We define a new directed order $<^*$ on Λ by putting $\lambda <^* \lambda'$ whenever $\lambda < \lambda'$ and there is a morphism $g_{\lambda\lambda'}: Y_{\lambda'} \to X_\lambda$ satisfying (10). Let $\lambda \leqslant^* \lambda'$ if either $\lambda <^* \lambda'$ or $\lambda = \lambda'$. To show transitivity of $<^*$ let $\lambda <^* \lambda' <^* \lambda''$. Then we have $\lambda < \lambda' < \lambda''$ and $\lambda < \lambda''$ (by transitivity of $<$). Furthermore, the composition $p_{\lambda\lambda'} g_{\lambda'\lambda''}: Y_{\lambda''} \to X_\lambda$, using (10), gives

$$p_{\lambda\lambda'} g_{\lambda'\lambda''} i_{\lambda''} = p_{\lambda\lambda''}$$

showing $\lambda <^* \lambda''$.

Next we have to show the directedness of $\leqslant^*$. For $\lambda, \lambda' \in \Lambda$ let $\lambda'' \geqslant \lambda, \lambda'$ and choose a $\lambda^* \in \Lambda$ such that $\lambda'' <^* \lambda^*$ (such a λ^* exists by the assumptions of the theorem). Then $\lambda^* > \lambda, \lambda'$ and for the morphisms $p_{\lambda\lambda'} g_{\lambda''\lambda^*}: Y_{\lambda^*} \to X_\lambda$ and $p_{\lambda'\lambda''} g_{\lambda''\lambda^*}: Y_{\lambda^*} \to X_{\lambda'}$ we have

$$p_{\lambda\lambda''} g_{\lambda''\lambda^*} i_{\lambda^*} = p_{\lambda\lambda^*}$$
$$p_{\lambda'\lambda''} g_{\lambda''\lambda^*} i_{\lambda^*} = p_{\lambda'\lambda^*}$$

showing $\lambda^* {}^*\!> \lambda, \lambda'$.

Finally, by the assumptions of the theorem, we see that the directed set $(\Lambda, \leqslant^*)$ is cofinal in $(\Lambda, \leqslant)$. Namely, for $\lambda \in \Lambda$ there is a $\lambda' > \lambda$ and a morphism $g_{\lambda\lambda'}$ satisfying (10). Hence $\lambda <^* \lambda'$ showing the cofinality.

Therefore the inverse systems $\mathbf{X} = \mathbf{X}(\Lambda, \leqslant)$ and $\mathbf{X}(\Lambda, \leqslant^*)$ are isomorphic. In order to simplify the notation we assume that the original ordering $<$ enjoys the property that for every $\lambda < \lambda'$ there is a morphism $g_{\lambda\lambda'}: Y_{\lambda'} \to X_\lambda$ satisfying (10).

For $\lambda < \lambda'$ define $q_{\lambda\lambda'}: Y_{\lambda'} \to Y_\lambda$ by

$$q_{\lambda\lambda'} := i_\lambda g_{\lambda\lambda'}. \tag{11}$$

Then for $\lambda < \lambda' < \lambda''$ the morphism $q_{\lambda\lambda''}$ need not coincide with $q_{\lambda\lambda'} q_{\lambda'\lambda''}$, but for $\lambda < \lambda' < \lambda'' < \lambda^*$ we have

$$\begin{aligned} q_{\lambda\lambda'} q_{\lambda'\lambda''} q_{\lambda''\lambda^*} &\overset{(11)}{=} i_\lambda (g_{\lambda\lambda'} i_{\lambda'})(g_{\lambda'\lambda''} i_{\lambda''}) g_{\lambda''\lambda^*} = \\ &\overset{(10)}{=} i_\lambda p_{\lambda\lambda'} p_{\lambda'\lambda''} g_{\lambda''\lambda^*} = i_\lambda p_{\lambda\lambda''} g_{\lambda''\lambda^*} = \\ &\overset{(10)}{=} (i_\lambda g_{\lambda\lambda''})(i_{\lambda''} g_{\lambda''\lambda^*}) \overset{(11)}{=} q_{\lambda\lambda''} q_{\lambda''\lambda^*} \end{aligned}$$

and therefore $\mathbf{Y} = \{Y_\lambda, q_{\lambda\lambda'}, \Lambda\}$ is a T-system over $\mathscr{C}$.

Note that thy family of morphisms $\{i_\lambda: X_\lambda \to Y_\lambda: \lambda \in \Lambda\}$ is a level map of the inverse system $\mathbf{X}$ into the T-system $\mathbf{Y}$, since by (10) and (11) we have

$$i_\lambda p_{\lambda\lambda'} = i_\lambda g_{\lambda\lambda'} i_{\lambda'} = q_{\lambda\lambda'} i_{\lambda'}.$$

Furthermore, for $\lambda < \lambda'$ there exists a morphism $g_{\lambda\lambda'}: Y_{\lambda'} \to X_\lambda$ satisfying (10) and (11), which by Theorem 2 implies that the morphism $\mathbf{i}: \mathbf{X} \to \mathbf{Y}$ induced by $\{i_\lambda\}$ is an isomorphism in pro-$\mathscr{C}$.

Finally we apply Theorem 1 to find an inverse system $\mathbf{Z} = \{Z_\mu, r_{\mu\mu'}, M\} \in$ pro-$\mathscr{C}$ which is isomorphic to the T-system $\mathbf{Y}$, and therefore to the inverse system $\mathbf{X}$, and every Z_μ is some Y_λ, proving the theorem.

We close this section by the following theorem:

THEOREM 4. *Let $\mathscr{C}$ be a category and $\mathbf{Y} = \{Y_\lambda, q_{\lambda\lambda'}, \Lambda\}$ an inverse system over $\mathscr{C}$ which dominates the inverse sequence $\mathbf{X} = \{X_n, p_{n,n+1}, N\}$. Then there is an inverse sequence $\mathbf{Z} = \{Z_n, r_{n,n+1}, N\}$ which is isomorphic to $\mathbf{X}$ in pro-$\mathscr{C}$ and each Z_n is some Y_λ.*

Proof. Let $(f, f_\lambda): \mathbf{X} \to \mathbf{Y}$ and $(g, g_n): \mathbf{Y} \to \mathbf{X}$ be maps of inverse systems such that $\mathbf{g\,f} = \mathbf{1_X}$, and let $n_1 := 1$ and $\lambda_1 := g(n_1)$. For $k \in N$ we define $n_{k+1} \in N$, and $\lambda_{k+1} \in \Lambda$ by induction, so that $n_{k+1} > n_k$, $f(\lambda_k)$ and

$$g_{n_k} f_{\lambda_k} p_{f(\lambda_k) n_{k+1}} = p_{n_k n_{k+1}}. \tag{12}$$

The existence of such n_{k+1} follows from $\mathbf{g\,f} = 1_\mathbf{X}$. Finally let $\lambda_{k+1} := g(n_{k+1})$.

Let $\mathbf{X}' := \{X_{n_k}, p_{n_k n_{k+1}}, N\}$. The inverse sequence $\mathbf{X}'$ is cofinal in $\mathbf{X}$ since $n_k < n_{k+1}$ for all k, hence $\mathbf{X}$ and $\mathbf{X}'$ are isomorphic in pro-$\mathscr{C}$.

Define the inverse sequence $\mathbf{W} = \{W_i, s_{i,i+1}, N\}$ by

$$W_i := \begin{cases} X_{n_k} & \text{for } i = 2k-1 \\ Y_{\lambda_k} & \text{for } i = 2k \end{cases}$$

$$s_{i,i+1} = \begin{cases} g_{n_k} & \text{for } i = 2k-1 \\ f_{\lambda_k} p_{f(\lambda_k) n_{k+1}} & \text{for } i = 2k. \end{cases}$$

For $i = 2k - 1$ using (12) we find

$$s_{i,i+1}\, s_{i+1,i+2} = g_{n_k} f_{\lambda_k} p_{f(\lambda_k) n_{k+1}} = p_{n_k n k+1}: X_{n_{k+1}} \to X_{n_k}$$

showing that $\mathbf{X}'$ is a cofinal subsequence of $\mathbf{W}$, and thus $\mathbf{X}'$ and $\mathbf{W}$ are isomorphic.

Finally put $Z_k := Y_{\lambda_k}$ and

$$r_{k,k+1} := f_{\lambda_k} p_{f(\lambda_k) n_{k+1}}\, g_{n_{k+1}}: Z_{k+1} \to Z_k, \quad k \in N.$$

Since $Z_k = W_{2k}$ and

$$r_{k,k+1} = (f_{\lambda_k} p_{f(\lambda_k) n_{k+1}})\, g_{n_{k+1}} = s_{2k,2k+1}\, s_{2k+1,2k+2} = s_{2k,2k+2}$$

we conclude that the inverse sequence $\mathbf{Z} = \{Z_k, r_{k,k+1}, N\}$ is cofinal in $\mathbf{W}$, and therefore isomorphic to $\mathbf{Z}$, which shows that $\mathbf{Z}$ and $\mathbf{X}$ are isomorphic inverse sequences and every Z_k is some Y_λ, proving the theorem.

3. *n*-connectedness of inverse systems

In this section we show that an n-connected inverse system of CW-complexes can be replaced by an inverse system having n-connected terms. Relative case is discused in details.

THEOREM 1. *Let $\mathscr{C}$ be one of the categories CW_0^2 or HCW_0^2 and let $(\mathbf{X}, \mathbf{A}, \mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an n-connected inverse system over $\mathscr{C}$, i. e.* pro-$\pi_k(\mathbf{X}, \mathbf{A}, \mathbf{x}) \cong 0$ *for* $1 \leqslant k \leqslant n$ and X_λ, A_λ *are connected. Then there exists an inverse system $(\mathbf{Y}, \mathbf{B}, \mathbf{y}) = \{(Y, B, y)_\mu, q_{\mu\mu'}, M\}$ which is isomorphic to $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ in* pro-$\mathscr{C}$ *and having all the terms $(Y, B, y)_\mu$ n-connected.*

Proof. For each $\lambda \in \Lambda$ define a pair of CW-complexes $(Y, B, y)_\lambda$ by

$$\begin{aligned} Y_\lambda &:= (X_\lambda^n \cup A_\lambda) \times I \cup X_\lambda \times \{0\} \\ B_\lambda &:= A_\lambda \times I \cup X_\lambda^n \times \{1\} \\ y_\lambda &:= x_\lambda. \end{aligned}$$

and let $i_\lambda: (X, A, x)_\lambda \to (Y, B, y)_\lambda$ denote the inclusion. First we prove the following lemma.

LEMMA 1. Under the assumptions of Theorem 1, for each $\lambda \in \Lambda$ there exist a $\lambda' > \lambda$ and a continuous map (or a homotopy class of continuous maps if $\mathscr{C}$ was the homotopy category)

$$g_{\lambda\lambda'}: (Y, B, y)_{\lambda'} \to (X, A, x)_\lambda$$

such that

$$g_{\lambda\lambda'}\, i_{\lambda'} = p_{\lambda\lambda'} \tag{1}$$

and

$$g_{\lambda\lambda'}(x, t) = p_{\lambda\lambda'}(x) \text{ for } x \in A_{\lambda'},\ t \in I. \tag{2}$$

Proof of Lemma 1. For each $\lambda \in \Lambda$ there is a sequence

$$\lambda =: \lambda_n < \lambda_{n-1} < \ldots < \lambda_1 < \lambda_0 =: \lambda'$$

such that

$$p_{\lambda_k \lambda_{k-1}\#}: \pi_k(X, A, x)_{\lambda_{k-1}} \to \pi_k(X, A, x)_{\lambda_k} \tag{3}$$

is the null-homomorphism, $1 \leqslant k \leqslant n$. Let

$$\begin{aligned} Y_k &:= (X_{\lambda'}^k \cup A_{\lambda'}) \times I \cup X_{\lambda'} \times \{0\} \\ B_k &:= A_{\lambda'} \times I \cup X_{\lambda'}^k \times \{1\} \\ y_k &:= x_{\lambda'}, \qquad\qquad 1 \leqslant k \leqslant n. \end{aligned}$$

Observe that $(Y, B, y)_{k-1} \subset (Y, B, y)_k \subset (Y, B, y)_n = (Y, B, y)_{\lambda'}$ for all $1 < k < n$.

We define a sequence of maps

$$f_k\colon (Y, B, y)_k \to (X, A, x)_{\lambda_k}$$

satisfying

(*i*) $f_k \mid X_{\lambda'} \times \{0\} = p_{\lambda_k \lambda'}$

(*ii*) $f_k \mid Y_{k-1} = p_{\lambda_k \lambda_{k-1}} f_{k-1}$

(*iii*) $f_k(B_k) \subset A_{\lambda_k}$.

First define $f_0\colon (Y, B, y)_0 \to (X, A, x)_{\lambda'}$ by

$$f_0(x, 0) = x, \ x \in X_{\lambda'}$$
$$f_0(x, t) = x, \ x \in A_{\lambda'}, \ t \in I$$
$$f_0(x, 1) = x_{\lambda'}, \ x \in X^0_{\lambda'} - A^0_{\lambda'},$$

and for $f_0 \mid x \times I : x \times I \to X_{\lambda'}$, $x \in X^0_{\lambda'} - A^0_{\lambda'}$, take any path joining x and $x_{\lambda'}$.

f_0 clearly satisfies (*i*) and (*iii*). We define f_k by induction as follows.

Put $f_k \mid Y_{k-1} = p_{\lambda_k \lambda_{k-1}} f_{k-1}$. For a k-cell $E \subset X^k_{\lambda'} - A_{\lambda'}$ we have $\partial E \subset X^{k-1}_{\lambda'}$ and the map $f_{k-1} \mid E \cup (\partial E \times I)$ represents an element of $\pi_k(X, A, x)_{\lambda_{k-1}}$. Because of (3) the composition $p_{\lambda_k \lambda_{k-1}} f_{k-1} \mid E \cup (\partial E \times I)$ represents the trivial element of $\pi_k(X, A, x)_{\lambda_k}$ and it can be extended to $E \times I$ in such a way that the image of $E \times \{1\}$ lies in A_{λ_k}. If we do this for all k-cells in $X^k_{\lambda'} - A_{\lambda'}$ we get the map f_k satisfying (*i*) – (*iii*).

Finally we define

$$g_{\lambda\lambda'} := f_n\colon (Y, B, y)_{\lambda'} \to (X, A, x)_{\lambda}.$$

Since f_n satisfies (*i*), (1) holds. Furthermore, because of (*ii*) and the definition of f_0, for $x \in A_{\lambda'}$ and $t \in I$ we obtain

$$\begin{aligned} g_{\lambda\lambda'}(x, t) &= f_n(x, t) = p_{\lambda\lambda_{n-1}} f_{n-1}(x, t) = \\ &= p_{\lambda\lambda_{n-1}} p_{\lambda_{n-1}\lambda_{n-2}} f_{n-2}(x, t) = \\ &= p_{\lambda\lambda_{n-2}} f_{n-2}(x, t) = p_{\lambda\lambda'} f_0(x, t) = \\ &= p_{\lambda\lambda'}(x) \end{aligned}$$

proving the lemma.

We continue the proof of Theorem 1.

By Lemma 1 we are able to apply Theorem 2.3.*) to our situation. Therefore the inverse system $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ is isomorphic to the inverse system $(\mathbf{Y}, \mathbf{B}, \mathbf{y})$ consisting of just constructed terms $(Y, B, y)_\lambda$. But, using the cellular approximation theorem and the fact that B_λ is a strong deformation retract of Y^n_λ, we conclude that all the pairs $(Y, B, y)_\lambda$ are n-connected, which proves the theorem.

Remark. In performing the construction in the proof of Theorem 1 we implicitly assumed that Λ had no maximal element. Nevertheless, there was no loss of generality, since otherwise the inverse system would be isomorphic to its »initial« term (i.e. the one with the maximal index) which, by n-connectedness of the inverse system, would have been n-connected itself. The same assumption is also needed in all the other theorems of this section, and therefore in the rest of the sections as well. Otherwise these theorems reduce to known theorems in the homotopy theory, and therefore we won't point the assumption (of nonexistence of the maximal element in Λ) out any more.

In the absolute case we have

THEOREM 2. *Let $\mathscr{C}$ be one of the categories CW_0 or HCW_0 and let $(\mathbf{X}, \mathbf{x}) = \{(X, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an n-connected inverse system over $\mathscr{C}$. Then there is an isomorphic inverse system $(\mathbf{Y}, \mathbf{y})$ having all the terms n-connected.*

Proof. This theorem is not a consequence of Theorem 1. Namely if we consider $(\mathbf{X}, \mathbf{x})$ as the inverse system of pairs $(\mathbf{X}, \{\mathbf{x}\}, \mathbf{x})$ and apply Theorem 1, we get an inverse system $(\mathbf{Y}, \mathbf{B}, \mathbf{y})$ having n-connected terms, but the B's are not singletons any more, so $(\mathbf{Y}, \mathbf{B}, \mathbf{y})$ fails to be a system in pro-$\mathscr{C}$. Therefore we need a separate proof for Theorem 2.

For $\lambda \in \Lambda$ denote

$$Y_\lambda := X_\lambda \cup CX^n_\lambda \qquad (cone)$$
$$y_\lambda := x_\lambda,$$

and let $i_\lambda\colon X_\lambda \to Y_\lambda$ be the inclusion.

LEMMA 2. *Under the assumptions of Theorem 2, for each $\lambda \in \Lambda$ there exist a $\lambda' > \lambda$ and a map (or a homotopy class of continuous maps if $\mathscr{C}$ was the homotopy category) $h_{\lambda\lambda'}\colon (Y, y)_{\lambda'} \to (X, x)_\lambda$ such that*

$$h_{\lambda\lambda'} i_{\lambda'} = p_{\lambda\lambda'}. \tag{4}$$

The proof of this lemma is similar to the proof of Lemma 1 and therefore we omit of the details.

The proof of Theorem 2 is now completed by applying Theorem 2.3. as in the proof of Theorem 1.

With the aid of Lemma 2 we can now improve Theorem 1.

*) Theorem 2.3. means Theorem 3 of Section 2. The same notation is used throughout the paper.

THEOREM 3. *Let* $(\mathbf{X},\mathbf{A},\mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *be an n-connected inverse system in* pro-CW_0^2 *or* pro-HCW_0^2, *and let the restricted system* $(\mathbf{A}, \mathbf{x}) = \{(A, x)_\lambda, p_{\lambda\lambda'} \mid A_{\lambda'}, \Lambda\}$ *be m-connected. Then there is an inverse system* $(\mathbf{Y}, \mathbf{B}, \mathbf{y}) = \{(Y, B, y)_\mu, q_{\mu\mu'}, M\}$ *isomorphic to* $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ *in* pro-CW_0^2 *or* pro-HCW_0^2 *respectively, having all the terms* $(Y, B, y)_\mu$ *n-connected and such that the restricted system* $(\mathbf{B}, \mathbf{y})$ *is isomorphic to* $(\mathbf{A}, \mathbf{x})$ *and all terms* $(B, y)_\mu$ *are m-connected.*

Proof. By Theorem 1 we can assume that the terms $(X, A, x)_\lambda$ are n-connected. For $\lambda \in \Lambda$ denote

$$Y_\lambda := X_\lambda \cup CA_\lambda^m$$
$$B_\lambda := A_\lambda \cup CA_\lambda^m$$
$$y_\lambda := x_\lambda,$$

and let $i_\lambda : (X, A, x)_\lambda \to (Y, B, y)_\lambda$ be the inclusion.

Applying Lemma 2 to the m-connected system $(\mathbf{A}, \mathbf{x})$ and the CW-complexes $(B, y)_\lambda$, we conclude that for each $\lambda \in \Lambda$ there exist $\lambda' > \lambda$ and a map $h_{\lambda\lambda'} : (B, y)_{\lambda'} \to (A, x)_\lambda$ such that

$$h_{\lambda\lambda'}\, i_{\lambda'} \mid A_{\lambda'} = p_{\lambda\lambda'} \mid A_{\lambda'}. \tag{5}$$

Define the maps $g_{\lambda\lambda'} : Y_{\lambda'} \to X_\lambda$ by

$$g_{\lambda\lambda'}(y) = \begin{cases} h_{\lambda\lambda'}(y), & y \in B_{\lambda'} \\ p_{\lambda\lambda'}(x), & y = i_{\lambda'}(x),\ x \in X_{\lambda'}. \end{cases}$$

By (5) the maps $g_{\lambda\lambda'}$ are well defined, and since $g_{\lambda\lambda'}(B_{\lambda'}) \subset A_\lambda$, we have

$$g_{\lambda\lambda'} : (Y, B, y)_{\lambda'} \to (X, A, x)_\lambda$$

and

$$g_{\lambda\lambda'}\, i_{\lambda'} = p_{\lambda\lambda'}.$$

Hence we can apply Theorem 2.3. to conclude that the original system $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ is isomorphic to a system consisting of $(Y, B, y)_\lambda$'s. By the construction the terms $(B, y)_\lambda$ are obviously m-connected so we need only to check that we did not make the pairs worse, i. e. that each $(Y, B, y)_\lambda$ is still n-connected. The pair $(X, A, x)_\lambda$ is n-connected and from the exact sequence

$$\pi_1(B, x)_\lambda \to \pi_1(B, A, x)_\lambda \to \pi_0(A, x)_\lambda$$

and the connectivity of A_λ we obtain that the pair $(B, A, x)_\lambda$ is 1-connected for $m \geq 1$. Therefore by the Blakers-Massey Theorem [1] for the triad $(Y; X, B; A)_\lambda$ the homomorphism

$$\pi_k(X, A, x)_\lambda \to \pi_k(Y, B, y)_\lambda$$

induced by the inclusion is an isomorphism for $1 \leq k \leq n$, showing n-connectivity of the pair $(Y, B, y)_\lambda$, which proves the theorem.

Applying the theorems of this section to topological spaces, we obtain

COROLLARY 1. *If* (X, A, x_0) *is a shape n-connected pair of topological spaces, then there exists an associated inverse system* $(\mathbf{X}, \mathbf{A}, \mathbf{x}) \in$ $\in$ pro-HCW_0^2 *having n-connected terms. If in addition A is shape m-connected and P-embedded in X, one can moreover achieve that the* A_λ*'s are m-connected.*

COROLLARY 2. *If* (X, A, x_0) *is a shape n-connected pair of Hausdorff continua, then there exists an inverse system* $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ *of n-connected polyhedral pairs having* (X, A, x_0) *as the inverse limit. If in addition A is shape m-connected, one can moreover achieve m-connected* A_λ*'s.*

COROLLARY 3. *If the topological space X has trivial shape, than there exist an associated inverse system of contractible polyhedra.*

Proof. This corollary is not a consequence of Theorem 2, but the proof is similar. Namely, if the inverse system $\mathbf{X} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$ is associated to the space X having trivial shape, then for each $\lambda \in \Lambda$ there is a $\lambda' > \lambda$ such that $p_{\lambda\lambda'} \simeq 0$. Therefore $p_{\lambda\lambda'}$ can be extended over the cone to get a map $q_{\lambda\lambda'} : CX_{\lambda'} \to X_\lambda$. By Theorem 2.3. the system $\mathbf{X}$ is isomorphic to an inverse system having some CX_λ's for terms, which proves the corollary.

In the case X being a Hausdorff continuum, the proof of Corollary 3 gives an inverse system of contractible polyhedra having X as its inverse limit, which proves a theorem of Felt [3]. In the metric case this result was proved by Hyman [6].

Remark. If X is shape n-connected metric continuum, then by the absolute version of Corollary 2 it can be obtained as the limit of an inverse system of n-connected polyhedra. At this point we apparently lose the fact that X is metric, i. e. that it can be obtained as the limit of an inverse sequence. Namely, if we apply the procedure yielding Corollary 2 we obtain an inverse system of n-connected polyhedra having X as its inverse limit, but the index set is certainly not $\mathbf{N}$ any more. But we can remove this inconvenience by applying Theorem 2.4. to get an inverse sequence of n-connected polyhedra having X as its inverse limit.

Similar remarks hold for Corollaries 1 and 3.

4. Hurewicz theorem in pro-homotopy and shape theory

Using the results of previous sections we give a simple proof of a version of the Hurewicz theorem in pro-homotopy and shape theory. Our theorems imply those of [10] and [12].

Let (X, A, x_0) be a pointed pair of topological spaces and denote by $G \subset \pi_n(X, A, x_0)$ the subgroup generated by the elements of the form $h_\alpha(\beta)\beta^{-1}$ for $\alpha \in \pi_1(A, x_0)$, $\beta \in \pi_n(X, A, x_0)$, where h_α denotes the action of α on $\pi_n(X, A, x_0)$. G is a normal subgroup and $\varphi_n(G) = 0$, where $\varphi_n: \pi_n(X, A, x_0) \to H_n(X, A)$ denotes the Hurewicz homomorphism. Therefore we have the groups $\pi'_n(X, A, x_0) := \pi_n(X, A, x_0)/G$ and the homomorphisms $\varphi'_n: \pi'_n(X, A, x_0) \to H_n(X, A)$, and π'_n is a functor.

In the absolute case one similarly defines $\pi'_n(X, x_0)$ and $\varphi'_n: \pi'_n(X, x_0) \to H_n(X)$.

Let $(\mathbf{X}, \mathbf{A}, \mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ be an inverse system over HCW_0^2. For each $\lambda' \in \Lambda$ let $G_{\lambda'} \subset \pi_n(X, A, x)_{\lambda'}$ be the subgroup generated by the elements $h_\alpha(\beta)\beta^{-1}$ for $\alpha \in \pi_1(A, x)_{\lambda'}$, $\beta \in \pi_n(X, A, x)_{\lambda'}$. For $\lambda < \lambda'$

$$p_{\lambda\lambda'\#}(h_\alpha(\beta)\beta^{-1}) = h_{p_{\lambda\lambda'}\alpha}(p_{\lambda\lambda'\#}\beta)(p_{\lambda\lambda'\#}\beta)^{-1} \in G_\lambda$$

and therefore we get the pro-group

$$\text{pro-}\pi'_n(\mathbf{X}, \mathbf{A}, \mathbf{x}) := \{\pi'_n(X, A, x)_\lambda, p_{\lambda\lambda'\#}, \Lambda\}$$

and the morphism

$$\text{pro-}\varphi'_n: \text{pro-}\pi'_n(\mathbf{X}, \mathbf{A}, \mathbf{x}) \to \text{pro-}H_n(\mathbf{X}, \mathbf{A}).$$

The homomorphisms which $p_{\lambda\lambda'\#}$ induce on the quotients are again denoted by $p_{\lambda\lambda'\#}$.

PROPOSITION 1. *Let* $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ *be an inverse system over* HCW_0^2 *If* pro-$\pi_1(\mathbf{A}, \mathbf{x}) \cong 0$, *then* pro-$\pi'_n(\mathbf{X}, \mathbf{A}, \mathbf{x}) \cong$ pro-$\pi_n(\mathbf{X}, \mathbf{A}, \mathbf{x})$, *i. e.* $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ *is n-simple, for all n.*

Proof. By theorem 3.3. we can assume that all $(A, x)_\lambda$ are 1-connected, hence $\pi'_n(X, A, x)_\lambda \cong \pi_n(X, A, x)_\lambda$, for all λ, which proves the proposition.

Let $(\mathbf{X}, \mathbf{A}, \mathbf{x}) \in$ pro-HCW_0^2. Because of the naturality of $\varphi_n: \pi_n \to H_n$ and $\varphi'_n: \pi'_n \to H_n$ we have well defined Hurewicz morphisms pro-φ_n: pro-$\pi_n(\mathbf{X}, \mathbf{A}, \mathbf{x}) \to$ pro-$H_n(\mathbf{X}, \mathbf{A})$ and pro-φ_n: pro-$\pi'_n(\mathbf{X}, \mathbf{A}, \mathbf{x}) \to$ pro-$H_n(\mathbf{X}, \mathbf{A})$.

We are now ready to prove Hurewicz theorem in pro-HCW_0^2.

THEOREM 1. Let

(*i*) $(\mathbf{X}, \mathbf{A}, \mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *be an* $(n-1)$*-connected inverse system over* HCW_0^2, $n \geq 2$.

(*a*) *Then*

$$\text{pro-}\varphi'_k: \text{pro-}\pi'_k(\mathbf{X}, \mathbf{A}, \mathbf{x}) \to \text{pro-}H_k(\mathbf{X}, \mathbf{A})$$

are isomorphisms for $1 \leq k \leq n$.

(*b*) *If* $n = 2$ *and* pro-$\pi_1(\mathbf{X}, \mathbf{x}) \cong 0$, then Ker pro-$\varphi_2$ *is isomorphic to the pro-group* $\{K_\lambda, p_{\lambda\lambda'\#} \mid K_{\lambda'}, \Lambda\}$ *where* K_λ *is the commutator subgroup of* $\pi_2(X, A, x)_\lambda$.

(*c*) *If in addition to* (*i*) pro-$\pi_1(\mathbf{A}, \mathbf{x}) \cong 0$ *then*

$$\text{pro-}\varphi_{n+1}: \text{pro-}\pi_{n+1}(\mathbf{X}, \mathbf{A}, \mathbf{x}) \to \text{pro-}H_{n+1}(\mathbf{X}, \mathbf{A})$$

is an epimorphism of pro-groups.

Proof. (*a*) By Theorem 3.1. we may assume that the terms $(X, A, x)_\lambda$ are $(n-1)$-connected, and applying the classical Hurewicz theorem (e. g. [13]) we conclude that $\varphi'_{k\lambda}: \pi'_k(X, A, x)_\lambda \to H_k(X, A)_\lambda$ are isomorphisms for $1 \leq k \leq n$. Since we are dealing with level morphisms, we get the result for pro-H_k and pro-φ'_n.

(*b*) By Theorem 3.1. we may assume $\pi_1(X, A, x)_\lambda = 0$ for all $\lambda \in \Lambda$. Let

$$Y_\lambda := X_\lambda \cup CX_\lambda^1, \quad \lambda \in \Lambda.$$

Since pro-$\pi_1(\mathbf{X}, \mathbf{x}) \cong 0$ we have for every λ a $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'\#}: \pi_1(X, x)_{\lambda'} \to \pi_1(X, x)_\lambda$ is the null-homomorphism, and therefore $p_{\lambda\lambda'}$ can be extended to $g_{\lambda\lambda'}: (Y, A, x)_{\lambda'} \to (X, A, x)_\lambda$ giving

$$g_{\lambda\lambda'} i_{\lambda'} = p_{\lambda\lambda'}$$

where $i_{\lambda'}: (X, A, x)_{\lambda'} \to (Y, A, x)_{\lambda'}$ denotes the inclusion. By Theorem 2.3. we conclude that the original system $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ is isomorphic to an inverse system $(\mathbf{W}, \mathbf{B}, \mathbf{w}) = \{(W, B, w)_\mu, r_{\mu\mu'}, M\}$ where each $(W, B, w)_\mu$ is some $(Y, A, x)_\lambda$. Therefore $\pi_1(W, w)_\mu = \pi_1(W, B, w)_\mu = 0$ for every μ. Namely, $\pi_1(Y, x)_\lambda = 0$, $\lambda \in \Lambda$, by the construction of Y_λ, and from the exact sequence

$$\pi_1(Y, x)_\lambda \to \pi_1(Y, A, x)_\lambda \to \pi_0(A, x)_\lambda$$

and connectivity of A_λ we obtain $\pi_1(Y, A, x)_\lambda = 0$, $\lambda \in \Lambda$.

By applying the classical result on Ker φ_2, we obtain the assertion on Ker pro-φ_2.

(*c*) If we also have pro-$\pi_1(\mathbf{A}, \mathbf{x}) \cong 0$ then by Theorem 3.3. we can, in addition to $(n-1)$-connectedness of the pairs $(X, A, x)_\lambda$, assume 1-connectedness of $(A, x)_\lambda$. Applying a version of the classical

Hurewicz theorem (e. g. [5]) we conclude that $\varphi_{n+1,\lambda}: \pi_{n+1}(X, A, x)_\lambda \to \to H_{n+1}(X, A)_\lambda$ are epimorphisms, hence pro-φ_{n+1} is an epimorphism of pro-groups, which proves the theorem.

Note that Theorem 1 and Proposition 1 imply Theorem 1 of [10].

In the absolute case we obtain

THEOREM 2. *Let* $(\mathbf{X}, \mathbf{x}) = \{(X, x)_\lambda, p_{\lambda\lambda'}, \Lambda\} \in$ pro-HCW_0 *be an* $(n-1)$*-connected inverse system,* $n \geqslant 2$. Then

$$\text{pro-}\varphi_k: \text{pro-}\pi_k(\mathbf{X}, \mathbf{x}) \to \text{pro-}H_k(\mathbf{X})$$

are isomorphisms for $1 \leqslant k \leqslant n$, *and*

$$\text{pro-}\varphi_{n+1}: \text{pro-}\pi_{n+1}(\mathbf{X}, \mathbf{x}) \to \text{pro-}H_{n+1}(\mathbf{X})$$

is an epimorphism of pro-groups.

If $n = 1$ *and all* X_λ *are connected, then*

$$\text{pro-}\varphi_1': \text{pro-}\pi_1'(\mathbf{X}, \mathbf{x}) \to \text{pro-}H_1(\mathbf{X})$$

is an isomorphism of pro-groups. Ker pro-φ_1 *is isomorphic to the pro--group* $\{K_\lambda, p_{\lambda\lambda'\#} \mid K_{\lambda'}, \Lambda\}$, *where* K_λ *denotes the commutator subgroup of* $\pi_1(X, x)_\lambda$, $\lambda \in \Lambda$.

Proof. In case $n \geqslant 2$ we may (Theorem 3.2.) assume $(n-1)$-connectedness of the terms $(X, x)_\lambda$ and apply the classical Hurewicz theorem (e. g. [5]). The case $n = 1$ follows from the definition of pro-$\pi_1'(\mathbf{X}, \mathbf{x})$ and the Poincaré theorem for φ_1.

The previous theorem is a slightly improved version of theorem 2 in [10].

From Theorems 1 and 2 we get the following results for topological spaces:

COROLLARY 1. *Let* (X, A, x_0) *be a shape* $(n-1)$*-connected pair of topological spaces, where A is P-embedded in X,* $n \geqslant 2$. *Then* pro-$H_k(X, A) \cong 0$ *and* $H_k(X, A) = 0$ *for* $1 \leqslant k \leqslant n-1$. *Furthermore*

$$\text{pro-}\varphi_n': \text{pro-}\pi_n'(X, A, x_0) \to \text{pro-}H_n(X, A)$$

and

$$\check{\varphi}_n': \check{\pi}_n'(X, A, x_0) \to H_n(X, A)$$

are isomorphisms of pro-groups and groups respectively.

If in addition (A, x_0) is shape 1-connected, then

$$\text{pro-}\varphi_{n+1}: \text{pro-}\pi_{n+1}(X, A, x_0) \to \text{pro-}H_{n+1}(X, A)$$

is an epimorphism of pro-groups.

H_k denotes the Čech homology functor, $\check{\pi}_n'(X, A, x_0) = = \varprojlim \text{pro-}\pi_n'(X, A, x_0)$ and $\check{\varphi}_n = \varprojlim \text{pro-}\varphi_n'$.

COROLLARY 2. *Let* (X, x_0) *be a shape* $(n-1)$*-connected topological space,* $n \geqslant 2$. *Then*

$$\text{pro-}H_k(X) \cong 0 \text{ and } H_k(X) = 0 \text{ for } 1 \leqslant k \leqslant n-1,$$

$$\text{pro-}\varphi_n: \text{pro-}\pi_n(X, x_0) \to \text{pro-}H_n(X)$$

and

$$\check{\varphi}_n: \check{\pi}_n(X, x_0) \to H_n(X)$$

are isomorphisms, and

$$\text{pro-}\varphi_{n+1}: \text{pro-}\pi_{n+1}(X, x_0) \to \text{pro-}H_{n+1}(X)$$

is an epimorphism of pro-groups.

If $n = 1$, *then*

$$\text{pro-}\varphi_1': \text{pro-}\pi_1'(X, x_0) \to \text{pro-}H_1(X)$$

and

$$\check{\varphi}_1': \check{\pi}_1'(X, x_0) \to H_1(X)$$

are isomorphisms.

5. Blakers-Massey theorem in pro-homotopy and shape theory

In calculating homology groups one often finds the excision axiom very useful. Homotopy groups enjoy this property only to certain extent described by the Blakers-Massey theorem [1], [14]. Here we prove the analogous theorems for pro-groups and shape groups.

THEOREM 1. *Let*

$$(\mathbf{X}; \mathbf{X}_1, \mathbf{X}_2; \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x}) = \{(X; X_1, X_2; X_1 \cap X_2, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

be an object in pro-HCW_0^{triad} *such that the restriction*

$$(\mathbf{X}_1, \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x}) = \{(X_1, X_1 \cap X_2, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

is n-connected in pro-HCW_0^2 *and*

$$(\mathbf{X}_2, \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x}) = \{(X_2, X_1 \cap X_2, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

is m-connected in pro-HCW_0^2, $n, m \geqslant 1$. *Then*

$$\mathbf{j}_\#: \text{pro-}\pi_k(\mathbf{X}_1, \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x}) \to \text{pro-}\pi_k(\mathbf{X}, \mathbf{X}_2, \mathbf{x})$$

is an isomorphism for $1 \leqslant k \leqslant n+m-1$ *and an epimorphism for* $k = n+m$, *where the morphism*

$$\mathbf{j}: (\mathbf{X}_1, \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x}) \to (\mathbf{X}, \mathbf{X}_2, \mathbf{x})$$

is induced by appropriate inclusions.

We prove first a lemma.

LEMMA 1. Let $(X; X_1, X_2; X_1 \cap X_2)$ be a CW-triad and let

$$Y_1 := X_1 \times \{0\} \cup (X_1^n \cup (X_1 \cap X_2)) \times I \cup X_2^m \times \{1\}$$
$$Y_2 := X_2 \times \{0\} \cup (X_2^m \cup (X_1 \cap X_2)) \times I \cup X_1 \times \{1\}.$$

Then the pair $(Y_1, Y_1 \cap Y_2)$ is n-connected and $(Y_2, Y_1 \cap Y_2)$ is m-connected.

Proof. Note first that

$$Y_1 \cap Y_2 = (X_1 \cap X_2) \times I \cup (X_1^n \cup X_2^m) \times \{1\}.$$

We show that the pair $(Y_1, Y_1 \cap Y_2)$ is n-connected.

Let the map

$$f: (I^r, I^{r-1}, J^{r-1}) \to (Y_1, Y_1 \cap Y_2, x)$$

represent an element in $\pi_r(Y_1, Y_1 \cap Y_2, x)$, $r \leqslant n$. By the cellular approximation theorem we can choose f to be cellular, i. e. $f((I^r)^p) \subset Y_1^p, p \leqslant r$. In particular

$$f(I^r) \subset Y_1^n = (X_1^n \times \{0\}) \cup (X_1^{n-1} \times I) \cup (X_1 \cup X_2^m)^n \times \{1\} \subset$$
$$\subset (X_1^n \times I) \cup (Y_1 \cap Y_2).$$

By a standard argument one can see that there exists a strong deformation retraction with the final stage

$$d: (X_1^n \times I) \cup (Y_1 \cap Y_2) \to Y_1 \cap Y_2.$$

Since $f \simeq df: (I^r, I^{r-1}, J^{r-1}) \to (Y_1, Y_1 \cap Y_2, x)$ and $df(I^r) \subset Y_1 \cap Y_2$, we have $[f] = 0 \in \pi_r(Y_1, Y_1 \cap Y_2, x)$.

Proof of Theorem 1. For each $\lambda \in \Lambda$ let let $Y_{1\lambda}$ and $Y_{2\lambda}$ be defined as in Lemma 1, using $X_{1\lambda}$, and $X_{2\lambda}$, and let $Y_\lambda := Y_{1\lambda} \cup Y_{2\lambda}$, $y_\lambda := x_\lambda \times \{0\}$. We shall show that for each $\lambda \in \Lambda$ there exist a $\lambda' > \lambda$ and a map

$$h_{\lambda\lambda'}: (Y; Y_1, Y_2; Y_1 \cap Y_2, y)_{\lambda'} \to (X; X_1, X_2; X_1 \cap X_2, x)_\lambda$$

such that

$$h_{\lambda\lambda'} i_{\lambda'} = p_{\lambda\lambda'} \tag{1}$$

where $i_{\lambda'}: X_{\lambda'} \to Y_{\lambda'}$ denotes the inclusion of triads.

Consider the restricted system $(\mathbf{X}_1, \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x})$ which is n-connected. By Lemma 3.1. for each $\lambda \in \Lambda$ there exist a $\lambda_1 > \lambda$ and a map

$$g_{\lambda\lambda_1}: ((X_1^n \cup (X_1 \cap X_2))_{\lambda_1} \times I \cup X_{1\lambda_1} \times \{0\}, (X_1 \cap X_2)_{\lambda_1} \times I \cup X_{1\lambda_1}^n \times \{1\}) \to$$
$$\to (X_1, X_1 \cap X_2)_\lambda$$

such that

$$g_{\lambda\lambda_1} i_{\lambda_1} = p_{\lambda\lambda_1} \tag{2}$$

and

$$g_{\lambda\lambda_1}(x, t) = p_{\lambda\lambda_1}(x) \text{ for } x \in (X_1 \cap X_2)_{\lambda_1}, \ t \in I. \tag{3}$$

Note that for each $\lambda' \geqslant \lambda_1$ there exists a map $g_{\lambda\lambda'}$ with the same property. Namely, if we define

$$g_{\lambda\lambda'} := g_{\lambda\lambda_1}(p_{\lambda_1\lambda'} \times 1_I): (X_1^n \cup (X_1 \cap X_2))_{\lambda'} \times I \cup X_{1\lambda'} \times \{0\} \to X_{1\lambda}$$

and choosing $p_{\lambda_1\lambda'}$ cellular, we have

$$g_{\lambda\lambda'}((X_1 \cap X_2)_{\lambda'} \times I \cup X_{1\lambda'}^n \times \{1\}) =$$
$$= g_{\lambda\lambda_1}(p_{\lambda_1\lambda'}(X_1 \cap X_2)_{\lambda'} \times I \cup p_{\lambda_1\lambda'}(X_{1\lambda'}^n) \times \{1\}) \subset$$
$$\subset g_{\lambda\lambda_1}((X_1 \cap X_2)_{\lambda_1} \times I \cup X_{1\lambda_1}^n \times \{1\}) \subset (X_1 \cap X_2)_\lambda.$$

Furthermore

$$g_{\lambda\lambda'} i_{\lambda'} = g_{\lambda\lambda_1}(p_{\lambda_1\lambda'} \times 1_I) i_{\lambda'} = g_{\lambda\lambda_1} i_{\lambda_1} p_{\lambda_1\lambda'} =$$
$$\overset{(2)}{=} p_{\lambda\lambda_1} p_{\lambda_1\lambda'} = p_{\lambda\lambda'}$$

and for $x \in (X_1 \cap X_2)_{\lambda'}$, $t \in I$ we have

$$g_{\lambda\lambda'}(x, t) = g_{\lambda\lambda_1}(p_{\lambda_1\lambda'} \times 1_I)(x, t) = g_{\lambda\lambda_1}(p_{\lambda_1\lambda'}(x), t) =$$
$$\overset{(3)}{=} p_{\lambda\lambda_1} p_{\lambda_1\lambda'}(x) = p_{\lambda\lambda'}(x)$$

and therefore $g_{\lambda\lambda'}$ is a map of pairs satisfying (2) and (3).

Similarly, beacuse of the m-connectedness of the system $(\mathbf{X}_2, \mathbf{X}_1 \cap \mathbf{X}_2, x)$, for each $\lambda \in \Lambda$ there exists a $\lambda_2 > \lambda$ such that for each $\lambda' \geqslant \lambda_2$ there is a map

$$f_{\lambda\lambda'}: ((X_2^m \cup (X_1 \cap X_2))_{\lambda'} \times I \cup X_{2\lambda'} \times \{0\}, (X_1 \cap X_2)_{\lambda'} \times I \cup$$
$$\cup X_{2\lambda'}^m \times \{1\}) \to (X_2, X_1 \cap X_2)_\lambda$$

satisfying

$$f_{\lambda\lambda'} i_{\lambda'} = p_{\lambda\lambda'} \tag{4}$$

and

$$f_{\lambda\lambda'}(x, t) = p_{\lambda\lambda'}(x) \text{ for } x \in (X_1 \cap X_2)_{\lambda'}, \quad t \in I. \tag{5}$$

For $\lambda \in \Lambda$ let $\lambda_1, \lambda_2 > \lambda$ be as above and choose $\lambda' \geqslant \lambda_1, \lambda_2$. Then

$$((X_1^n \cup (X_1 \cap X_2))_{\lambda'} \times I \cup X_{1\lambda'} \times \{0\}) \cup ((X_2^m \cup (X_1 \cap X_2))_{\lambda'} \times I \cup X_{2\lambda'} \times \{0\}) = Y_{\lambda'}$$

and hence we define $h_{\lambda\lambda'}\colon Y_{\lambda'} \to X_\lambda$ by

$$h_{\lambda\lambda'}(x,t) = \begin{cases} g_{\lambda\lambda'}(x,t) & x \in X_{1\lambda'}, \quad t \in I \\ f_{\lambda\lambda'}(x,t) & x \in X_{2\lambda'}, \quad t \in I. \end{cases}$$

By (3) and (5) $h_{\lambda\lambda'}$ is well defined and $h_{\lambda\lambda'}(Y_{1\lambda'}) \subset X_{1\lambda}$ and $h_{\lambda\lambda'}(Y_{2\lambda'}) \subset X_{2\lambda}$. Therefore $h_{\lambda\lambda'}$ is a map of triads

$$h_{\lambda\lambda'}\colon (Y;\, Y_1,\, Y_2;\, Y_1 \cap Y_2, y)_{\lambda'} \to (X;\, X_1, X_2;\, X_1 \cap X_2, x)_\lambda$$

satisfying (1).

Applying Theorem 2.3. we obtain the inverse system $(\mathbf{Y}; \mathbf{Y}_1, \mathbf{Y}_2; \mathbf{Y}_1 \cap \mathbf{Y}_2, \mathbf{y})$ consisting of the triads constructed above, which satisfy, by Lemma 1, the assumptions of the classical Blakers-Massey theorem. Therefore

$$j_{\mu\#}\colon \pi_k(Y_1, Y_1 \cap Y_2, y)_\mu \to \pi_k(Y, Y_2, y)_\mu$$

are isomorphisms for $1 \leqslant k \leqslant n+m-1$ and epimorphisms for $k = n+m$, for all μ.

The theorem now follows from the commutative diagram of pro-groups

$$\begin{array}{ccc} \text{pro-}\pi_k(\mathbf{X}_1, \mathbf{X}_1 \cap \mathbf{X}_2, \mathbf{x}) & \longrightarrow & \text{pro-}\pi_k(\mathbf{X}, \mathbf{X}_2, \mathbf{x}) \\ \downarrow \cong & & \downarrow \cong \\ \text{pro-}\pi_k(\mathbf{Y}_1, \mathbf{Y}_1 \cap \mathbf{Y}_2, \mathbf{y}) & \xrightarrow{\cong} & \text{pro-}\pi_k(\mathbf{Y}, \mathbf{Y}_2, \mathbf{y}). \end{array}$$

For topological spaces we have

COROLLARY 1. *Let $X = X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$ and assume all the inclusions $X_1 \cap X_2 \subset X_i \subset X$, $i = 1, 2$ are P-embeddings. If the pair $(X_1, X_1 \cap X_2, x_0)$ is shape n-connected and the pair $(X_2, X_1 \cap X_2, x_0)$ shape m-connected, $n, m \geqslant 1$, then the morphism*

$$\mathbf{j}_\#\colon \text{pro-}\pi_k(X_1, X_1 \cap X_2, x_0) \to \text{pro-}\pi_k(X, X_2, x_0)$$

induced by the inclusion, is an isomorphism for $k \leqslant n+m-1$ and an epimorphism for $k = n+m$, and

$$j_\#\colon \check{\pi}_k(X_1, X_1 \cap X_2, x_0) \to \check{\pi}_k(X, X_2, x_0)$$

is an isomorphism for $k \leqslant n+m-1$.

In the movable metric case one has

COROLLARY 2. *Let $(X; X_1, X_2; X_1 \cap X_2, x_0)$ be a movable pointed triad of metric continua such that*

$$\check{\pi}_k(X_1, X_1 \cap X_2, x_0) = 0, \quad 1 \leqslant k \leqslant n$$
$$\check{\pi}_k(X_2, X_1 \cap X_2, x_0) = 0, \quad 1 \leqslant k \leqslant m$$

and let $j\colon (X_1, X_1 \cap X_2, x_0) \to (X, X_2, x_0)$ be the inclusion. Then the induced morphisms

$$\mathbf{j}_\#\colon \text{pro-}\pi_k(X_1, X_1 \cap X_2, x_0) \to \text{pro-}\pi_k(X, X_2, x_0)$$

and

$$j_\#\colon \check{\pi}_k(X_1, X_1 \cap X_2, x_0) \to \check{\pi}_k(X, X_2, x_0)$$

are isomorphisms for $1 \leqslant k \leqslant n+m-1$ and epimorphisms for $k = n+m$, of pro-groups and groups respectively.

In the homotopy theory, using Blakers-Massey theorem, one usually proves the following theorem (e. g. [14]): If $A \subset X$ is a cofibration, A m-connected, (X, A) n-connected, then the identification map $\varrho\colon (X, A) \to (X/A, x_0)$ induces isomorphisms $\varrho_\#\colon \pi_k(X, A, x_0) \to \pi_k(X/A, x_0)$ for $k \leqslant n+m$ and an epimorphism for $k = n+m+1$. Before proving the analogous theorem for pro-groups, we need some lemmas.

LEMMA 2. *Let (X, A, x_0) be a pointed pair of topological spaces and let $\varrho\colon (X, A, x_0) \to (X/A, x_0, x_0)$ be the identification map, where the class A/A of the base point is again denoted by x_0. Then the homotopy class q defined by ϱ is an epimorphism in the category $HTOP_0$.*

Proof. Let $\varphi, \psi\colon (X/A, x_0) \to (Y, y_0)$ be two maps of pointed topological spaces such that

$$\varphi\varrho \simeq \psi\varrho\colon (X, A, x_0) \to (Y, y_0, y_0)$$

and let $H\colon (X \times I, A \times I, x_0 \times I) \to (Y, y_0, y_0)$ be the homotopy from $\varphi\varrho$ to $\psi\varrho$, i. e.

$$\begin{aligned} H(x, 0) &= \varphi\varrho(x) \\ H(x, 1) &= \psi\varrho(x), \quad x \in X. \end{aligned} \tag{6}$$

Define the homotopy $H'\colon ((X/A) \times I, x_0 \times I) \to (Y, y_0)$ by

$$H'([x], t) = H(x, t). \tag{7}$$

Since I is compact, the map

$$\varrho \times 1_I\colon (X \times I, A \times I, x_0 \times I) \to ((X/A) \times I, x_0 \times I, x_0 \times I)$$

is an identification, and from the commutative diagram

$$\begin{array}{ccc} (X \times I, A \times I, x_0 \times I) & & \\ \downarrow{\scriptstyle \varrho \times 1_I} & \overset{H}{\searrow} & (Y, y_0, y_0) \\ ((X/A) \times I, x_0 \times I, x_0 \times I) & \underset{H'}{\nearrow} & \end{array}$$

it follows that H' is continuous.

From (7) and (6) it is clear that H' is the required homotopy between φ and ψ.

COROLLARY 3. (cf. [7]). *Let* $\varphi \simeq \psi : (X, A, x_0) \to (Y, B, y_0)$ *be two homotopic maps between pointed pairs of topological spaces. Then the maps* $\tilde{\varphi}, \tilde{\psi} : (X/A, x_0) \to (Y/B, y_0)$ *induced on the identification spaces by* φ *and* ψ *respectively, are again homotopic. Therefore, every homotopy class* $f : (X, A, x_0) \to (Y, B, y_0)$ *induces a homotopy class* $\tilde{f} : (X/A, x_0) \to$ $\to (Y/B, y_0)$ *and*

$$\tilde{f}\, q_X = q_Y f : (X, A, x_0) \to (Y/B, y_0, y_0)$$

where q_X *and* q_Y *denote the homotopy classes of the appropriate identification maps.*

Proof. Let ϱ_X and ϱ_Y be the identification maps. Then

$$\tilde{\varphi}\, \varrho_X = \varrho_Y \varphi \simeq \varrho_Y \psi = \tilde{\psi}\, \varrho_X : (X, A, x_0) \to (Y/B, y_0, y_0)$$

and by Lemma 2 we get

$$\tilde{\varphi} \simeq \tilde{\psi} : (X/A, x_0) \to (Y/B, y_0).$$

As a consequence we obtain

COROLLARY 4. *Let* $(\mathbf{X}, \mathbf{A}, \mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *be an inverse system over* HCW_0^2. *Then there is a well defined inverse system* $(\mathbf{X}/\mathbf{A}, \mathbf{x}) = \{(X/A, x)_\lambda, \tilde{p}_{\lambda\lambda'}, \Lambda\} \in$ pro-HCW_0 *and a level morphism* $\mathbf{q} : (\mathbf{X}, \mathbf{A}, \mathbf{x}) \to (\mathbf{X}/\mathbf{A}, \mathbf{x}, \mathbf{x})$ *induced by the map of systems* $(1_\Lambda, q_\lambda)$.

LEMMA 3. *Let the inverse systems* $(\mathbf{X}, \mathbf{A}, \mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\}$ *and* $(\mathbf{Y}, \mathbf{B}, \mathbf{y}) = \{(Y, B, y)_\mu, q_{\mu\mu'}, M\}$ *be isomorphic in* pro-HCW_0^2. *Then the systems* $(\mathbf{X}/\mathbf{A}, \mathbf{x})$ *and* $(\mathbf{Y}/\mathbf{B}, \mathbf{y})$ *are isomorphic in* pro-HCW_0.

Proof. Let the morphisms $\mathbf{f} : (\mathbf{X}, \mathbf{A}, \mathbf{x}) \to (\mathbf{Y}, \mathbf{B}, \mathbf{y})$ and $\mathbf{g} : (\mathbf{Y}, \mathbf{B}, \mathbf{y}) \to (\mathbf{X}, \mathbf{A}, \mathbf{x})$ be induced by the maps of systems (f, f_μ) and (g, g_λ) respectively, and let $\mathbf{f}\,\mathbf{g} = \mathbf{1}$ and $\mathbf{g}\,\mathbf{f} = \mathbf{1}$.

Let $\mu \leqslant \mu'$ in M. Since (f, f_μ) is a map of systems, there exists a $\lambda \geqslant f(\mu), f(\mu')$ such that

$$f_\mu\, p_{f(\mu)\lambda} = q_{\mu\mu'}\, f_{\mu'}\, p_{f(\mu')\lambda},$$

hence by Corollary 3

$$\tilde{f}_\mu\, \tilde{p}_{f(\mu)\lambda} = \tilde{q}_{\mu\mu'}\, \tilde{f}_{\mu'}\, \tilde{p}_{f(\mu')\lambda}.$$

Therefore $(f, \tilde{f}_\mu) : (\mathbf{X}/\mathbf{A}, \mathbf{x}) \to (\mathbf{Y}/\mathbf{B}, \mathbf{y})$ is a map of systems, and similarly for $(g, \tilde{g}_\mu) : (\mathbf{Y}/\mathbf{B}, \mathbf{y}) \to (\mathbf{X}/\mathbf{A}, \mathbf{x})$.

Using Corollary 3 again, one easily sees that the induced morphism $\tilde{f} : (\mathbf{X}/\mathbf{A}, \mathbf{x}) \to (\mathbf{Y}/\mathbf{B}, \mathbf{y})$ is an isomorphism.

Results similar to Corollary 3 and Lemma 2 are proved in [7].

We are now ready to prove

THEOREM 2. *Let* $(\mathbf{X}, \mathbf{A}, \mathbf{x})$ *be an* n*-connected inverse system in* pro-HCW_0^2 *and let the restricted system* $(\mathbf{A}, \mathbf{x})$ *be* m*-connected,* $n, m \geqslant 1$. *Then the morphism*

$$\mathbf{q} : (\mathbf{X}, \mathbf{A}, \mathbf{x}) \to (\mathbf{X}/\mathbf{A}, \mathbf{x}, \mathbf{x})$$

nduces the morphisms of pro-groups

$$\mathbf{q}_{\#} : \text{pro-}\pi_k(\mathbf{X}, \mathbf{A}, \mathbf{x}) \to \text{pro-}\pi_k(\mathbf{X}/\mathbf{A}, \mathbf{x})$$

which are isomorphisms for $1 \leqslant k \leqslant n + m$ *and epimorphism for* $k = n + m + 1$.

Proof. By Theorem 3.3 choose an inverse system $(\mathbf{Y}, \mathbf{B}, \mathbf{y}) \in$ $\in$ pro-HCW_0^2 isomorphic to $(\mathbf{X}, \mathbf{A}, \mathbf{x})$, whose terms are n-connected and such that the restricted system $(\mathbf{B}, \mathbf{y})$ is isomorphic to $(\mathbf{A}, \mathbf{x})$ and the terms are m-connected. Because of Lemma 3 the systems $(\mathbf{X}/\mathbf{A}, \mathbf{x})$ and $(\mathbf{Y}/\mathbf{B}, \mathbf{y})$ are isomorphic, hence it suffices to prove the theorem for the system $(\mathbf{Y}, \mathbf{B}, \mathbf{y})$. But, by the classical theorem quoted before Lemma 2, the homotopy classes of the identification maps $q_\mu : (Y, B, y)_\mu \to (Y/B, y, y)_\mu$ induce isomorphisms of homotopy groups for $1 \leqslant k \leqslant n + m$, and epimorphisms for $k = n + m + 1$, for all μ, which proves the theorem.

To state such a theorem for topological spaces, we need

LEMMA 4. *Let the inverse system* $(\mathbf{X}, \mathbf{A}, \mathbf{x}) = \{(X, A, x)_\lambda, p_{\lambda\lambda'}, \Lambda\} \in$ $\in$ pro-HCW_0^2 *be associated to the pointed pair of topological spaces* (X, A, x_0). *Then the system* $(\mathbf{X}/\mathbf{A}, \mathbf{x}) = \{(X/A, x)_\lambda, \tilde{p}_{\lambda\lambda'}, \Lambda\}$ *is associated to the identification space* $(X/A, x_0)$.

Proof. We check the conditions of Definition 1.2. of [11]. For every $\lambda \in \Lambda$, by Corollary 3, we have a homotopy class of continuous maps $\tilde{p}_\lambda: (X/A, x_0) \to (X/A, x)_\lambda$ induced by $p_\lambda: (X, A, x_0) \to \to (X, A, x)_\lambda$, and $\tilde{p}_{\lambda\lambda'} \tilde{p}_{\lambda'} = \tilde{p}_\lambda$ for every $\lambda \leqslant \lambda'$.

Let $q: (X, A) \to (X/A, x_0)$ denote the homotopy class of the identification map and let $f: (X/A, x_0) \to (P, p_0)$ be a homotopy class of continuous maps into a pointed CW-complex. Since the system $(\mathbf{X}, \mathbf{A})$ is associated to the pair (X, A), the composition $f q: (X, A) \to (P, p_0)$ factors through the system, i.e. there is a $\lambda \in \Lambda$ and a homotopy class $f_\lambda: (X, A)_\lambda \to \to (P, p_0)$ such that $f_\lambda p_\lambda = f q$. For the homotopy class $\tilde{f}_\lambda: (X/A, x)_\lambda \to \to (P, p_0)$ induced by f_λ, we find

$$\tilde{f}_\lambda \tilde{p}_\lambda q = \tilde{f}_\lambda q_\lambda p_\lambda = f_\lambda p_\lambda = f q,$$

hence, by Lemma 2, $\tilde{f}_\lambda \tilde{p}_\lambda = f$.

Finally, for $\lambda \in \Lambda$, let $f_\lambda, g_\lambda: (X/A, x)_\lambda \to (P, p_0)$ be two homotopy classes such that $f_\lambda \tilde{p}_\lambda = g_\lambda \tilde{p}_\lambda$. Then

$$f_\lambda \tilde{p}_\lambda q = g_\lambda \tilde{p}_\lambda q: (X, A) \to (P, p_0)$$

and therefore $f_\lambda q_\lambda p_\lambda = g_\lambda q_\lambda p_\lambda$. Since the system $(\mathbf{X}, \mathbf{A})$ is associated to the pair (X, A), there is a $\lambda' \geqslant \lambda$ such that

$$f_\lambda q_\lambda p_{\lambda\lambda'} = g_\lambda q_\lambda p_{\lambda\lambda'}: (X, A)_{\lambda'} \to (P, p_0),$$

and therefore

$$f_\lambda \tilde{p}_{\lambda\lambda'} q_{\lambda'} = f_\lambda q_\lambda p_{\lambda\lambda'} = g_\lambda q_\lambda p_{\lambda\lambda'} = g_\lambda \tilde{p}_{\lambda\lambda'} q_{\lambda'}.$$

By Lemma 2 we thus have

$$f_\lambda \tilde{p}_{\lambda\lambda'} = g_\lambda \tilde{p}_{\lambda\lambda'}$$

proving the lemma.

In the case of Hausdorff compacta this result was proved in [7].

From Theorem 2 and Lemma 4 we obtain

COROLLARY 5. *Let (X, A, x_0) be a shape n-connected pointed pair of topological spaces, and let A be P-embedded in X and shape m-connected. Then the identification map $q: (X, A) \to (X/A, x_0)$ induces morphisms of pro-groups*

$$\mathbf{q}_\#: \text{pro-}\pi_k (X, A, x_0) \to \text{pro-}\pi_k (X/A, x_0)$$

which are isomorphisms for $1 \leqslant k \leqslant m + n$ and epimorphism for $k = n + m$, and isomorphisms of shape groups

$$q_\#: \check{\pi}_k (X, A, x_0) \to \check{\pi}_k (X/A, x_0)$$

for $1 \leqslant k \leqslant n + m$.

In the movable case we have

COROLLARY 6. *Let (X, A, x_0) be a movable pointed pair of metric continua such that $\check{\pi}_k(X, A, x_0) = 0$ for $1 \leqslant k \leqslant n$ and $\check{\pi}_k(A, x_0) = 0$ for $1 \leqslant k \leqslant m$. Then the identification map $q: (X, A) \to (X/A, x_0)$ induces morphisms of homotopy pro-groups*

$$\mathbf{q}_\#: \text{pro-}\pi_k (X, A, x_0) \to \text{pro-}\pi_k (X/A, x_0)$$

and homomorphisms of shape groups

$$q_\#: \check{\pi}_k (X, A, x_0) \to \check{\pi}_k (X/A, x_0)$$

which are isomorphisms for $1 \leqslant k \leqslant n + m$ and epimorphisms for $k = n + m + 1$.

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