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Coherent States for Hopf Algebras

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Abstract. Families of Perelomov coherent states are defined axiomatically in the context of unitary representations of Hopf algebras. A global geometric picture involving locally trivial noncommutative fibre bundles is involved in the construction. If, in addition, the Hopf algebra has a left Haar integral, then a formula for noncommutative resolution of identity in terms of the family of coherent states holds. Examples come from quantum groups.

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Perelomov's construction of coherent states [20,21] here extended to a class of Hopf algebras, classically goes as follows: given a real Lie group G, and a unitary irreducible representation $T: G \to \operatorname{Aut} V$ on a complex Hilbert space V, fix a vector v_0 in V with projective isotropy subgroup $H \subset G$ (i.e. $h \in H$ iff hv_0 equals v_0 up to a constant phase). There is a unitary character $\chi : H \to S^1$ such that $hv_0 = \chi(h)v_0$ for each $h \in H$. For G compact, the representation T extends to a representation of the complexification $G^{\mathbb{C}}$ of G.

A **family of Perelomov coherent vectors** in V is a family of vectors $\{C(u), u \in G/H\}$, such that $C([g]) = T(g)v_0$ up to a phase. Coherent *states* are projective classes (rays) of coherent *vectors*, but in practice one often says "coherent states" for both notions. If V is constructed by the method of geometric quantization, i.e. as the space of holomorphic sections ΓL of the corresponding quantization line bundle L over G/H, then the coherent vectors may be defined invariantly in terms of that line bundle [22]. For G a compact form of a semisimple Lie group $G^{\mathbb{C}}$, the details are in Sect. 3 below.

Hopf algebras appear in physics as symmetries of noncommutative and quantum spaces [6,15–18,37]. Algebra $\mathcal{O}(G) = \Gamma\mathcal{O}_G$ of regular functions on affine algebraic group G are commutative examples of Hopf algebras [11] with coproduct $\Delta: \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$ given by $(\Delta f)(g_1, g_2) = f(g_1 \cdot g_2)$. In fact, the category of commutative Hopf algebras is antiequivalent to the category of affine group schemes [11,26]. Hence, the noncommutative Hopf algebras are thought of as (duals to) noncommutative affine group schemes ([8,26]; drawback: $\otimes$ is not a categorical product of noncommutative rings). Actions of affine group schemes generalize then to the coactions of Hopf algebras, which can furthermore be "structure groups" of noncommutative fibre bundles. The total space of such a bundle is either a single algebra (affine case) or a more complicated system of algebras or categories with gluing or localizing mechanism to pass between global and local description. Noncommutative fibre bundles with coacting Hopf algebras playing the role of a structure group first appeared in now classical work on smash products and Hopf–Galois extensions.

Then, H-J. Schneider introduced in [27] a crucial descent theorem supporting the geometric torsor intuition for faithfully flat Hopf–Galois extensions. In a study of noncommutative algebras equipped with differential calculi, Majid and Brzeziński [5] discovered a remarkable condition on differential calculi which enters the natural definition of principal bundles in that case. The coherent states on noncommutative projective homogeneous spaces, exhibited in the present work, seem to need a bundle theory extended in a different direction. To this aim, the present author has extended the concepts of Zariski *locally trivial* principal fibre bundles ([28,29, Škoda, Z. in Quantum bundles using coactions and localization, in preparation], and [33], Part I) to the setup where both total and base space are noncommutative (described locally by noncommutative algebras) and *not necessarily affine*.

Every complex semisimple Lie group $G^{\mathbb{C}}$ is an affine algebraic $\mathbb{C}$-group, and $G^{\mathbb{C}} \to G^{\mathbb{C}}/B$ is an algebraic principal fibration Zariski locally trivialized in a cover by shifts by action of Weyl group W of the main Bruhat cell [11]. Noncommutative analogues of such fibrations, derived from quantum matrix groups $\mathcal{G}_q$, are recently exhibited [28,29]. The fibrations trivialize in coaction-compatible Ore localizations $S_w^{-1}\mathcal{G}_q$ labeled by the elements w of the Weyl group W. The trivializations are explicitly computed using an elaborate *Ansatz* involving q-w-Gauss decompositions ([29], Theorems 9–12; proofs in [28] and [33], II).

In noncommutative case, it is not appropriate to seek for individual coherent vectors or rays in representation space V. A *family* of coherent vectors C should be a section of a noncommutative bundle $V \otimes L_\chi$ over a noncommutative "coset" space X "parametrizing would-be individual" coherent states, where the fiber $V = V_\chi = \Gamma L_\chi = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} \mathbb{C}_\chi$ is an analogue of a holomorphically induced representation space, χ is an analogue of a character of the inducing subgroup B and L_χ is an analogue of the Borel–Weil line bundle. Our noncommutative coset spaces are patched from charts. Local descriptions of X and C in different *covers* by charts are naturally equivalent. Earlier studies of coherent states for quantum groups [14, 25] used computations in a single local chart. One of our goals was to show that states locally computed in [25] may be defined *a priori*, regardless coordinate choices. The main goal was to find a resolution of unity in terms of coherent states of compact quantum groups.

Notation for Hopf algebras: unit map η, counit ϵ, multiplication μ, coproduct Δ, antipode S (do not confuse with S and T sometimes used for generic Ore subsets in a ring). We use SWEEDLER*'s notation*: for coproduct $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$; for (say right) coactions $\rho(v) = \sum v_{(0)} \otimes v_{(1)}$; and their multiplace extensions ([17]). Ground ring $\mathbf{k}$ is any commutative unital (in Sections 1–3, later $\mathbf{k} = \mathbb{C}$); the category of left $\mathcal{E}$-modules for a $\mathbf{k}$-algebra $\mathcal{E}$ is denoted ${}_{\mathcal{E}}\mathcal{M}$. The $\mathcal{B}$-comodule analogue has a superscript (${}^{\mathcal{B}}\mathcal{M}$). The right-hand versions have a *right* sub/super-script instead (e.g. $\mathcal{M}^{\mathcal{B}}$), and for bi(co)modules we use combinations.

1. Quantum Principal Bundles Using Ore Localizations

Let $\mathcal{B}$ be a Hopf algebra. An algebra $\mathcal{E}$ is a **$\mathcal{B}$–comodule algebra** if it is given with a $\mathcal{B}$-coaction ρ which is an algebra map [15, 17, 19]. For commutative $\mathcal{E}$ and $\mathcal{B}$ this means that the affine scheme $E = \mathrm{Spec}\,\mathcal{E}$ is given a regular action of an affine algebraic group $B = \mathrm{Spec}\,\mathcal{B}$.

For basics of Ore localization and covers in the context of noncommutative geometry see our survey [31] and Section 1 of the earlier arXiv version v4 of this article. An Ore localization $S^{-1}\mathcal{E}$ is **ρ–compatible** [29] if there is a (unique) $\mathcal{B}$-coaction ρ_S on $S^{-1}\mathcal{E}$, making $S^{-1}\mathcal{E}$ a $\mathcal{B}$-comodule algebra such that the localization map $\iota_{\mathcal{E}} : \mathcal{E} \to S^{-1}\mathcal{E}$ is a map of $\mathcal{B}$-comodule algebras. In commutative case, this means that $\mathrm{Spec}\, S^{-1}\mathcal{E}$ is a B-invariant Zariski open subscheme of $\mathrm{Spec}\,\mathcal{E}$. **Localized coinvariants** are those e in $S^{-1}\mathcal{E}$ for which $\rho_S(e) = e \otimes 1$ i.e. the coinvariants for the "localized" coaction ρ_S. They form subalgebra $(S^{-1}\mathcal{E})^{\mathrm{co}\mathcal{B}} \subset S^{-1}\mathcal{E}$.

An $(\mathcal{E}, \mathcal{B})$**-Hopf module** is an $\mathcal{E}$-module M, with $\mathcal{B}$-coaction ρ_M, so that $\rho_M(em) = \rho(e)\rho_M(m)$ for all $e \in \mathcal{E}$ and $m \in M$. In commutative case, Hopf modules correspond to B-equivariant quasicoherent sheaves over $\mathrm{Spec}\,\mathcal{E}$. They form a category commonly denoted by ${}_{\mathcal{E}}\mathcal{M}^{\mathcal{B}}$.

A flat localization functor Q on $\mathcal{E} - \mathrm{Mod}$ is ρ-compatible if there is a (unique) functor $Q^{\mathcal{B}}$ on the category ${}_{\mathcal{E}}\mathcal{M}^{\mathcal{B}}$ agreeing with Q after forgetting the comodule structures.

DEFINITION 1. [29, Škoda, Z. in Quantum bundles using coactions and localization, in preparation] A **Zariski locally trivial principal $\mathcal{B}$-bundle** is an $\mathcal{E}$-comodule algebra $(\mathcal{E}, \rho)$ for which there exists a Zariski local trivialization. A Zariski **local trivialization** of $(\mathcal{E}, \rho)$ consists of

- a finite **cover** $\{(\iota_\lambda, S_\lambda^{-1}\mathcal{E})\}_{\lambda \in \Lambda}$ of $\mathcal{E}$ by **ρ-compatible Ore localizations**, and
- a family $\{\gamma_\lambda : \mathcal{B} \to S_\lambda^{-1}\mathcal{E}\}_{\lambda \in \Lambda}$ of $\mathcal{B}$-comodule algebra maps.

Here the $\mathcal{B}$-comodule structure on $\mathcal{E}_\lambda$ is the one induced by ρ-compatibility. Maps γ_λ are, in commutative case, induced by trivializing sections (cf. (2) in [29]), and we view γ_λ as an algebraic replacement for trivializing sections. We discuss some generalizations in (Škoda, Z. in Quantum bundles using coactions and localization, in preparation), where we also show how these bundles indeed may be viewed as sheaves over a quantum quotient space (cf. arXiv version v4 of this article and [30]; and for general background [23,24,31]).

2. Quantum Associated Bundles

For any **k**-coalgebra C (e.g. $C=\mathcal{B}$), denote by $\mathcal{M}^C$ (*resp.*${}^C\mathcal{M}$) the category of right (left) C-*co*modules. **Cotensor product** is a bifunctor $\square=\square^C:\mathcal{M}^C\times{}^C\mathcal{M}\to{}_{\mathbf{k}}\mathcal{M}$ which is given on objects as $N\square M:=\ker(\rho_N\otimes\mathrm{id}_M-\mathrm{id}_N\otimes\rho_M)$. The same formula defines the bifunctor $\square:{}_{\mathcal{E}}\mathcal{M}^{\mathcal{B}}\times{}^{\mathcal{B}}\mathcal{M}\to{}_{\mathcal{E}}\mathcal{M}$. If D is *flat* as a **k**-module (e.g. **k** is a field), and N a left D- right C-bicomodule, then the cotensor product $N\square M$ is a D-subcomodule of $N\otimes_{\mathbf{k}}M$. In particular, if $\pi:D\to C$ is a surjection of coalgebras then D is a left D- right C-bicomodule via Δ_D and $(\mathrm{id}\otimes\pi)\circ\Delta_D$ respectively, hence $\mathrm{Ind}_C^D:=D\square^C\_$ is a functor from left C- to left D-comodules called the **induction** from C to D.

Consider for a moment functor $\mathcal{E}\otimes_{\mathbf{k}}\_:{}^{\mathcal{B}}\mathcal{M}\to{}_{\mathbf{k}}\mathcal{M}$ (the superscript is intended!). Given a map $\gamma:\mathcal{B}\to\mathcal{E}$ of $\mathcal{B}$-comodules for which there is a convolution–inverse γ^{-1}, define natural transformations of functors $\kappa^\gamma,\bar\kappa^\gamma:\mathcal{E}\otimes_{\mathbf{k}}\_\to\mathcal{E}\otimes_{\mathbf{k}}\_$ by $\kappa^\gamma_M(\sum_i e_i\otimes m_i)=\sum_i e_i\gamma(m_{i(-1)})\otimes m_{(0)}$ and $\bar\kappa^\gamma=\kappa_{\gamma^{-1}}$. If γ is a map of $\mathcal{B}$-comodule algebras, then $\gamma^{-1}=\gamma\circ S$. In that case, restrict the natural transformation κ^γ to the subfunctor $\mathcal{E}^{\mathrm{co}\mathcal{B}}\otimes_{\mathbf{k}}\_$ and $\bar\kappa^\gamma$ to the subfunctor $\mathcal{E}\square^{\mathcal{B}}\_$ and denote the restrictions simply by $\kappa^\gamma|$ and $\bar\kappa^\gamma|$. For any natural transformation of functors with values $\Phi:F\to G$ in a (say) Abelian category G, denote by $\mathrm{Im}\,\Phi:F\to G$ the functor $M\mapsto\mathrm{Im}\,\Phi_M(F(M))$.

LEMMA 1. (a) $\kappa^\gamma\circ\bar\kappa^\gamma=\mathrm{Id}_{\mathcal{E}\otimes_{\mathbf{k}}\_}=\bar\kappa^\gamma\circ\kappa^\gamma$;
(b) $\mathrm{Im}(\kappa^\gamma|)=\mathcal{E}\square^{\mathcal{B}}\_$ *and* $\mathrm{Im}(\bar\kappa^\gamma|)=\mathcal{E}^{\mathrm{co}\mathcal{B}}\otimes_{\mathbf{k}}\_$.

We prove this lemma in (Škoda, Z. in Quantum bundles using coactions and localization, in preparation) (and in arXiv version v4 of this article).

By abuse of notation, let $\kappa^\gamma|:\mathcal{E}^{\mathrm{co}\mathcal{B}}\otimes_{\mathbf{k}}\_\to\mathcal{E}\square\_$ denote also the corestriction of $\kappa^\gamma|$ onto the image functor $\mathcal{E}\square\_$, and alike for $\bar\kappa^\gamma$. The lemma easily implies that $\kappa^\gamma|$ is an equivalence of subfunctors with inverse $\bar\kappa^\gamma|$. That is, the pair of natural transformations $(\kappa^\gamma|,\bar\kappa^\gamma|)$ extends to a pair of mutually inverse natural autoequivalences of $\mathcal{E}\otimes_{\mathbf{k}}\_$, namely $(\kappa^\gamma,\bar\kappa^\gamma)$.

We apply this discussion to our localization picture. For any local trivialization $\Lambda=\{\iota_\lambda,\mathcal{E}_\lambda,\gamma_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{E}$, we have the natural transformations $\kappa_\lambda=\kappa^{\gamma_\lambda},\bar\kappa_\lambda=\kappa^{\gamma_\lambda\circ S}$ for all $\lambda\in\Lambda$, and the coproducts $\mathcal{E}_\lambda\square M$ are locally "identified" to $\mathcal{E}_\lambda^{\mathrm{co}\mathcal{B}}\otimes_{\mathbf{k}}M$. Introduce functor $\Gamma_\lambda\xi_\_:{}^{\mathcal{B}}\mathcal{M}\to{}_{\mathbf{k}}\mathcal{M}$ by $M\mapsto\Gamma_\lambda\xi_M=\mathcal{E}_\lambda^{\mathrm{co}\mathcal{B}}\otimes_{\mathbf{k}}M$ and similarly, for consecutive localizations, $\Gamma_{\lambda\mu\ldots}\xi_\_$. Define the natural transformations $\kappa^\lambda_{\lambda\lambda'}$ by the compositions

$$\kappa^\lambda_{\lambda\lambda' M}:\Gamma_\lambda\xi_M\xrightarrow{\kappa_{\lambda,M}}\mathcal{E}_\lambda\square M\xrightarrow{\iota^\lambda_{\lambda\lambda'}\square M}\mathcal{E}_{\lambda\lambda'}\square M.$$

and similarly define $\kappa^{\lambda'}_{\lambda\lambda'}$ using $\iota^{\lambda'}_{\lambda\lambda'}: \mathcal{E}_\lambda \to \mathcal{E}_{\lambda\lambda'} = S^{-1}_{\lambda'} S^{-1}_{\lambda}\mathcal{E}$. Finally natural transformations

$$\mathcal{K}_{\Lambda\Lambda} : \prod_{\lambda\in\Lambda} \Gamma_\lambda \xi_{-} \to \prod_{(\lambda,\lambda')\in\Lambda\times\Lambda} \mathcal{E}_{\lambda\lambda'}\Box_{-}$$

are defined by $\mathrm{pr}_{\lambda\lambda'} \circ \mathcal{K}_{\Lambda\Lambda} = \prod_{\lambda\in\Lambda} \kappa^{\lambda}_{\lambda\lambda'} \circ \mathrm{pr}_\lambda$ where $\mathrm{pr}_\mu : \prod_\lambda \Gamma_\lambda\xi_{-} \to \Gamma_\mu \xi_{-}$ are the natural projection transformations and $\mathrm{pr}_{\lambda\lambda'}$ alike; and similarly define $\mathcal{K}'_{\Lambda\Lambda}$ by using $\kappa^{\lambda'}_{\lambda\lambda'}$. The **global sections of associated vector bundle functor** $\Gamma_\Lambda \xi_{-}$ is the subfunctor of $\prod_{\lambda\in\Lambda}\Gamma_\lambda\xi_{-}$ such that the fork diagram

$$\Gamma_\Lambda\xi_{-} \xrightarrow{\mathrm{in}} \prod_{\lambda\in\Lambda}\Gamma_\lambda\xi_{-} \underset{\mathcal{K}'_{\Lambda\Lambda}}{\overset{\mathcal{K}_{\Lambda\Lambda}}{\rightrightarrows}} \prod_{(\lambda,\lambda')\in\Lambda\times\Lambda} \mathcal{E}_{\lambda\lambda'}\Box_{-}$$

is an equalizer diagram of natural transformations.

THEOREM 1. *Functors $\Gamma_\Lambda\xi_{-}$ are naturally equivalent for different local trivializations Λ. There is a natural equivalence $\mathcal{K}_\Lambda : \Gamma_\Lambda\xi_{-} \to \mathcal{E}\Box_{-}$ making the following diagram sequentially commute:*

$$\begin{array}{ccccc} \Gamma_\Lambda\xi_{-} & \xrightarrow{\mathrm{in}} & \prod_{\lambda\in\Lambda}\Gamma_\lambda\xi_{-} & \underset{\mathcal{K}'_{\Lambda\Lambda}}{\overset{\mathcal{K}_{\Lambda\Lambda}}{\rightrightarrows}} & \prod_{(\lambda,\lambda')\in\Lambda\times\Lambda}\mathcal{E}_{\lambda\lambda'}\Box_{-} \\ \downarrow \mathcal{K}_\Lambda & & \downarrow \prod_\lambda \kappa_\lambda & & \| \\ \mathcal{E}\Box_{-} & \xrightarrow{\iota_\Lambda\Box_{-}} & \prod_{\lambda\in\Lambda}\mathcal{E}_\lambda\Box_{-} & \underset{i_2\Box_{-}}{\overset{i_1\Box_{-}}{\rightrightarrows}} & \prod_{(\lambda,\lambda')\in\Lambda\times\Lambda}\mathcal{E}_{\lambda\lambda'}\Box_{-} \end{array} \tag{1}$$

Proof. The square on the right is manifestly commutative by the construction of the maps involved. Since upper and lower fork diagrams are equalizer diagrams, and the right vertical arrows natural equivalences, the transformation $\mathcal{K}_\Lambda$ exists and is uniquely defined by the rest of the diagram.

As a consequence, given two different local trivializations Λ, Λ' the transformation $\mathcal{K}_{\Lambda'} \circ \mathcal{K}_\Lambda : \Gamma_\Lambda\xi_{-} \xrightarrow{\cong} \Gamma_{\Lambda'}\xi_{-}$ is a canonical isomorphism of functors. Hence for fixed M, we may denote by $\Gamma\xi_M$ the equivalence class of pairs of the form $(\Lambda, \Gamma_\Lambda\xi_M)$.

Alternatively, we may construct the associated bundles by means of transition matrices (Škoda, Z. in Quantum bundles using coactions and localization, in preparation).

Quantum line bundles. Let us specialize now to $\mathbf{k} = \mathbb{C}$, choose a group-like element $\chi \in \mathcal{B}$ and consider the one-dimensional left comodule $M = \mathbb{C}_\chi$ given by $\rho_M(m) = m \otimes \chi$. Denote the "line bundle" ξ_M by L_χ. Its space of sections ΓL_χ can be identified with a $\mathbb{C}$-subspace of $\prod_\lambda \mathcal{E}_\lambda$. Namely, write explicitly maps $\mathcal{K}_{\Lambda\Lambda'}$ and $\mathcal{K}'_{\Lambda\Lambda'}$, and use the identifications $\mathcal{E}_\lambda \otimes M \cong \mathcal{E}_\lambda$ in expressing (1) to obtain

$$\Gamma L_\chi \cong \left\{ f = \prod_{\lambda\in\Lambda} f_\lambda \;\middle|\; \begin{array}{c} f_\lambda \gamma_\lambda(\chi) = f_{\lambda'}\gamma_{\lambda'}(\chi) \\ \forall \lambda, \lambda' \text{ in } \mathcal{E}_{\lambda\lambda'} \text{ and in } \mathcal{E}_{\lambda'\lambda} \end{array} \right\} \tag{2}$$

THEOREM 2. *ΓL_χ is naturally isomorphic to the cotensor product $\mathcal{E}\Box_{\mathcal{B}} M$ as a* $\mathbf{k}$-*vector space.*

3. Background: Perelomov Coherent States

Perelomov coherent states generalize the Schrödinger coherent states to the Lie group setting [20,21].

We use the geometric language of [22]; cf. also [4].

Let $G^{\mathbb{C}}$ be a complex connected semisimple Lie group with compact real form G, and a Borel subgroup B. We will often view these groups as affine algebraic groups over $\mathbb{C}$. Let $\chi : B \to \mathbb{C}$ be a character of B and $\mathbb{C}_\chi$ the corresponding one-dimensional B-module. The projection $p : G^{\mathbb{C}} \to G^{\mathbb{C}}/B$ defines a principal B-bundle. The associated bundle $L_\chi = G^{\mathbb{C}} \times_\chi \mathbb{C}_\chi$ with the projection $p_L : L_\chi \to G^{\mathbb{C}}/B$. The left action of G on $G^{\mathbb{C}}$ induces an action of G on L_χ and the formula $(g_* s)(x) = gs(g^{-1}x)$ defines an action of $G^{\mathbb{C}}$ on the space $V_\chi = \Gamma L_\chi$ of holomorphic sections of L_χ which is by Borel–Weil theorem, an irreducible unitarizable G-module. An invariant unitary product on ΓL_χ, antilinear in first and linear in second argument, is denoted $\langle | \rangle$.

Consider a (holomorphic) section $s \in \Gamma L_\chi$ and a nonzero point q in some fiber ${p_L}^{-1}(x)$. Then

$$s(x) = s(p_L(q)) = l_q(s) q,$$

for some number $l_q(s)$. The correspondence

$$s \mapsto l_q(s), \quad l_q : \Gamma L_\chi \to \mathbb{C},$$

is a continuous linear functional. Using Riesz's theorem, we infer the existence of an element

$$e_q \in \Gamma L_\chi \quad \text{such that} \quad l_q(s) = \langle e_q | s \rangle.$$

The vectors (sections) of the form $e_q \in \Gamma L_\chi$ are called **coherent vectors**. Corresponding projective classes are called coherent states.

PROPOSITION 1. ([22,28])

(i) *$e_{gq} = g_* e_q$ for all $g \in G^{\mathbb{C}}$.*
(ii) *$e_{cq} = \bar{c}^{-1} e_q$ for all $c \in \mathbb{C}$.*
(iii) *Coherent states i.e. the projective classes of all coherent vectors belong to the same projective orbit.*
(iv) *The set of all e_q where $q \in (p_L)^{-1}(1_G B)$ agrees with the set (ray) of all heighest weight vectors in V_χ for fixed B.*
(v) *The set of all e_q where $q \in (p_L)^{-1}(u)$ for fixed $u \in G^{\mathbb{C}}/B$ is the heighest weight space for some subgroup of $G^{\mathbb{C}}$ conjugated to B.*

COROLLARY 1. *Let $U \subset G^{\mathbb{C}}/B$ be an open set, $q \in L_\chi$ given, and $t : U \to G^{\mathbb{C}}$ a section of the principal B-bundle $p^{-1}(U) \to U$. For each $g \in p^{-1}(U)$ there is a* *unique decomposition* $g = t(gB)b$ *such that* $b \in B$. *The following "homogeneity" formula holds:*

$$ge_q = \chi^{-1}(b)e_{t(gB)q} \tag{3}$$

Proof. We have $g_* e_q = t(gB)_* b_* e_q = t(gB)_* \chi^{-1}(b)e_q$ by (5); taking into account that $\chi^{-1}(b)$ is a scalar, this equals to $\chi^{-1}(b)t(gB)_* e_q$, hence by *(i)* of the Proposition 1, also to $\chi^{-1}(b)e_{t(gB)q}$.

DEFINITION 2. The **local family of coherent vectors** corresponding to the triple (U, t, q) is the map

$$C_{(U,t,q)} : U \to V_\chi \equiv \Gamma L_\chi, \quad C_{(U,t,q)} : [g] \mapsto e_{t([g])q}. \tag{4}$$

For any w in the Weyl group W of G, there is a Zariski open subset $G^{\mathbb{C}}_w \subset G^{\mathbb{C}}$ consisting of all $g \in G^{\mathbb{C}}$ for which there exists (automatically unique) w-Gauss decomposition $g = wyb$ where $y \in G^{\mathbb{C}}$ belongs to the unipotent subgroup of the opposite Borel B', and $b \in B$. Set $G^{\mathbb{C}}_w$ is also B-invariant, hence a total space of the restricted fibration over a Zariski open subset $G^{\mathbb{C}}_w/B \subset G^{\mathbb{C}}/B$. Define the local section $t_w : G^{\mathbb{C}}_w/B \to G^{\mathbb{C}}_w \subset G^{\mathbb{C}}$ by $t_w([g]) = wy$ where $g = wyb$ as above. We denote

$$C_w := C_{(w,v_0)} := C_{(G^{\mathbb{C}}_w/B, t_w, q)}$$

where $v_0 = e_q$ is a fixed highest weight vector in V_χ. The collection of maps $\{C_w, w \in W\}$ will be generalized to the quantum group setting below. They can be viewed as $C_w \in \mathcal{O}(G^{\mathbb{C}}_w/B) \otimes V_\chi$ where $\mathcal{O}(G^{\mathbb{C}}_w/B)$ is the complex algebra of all algebraic functions on $G^{\mathbb{C}}_w/B$.

In this particular case, Corollary 1 becomes

PROPOSITION 2. *If* $g = wyb$ *is the Gauss decomposition in* G_w *then for all* $g \in G$

$$gv_0 = \chi^{-1}(b)C_w(gB), \tag{5}$$

and C_w *is the unique element in* $\mathcal{O}(G^{\mathbb{C}}_w/B) \otimes V_\chi$ *for which this holds.*

4. Quantum Coherent States and Localizations

DEFINITION 3. Let χ be a group–like element in a Hopf algebra $\mathcal{B}$, and (V, ρ) a right $\mathcal{B}$-comodule. A χ**-coinvariant** in V is an element $v_\chi \in V$ such that $\rho v_\chi = v_\chi \otimes \chi$.

Let $\pi : \mathcal{G} \to \mathcal{B}$ be a surjective homomorphism of Hopf algebras. We say that $(\pi, \mathcal{B})$ is a **quantum subgroup** of $\mathcal{G}$. Every $\mathcal{G}$-comodule (resp. comodule algebra) (V, ρ) is a $\mathcal{B}$-comodule (comodule algebra) via $\rho_{\mathcal{B}} = (\mathrm{id} \otimes \pi) \circ \rho$. In particular, $\mathcal{B}$ coacts on $\mathcal{G}$ by $(\mathrm{id} \otimes \pi) \circ \Delta_{\mathcal{G}}$ and this coaction makes $\mathcal{G}$ a left $\mathcal{B}$–comodule algebra and similarly for the right coactions. In particular, $\mathcal{G}$ can be viewed as left–right

$\mathcal{B}$-$\mathcal{G}$–bicomodule ${}^{\mathcal{B}}\mathcal{G}^{\mathcal{G}}$. Hence to each $\mathcal{B}$-comodule V one can attach an **induced** $\mathcal{G}$**–comodule** by the formula $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} V = V \square^{\mathcal{B}} \mathcal{G}$. This defines the induction functor $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}$ which is left adjoint to the restriction functor $(V, \rho) \mapsto (V, \rho_{\mathcal{B}})$ (Frobenius reciprocity for comodules).

DEFINITION 4. Let (V, ρ) be any $\mathcal{G}$-comodule and $(\pi, \mathcal{B})$ a quantum subgroup of $\mathcal{G}$. A **weight covector** of weight χ, is any χ-coinvariant for $\mathcal{B}$–coaction $\rho_{\mathcal{B}}$ in V i.e.

$$(\mathrm{id} \otimes \pi)\rho v_\chi = v_\chi \otimes \chi.$$

Let $(V_\chi, \rho) = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} \mathbb{C}_\chi$ be the induced right $\mathcal{G}$-comodule induced from the one-dimensional comodule $z \mapsto z \otimes \chi$.

DEFINITION 5. A $*$-involution on a $\mathbb{C}$-bialgebra H is an antilinear map $*: H \to H$, for which $(ab)^* = b^* a^*$, $\Delta(a^*) = \sum a^*_{(1)} \otimes a^*_{(1)}$ and $\epsilon(a^*) = \overline{\epsilon(a)}$. A pair $(H, *)$ is called a **real form** of H.

LEMMA 2. (*Schur's lemma for comodules*) *Let C be a coalgebra over $\mathbb{C}$ and (V, ρ) a right C-comodule. If (V, ρ) is finite-dimensional and simple* (*no coinvariant subspaces*), *then every C-comodule map $A: V \to V$ equals $\alpha \cdot \mathrm{id}_V$ for some $\alpha = \alpha_A \in \mathbb{C}$.*

DEFINITION 6. ([15]) Let H be a Hopf $*$-algebra. An inner product $\langle \cdot | \cdot \rangle$ on a right H-comodule V is a coinvariant inner product iff

$$\langle w | z \rangle 1_H = \sum \langle w_{(0)} | z_{(0)} \rangle z_{(1)} w^*_{(1)}$$

An H-comodule which is a Hilbert space via a coinvariant inner product will be called a right **unitary** H**-comodule**.

Consider a real form of a Hopf algebra $\mathcal{G}$ with the following data:

- (D1) A surjective map of Hopf algebras $\pi: \mathcal{G} \to \mathcal{B}$.
- (D2) A group–like element $\chi \in \mathcal{B}$.
- (D3) A coinvariant inner product on V_χ.
- (D4) A weight covector $v_\chi \in V_\chi$ with norm 1.
- (D5) A Zariski local trivialization $\Lambda = \{\lambda = (\iota_\lambda, S_\lambda^{-1}\mathcal{G}, \gamma_\lambda)\}_{\lambda \in \Lambda}$ of $\mathcal{G}$ as a right $\mathcal{B}$-comodule algebra.

From now on, V will be a comodule over the real form of $\mathcal{G}$ with fixed unitary equivalence $V \cong V_\chi$ which we often treat as an identification. Denote by V^{triv} the trivial $\mathcal{G}$-comodule with the same underlying vector space as V.

DEFINITION 7. Let (D1–5) be given and $\lambda \in \Lambda$. A (Zariski-) **local family of coherent vectors** in λ or a **polynomial coherent vector**[1] in λ is an element $C_\lambda \in V \otimes \mathcal{G}_\lambda^{\mathrm{co}\mathcal{B}}$ such that

$$\rho_\lambda v_\chi = C_\lambda \gamma_\lambda(\chi) \tag{6}$$

holds in $V \otimes \mathcal{G}_\lambda$ where $\gamma_\lambda(\chi)$ on the right multiplies the second tensor factor in C_λ and ρ_λ is the localized $\mathcal{B}$-coaction $(\mathrm{id} \otimes \iota_\lambda)\rho$. A **global family of coherent vectors** is an element C of $\Gamma(V^{\mathrm{triv}} \otimes L_\chi)$ such that $\mathcal{K}(C) = \mathcal{K}_\Lambda(C_\Lambda) = \rho v_\chi$ (for one, hence any, choice of Λ). Then $\kappa_\lambda(C) = \rho_\lambda v_\chi$.

Remark. Equality (6) is a generalization of the identity (5) and related to Proposition 5.11 in [14].

PROPOSITION 3. *The following are equivalent*:

(a) *There exists a global family of coherent states* C;
(b) *There exists a local trivialization* Λ *of* $\mathcal{E}$ *such that a local family of coherent states* C_λ *exists for each* λ *in* Λ;
(c) *For each local trivialization* Λ *of* $\mathcal{E}$ *and each* λ *in* Λ *there exists a local family* C_λ *of coherent states in* λ.

Since $\mathcal{K}_\Lambda$ *is a natural equivalence, if* (*a–c*) *are true, then the global family is unique. The same for the local family in any given local trivialization.*

Proof. An exercise to the reader: use the globalization lemma and the explicit description of $\mathcal{K}_\Lambda$. Notice though that given only one C_λ does not always suffice. Indeed, $C_\lambda \gamma_\lambda(\chi)\gamma_{\lambda'}(S\chi)$ is a candidate for $C_{\lambda'}$, but it does not need to extend to an element in $V \otimes \mathcal{G}_{\lambda'}$ in general.

Let us extend the product $\langle | \rangle$ *on* $V \cong V \otimes \mathbb{C} \subset V \otimes \mathcal{G}$ *to a sesquilinear form*

$$\langle | \rangle : (V \otimes \mathcal{G}) \otimes V \to \mathcal{G}, \quad \left\langle \sum_o w_i \otimes g_i \,\middle|\, v \right\rangle := \sum_i \langle w_i | v \rangle g_i,$$

and analogously define $\langle | \rangle_{\lambda\mu\ldots}$ (often skipping the subscripts) on $V \otimes \mathcal{G}_{\lambda\mu\ldots}$. In particular, for any $v \in V_\chi$ the expression $\langle C_\lambda | v \rangle_\lambda$ is an element in $\mathcal{G}_\lambda$. Let $|C_\lambda\rangle := C_\lambda$ in such context.

PROPOSITION 4. *For each* $v \in V$, $\prod_\lambda \langle C_\lambda | v \rangle_\lambda$ *is an element in* $\Gamma_\Lambda L_\chi$, *and hence, by Theorem 2, it determines an element in* $V_\chi \cong V$.

[1] Terminological remark. '*Polynomial*' because it is a polynomial in the generators of algebra $\mathcal{E}^{\mathrm{co}\mathcal{B}}$ of localized coinvariants decorated (tensored) with coefficients in Hilbert space. This terminology is occasionally used in physics literature (in the group case, as well as in the quantum group examples, e.g. [10], p. 1382).

Proof. By the definition, $\langle C_\lambda|v\rangle \in \mathcal{G}_\lambda^{\mathrm{co}\mathcal{B}}$. Hence, by (2), for each pair (λ,λ'), we have to check that $\langle C_\lambda|v\rangle_\lambda \gamma_\lambda(\chi) = \langle C_{\lambda'}|v\rangle_{\lambda'}\gamma_{\lambda'}(\chi)$ in both consecutive localizations. To that aim observe that

$$\langle \rho_\lambda v_\chi|v\rangle_\lambda = \langle C_\lambda \cdot (1\otimes \gamma_\lambda \chi)\,|v\rangle_\lambda = \langle C_\lambda|v\rangle_\lambda \gamma_\lambda(\chi). \tag{7}$$

Then observe, that symbol $\langle|v\rangle = \langle ,v\rangle \otimes \mathrm{id}$ commutes with the localizations, in the sense that $(\mathrm{id}\otimes \iota^\lambda_{\lambda,\lambda'})\circ\langle|v\rangle_\lambda = \langle|v\rangle_{\lambda\lambda'}\circ(\mathrm{id}\otimes\iota^\lambda_{\lambda,\lambda'})$. Hence the equality $\rho_\lambda v_\chi = \rho_{\lambda'}v_\chi$, which may be fully expanded as

$$(\mathrm{id}\otimes\iota^\lambda_{\lambda\lambda'})(\mathrm{id}\otimes\iota_\lambda)\rho v_\chi = (\mathrm{id}\otimes\iota^{\lambda'}_{\lambda\lambda'})(\mathrm{id}\otimes\iota_{\lambda'})\rho v_\chi \tag{8}$$

implies that $\langle\rho_\lambda v_\chi|v\rangle = \langle\rho_{\lambda'}v_\chi|v\rangle$, and by (7) this yields the wanted equality. The same way, using $\iota^\lambda_{\lambda'\lambda}$ and $\iota^{\lambda'}_{\lambda'\lambda}$ in (8) this time, check the identity in another consecutive localization.

5. Resolution of Unity

DEFINITION 8. A **left-invariant integral** (= left Haar integral, [17,19]) on a Hopf algebra H is a linear functional $\int$ on H such that

$$\langle h\otimes \int, \Delta(f)\rangle = \langle h,1\rangle\langle \int, f\rangle, \quad \forall h\in H^*.$$

A left Haar integral $\int$ is **normalized** if $\langle \int, 1\rangle = 1$.

Since linear functionals separate elements of H, the left invariance can be expressed as (dropping the evaluation brackets)

$$(\mathrm{id}\otimes \int)\Delta(a) = (\int a)\cdot 1_H, \quad \forall a\in H.$$

THEOREM 3. *Let $\int$ be a left integral on a Hopf $*$-algebra H, and $(V,\rho,\langle,\rangle)$ a simple unitary right H-comodule. Fix a vector $w\in V$. Define the operator $A: V\to V$ by*

$$A|v\rangle = \sum \langle w_{(0)}|v\rangle w_{(0)'} \int w^*_{(1)} w_{(1)'}$$

Then A is a scalar operator.

Proof. In the following, the primed Sweedler indices belong to another copy of the same variable, as in [17]. We compute directly

$$\rho A v = \sum \langle w_{(0)}|v\rangle w_{(0)'} \int w^*_{(1)} w_{(2)'} \otimes w_{(1)'}$$

On the other hand,

$$(A\otimes\mathrm{id})\rho v = \sum \langle w_{(0)}|v_{(0)}\rangle w_{(0)'} \int w^*_{(1)} w_{(1)'} \otimes v_{(1)},$$

what is by the left invariance of the integral equal to

$$\sum \langle w_{(0)} | v_{(0)} \rangle w_{(0)'} \int w^*_{(2)} w_{(2)'} \otimes v_{(1)} w^*_{(1)} w_{(1)'},$$

and, by the coinvariance of the inner product,

$$(A \otimes \mathrm{id})\rho v = \sum \langle w_{(0)} | v \rangle w_{(0)'} \int w^*_{(2)} w_{(2)'} \otimes w_{(1)'}$$

We conclude that $\rho A v = (A \otimes \mathrm{id})\rho v$. Hence the theorem follows from the Schur's lemma for comodules.

DEFINITION 9. $d\mu_\lambda(\chi) := \gamma_\lambda(\chi)(\gamma_\lambda(\chi))^*$ in $\mathcal{E}_\lambda$.

THEOREM 4. *Elements* $|C_\lambda\rangle d\mu_\lambda(\chi)\langle C_\lambda| := C_\lambda d\mu_\lambda(\chi) C^*_\lambda$ *do not depend on* λ (*agrees in all consecutive localization overlaps*). *Hence, by the globalization lemma, this family defines localized representatives of a unique expression* $|C\rangle d\mu(\chi)\langle C|$ *in* $V \otimes \mathcal{G} \otimes V^*$. *Taking a Haar integral in the tensor factor* $\mathcal{G}$ *yields a scalar operator* $\alpha \cdot \mathrm{id}$ *on* V (*we identify "states" in* $V \otimes V^*$ *with operators*).

Remark. While $\langle C_\lambda| = C^*_\lambda$ should live in $\mathcal{G}_\lambda \otimes V^*$, we may define it only as a part of the expressions of the form $f^* C^*_\lambda := (C_\lambda f)^*$ with $f \in \mathcal{G}$ such that $C_\lambda f \in V \otimes \iota_\lambda(\mathcal{G})$. Indeed, the involution $*$ is not defined on entire $\mathcal{G}_\lambda$, but only on $\mathcal{G}$, or if you wish, $\iota_\lambda(\mathcal{G})$.

Proof. Notice that in each local trivialization λ,

$$|C_\lambda\rangle \gamma_\lambda(\chi)(\gamma_\lambda(\chi))^* \langle C_\lambda | v \rangle = \sum \langle w_{(0)} | v \rangle w_{(0)'} w^*_{(1)} w_{(1)'}$$

where on the LHS we assume that the pairing between V^* and V is assumed (applied) and on the RHS we assume appropriate localization. Recall that the product $C_\lambda \gamma_\lambda(\chi)$ does NOT depend on the localization. By (2) and Theorem 2 the RHS reads (no localizations this time) the element in $V \otimes \mathcal{G}$ to integrate. Hence by Theorem 3 we see that

$$\int C_\lambda d\mu_\lambda(\chi) C^*_\lambda = \int |C_\lambda\rangle \gamma_\lambda(\chi)(\gamma_\lambda(\chi))^* \langle C_\lambda|$$

is a scalar operator.

In other words, if a family $\{C_\lambda\}_\lambda$ of coherent states exists, then the **coherent states make a resolution of unity**. This fact enables us to define an analogue of the Bargmann transform [3]. To a vector $v \in V$ (V is physically a space describing some quantum numbers of the system; or a sector in a decomposition of such a space) we assign $\langle C_\lambda | v \rangle \in \mathcal{G}^{\mathrm{co}\mathcal{B}}_\lambda$. If H is a linear operator on V, denote $H|C_\lambda\rangle := (\mathrm{id} \otimes H)|C_\lambda\rangle$. Suppose $\alpha \neq 0$ is the constant from Theorem 4. Then

$$H|v\rangle = \alpha^{-1} \int H\,|C_\lambda\rangle\, d\mu_\lambda(\chi)\, \langle C_\lambda | v \rangle.$$

One may proceed in usual way to obtain (a noncommutative version of) a reproducing "integral" kernel on a Hilbert space, and equations involving H (e.g. deformations of Schrödinger equation where H is a Hamiltonian) can be written down in this coherent state representation.

6. Examples from Quantum Groups

The simplest example concerns the coherent states for $\mathcal{G} = \mathcal{O}(SU_q(2))$. We will mainly follow the notation and conventions of [15]. $\mathcal{O}(SL_q(2))$ is a noncommutative Hopf algebra over $\mathbb{C}$ with four generators a, b, c, d, usually assembled in a matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with relations $ab = qba$, $ac = qca$, $bc = cb$, $bd = qdb$, $cd = qdc$, $ad - da = (q - q^{-1})bc$ and $\det_q T := ad - qbc = 1$. $\mathcal{O}(SU_q(2))$ is a real form of $\mathcal{O}(SL_q(2))$ determined by formulas $a^* = d$, $b^* = -qc$, $c^* = -q^{-1}b$, $d^* = a$. A vector space basis of $\mathcal{O}(SL_q(2))$ is $\{a^k b^r c^s\}_{k>0,r,s\geq 0} \cup \{b^r c^s d^t\}_{r,s,t\geq 0}$. In particular, $\mathcal{O}(SL_q(2))$ splits into a direct sum $\mathbb{C}[\zeta] \oplus \mathrm{compl}(\zeta)$ where $\mathbb{C}[\zeta]$ is the span of the basis elements of the form $(bc)^r$ and $\mathrm{compl}(\zeta)$ the span of the rest of basis. Notation $\mathbb{C}[\zeta]$ suggests that it is the algebra of polynomials in $\zeta = -qbc$, which will play major role below. $\mathcal{O}(SU_q(2))$ posses a unique Haar functional $\int$, found by Woronowicz. With respect to the direct sum decomposition above, $\int$ is nontrivial only on $\mathbb{C}[\zeta]$ where it is given by formulas involving Jackson's q-integral, or equivalently ([15])

$$\int \zeta^r = \frac{1 - q^{-2}}{1 - q^{-2(r+1)}}, \quad r = 0, 1, 2, \ldots$$

The lower quantum Borel subgroup $\mathcal{B}$ will be the quotient $\mathcal{O}(SL_q(2))/I$, where I is the two-sided ideal generated by b. I is a Hopf ideal, hence $\mathcal{B}$ is a Hopf algebra. The quotient map $\pi : \mathcal{G} \to \mathcal{B}$ is datum (D1) from Sec. 4. The images of generators are denoted $\lambda = \pi(a)$, $\xi = \pi(c)$, $\lambda^{-1} = \pi(d)$ and $\pi(b) = 0$. Manin plane $\mathcal{O}(\mathbb{C}_q^2)$ is an algebra with two generators x, y and a single relation $xy = qyx$. Elements of the form $x^r y^s$ form a basis of $\mathcal{O}(\mathbb{C}_q^2)$. The latter is a right $\mathcal{O}(SL_q(2))$-comodule algebra via

$$\rho(x^r y^s) = (x \otimes a + y \otimes c)^r (x \otimes b + y \otimes d)^s.$$

$\mathcal{O}(\mathbb{C}_q^2)$ splits into the homogeneous components $V_n = \oplus_{r+s=n} \mathbb{C} x^r y^s$ of dimension $n+1$, which are irreducible and unitary. Our datum (D2) will be $\chi = \lambda^{-n}$ in $\mathcal{B}$, (D3) $V_\chi = V_n$, and (D4) will be the weight vector $v_\chi = y^n$. Datum (D5) is given by 1) two localizations $\mathcal{G}_b = \mathcal{G}[b^{-1}]$ and $\mathcal{G}_d = \mathcal{G}[d^{-1}]$ at Ore sets multiplicatively generated by b and d respectively; 2) comodule algebra maps γ_b, γ_d obtained from the quantum Gauss decomposition. Let $u := bd^{-1} \in \mathcal{G}_d$. It is easy to show [28,33] that these localizations cover $\mathcal{G}$. Both localizations are $\rho_{\mathcal{B}}$-compatible, namely $\rho_{\mathcal{B}}$ extends by $\rho_{\mathcal{B}}(b^{-1}) = b^{-1} \otimes \lambda^{-1}$ and $\rho_{\mathcal{B}}(d^{-1}) = d^{-1} \otimes \lambda^{-1}$. The algebras of localized $\rho_{\mathcal{B}}$-coinvariants are given by $\mathcal{G}_b^{\mathrm{co}\mathcal{B}} = \mathbb{C}[u]$ and $\mathcal{G}_d^{\mathrm{co}\mathcal{B}} = \mathbb{C}[u']$ where $u = bd^{-1}$ and

$u' = db^{-1}$. A unique ("Gauss") decomposition of matrix T in the form wUA where w is a permutation matrix, U upper triangular unidiagonal and A lower triangular is possible in $\mathcal{G}_b$ with $w = \mathrm{id}$ and in $\mathcal{G}_d$ with $w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Map $\lambda \mapsto A^1_1$, $\lambda^{-1} \mapsto A^2_2$ $\xi \mapsto A^2_1$ uniquely extend to a $\mathcal{B}$-comodule algebra map $\gamma_d : \mathcal{B} \to \mathcal{G}_d$, or to $\gamma_b : \mathcal{B} \to \mathcal{G}_b$ in the latter case. Explicitly $\gamma_d(\lambda) = a - bd^{-1}c$, $\gamma_d(\xi) = c$, $\gamma_d(\lambda^{-1}) = d$, $\gamma_b(\chi_b) = d^n$; $\gamma_b(\xi) = c - db^{-1}a$, $\gamma_b(\lambda) = a$, $\gamma_b(\lambda^{-1}) = b$. Also $U^1_2 = u$ in $\mathcal{G}_d$ and $U^1_2 = u'$ in $\mathcal{G}_b$.

Analogously for general n, a cover of $\mathcal{O}(SL_q(n))$ by $n!\rho_{\mathcal{B}_n}$-compatible Ore localizations S_w (w in permutation group Σ_n) and $\mathcal{B}_n$-comodule algebra maps $\gamma_w : \mathcal{B}_n \to \mathcal{O}(SL_q(n))[S_w^{-1}]$ is a highly nontrivial fact which we have shown elsewhere ([29, 28]; using results on quantum minors in [32, 34]). It may be used to obtain the $\mathcal{O}(SU_q(n))$-coherent states.

Using the q-binomial theorem, one obtains (in $V_n \otimes \mathcal{G}_d$)

$$\rho(y^n) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}} q^{-\binom{i}{2}} x^i y^{n-i} \otimes u^i d^n$$

Basis vectors $v^n_i = \sqrt{\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_{q^{-2}}} x^i y^{n-i}$ are orthonormal. Thus

$$C_d := \sum_{i=0}^{n} q^{-\binom{i}{2}} \sqrt{\begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}}} v^n_i \otimes u^i,$$

satisfies (6). Similar formula defines C_b and the rest of requirements hold for these data. Thus

$$A = \int_{SU_q(2)} \sum_{i,j=0}^{n} \sqrt{\begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{-2}}} q^{\binom{i}{2} + \binom{j}{2}} v^n_i \otimes (v^n_j)^* \otimes u^i d^n (u^j d^n)^*$$

LEMMA 3.

$$\int_{SU_q(2)} u^i d^n (u^j d^n)^* = \begin{cases} 0, & i \neq j \\ \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]^{-1}_{q^{-2}} q^n q^{2\binom{i}{2}} [n+1]_q^{-1}, & i = j \end{cases}$$

Proof. $(u^j d^n)^* = q^{\binom{i}{2}} (d^*)^{n-j} (b^*)^j = q^{\binom{i}{2}} (-q)^j a^{n-j} c^j$. The identity

$$d^r a^r = (1 + q^{-1} bc)(1 + q^{-3} (bc)^2) \ldots (1 + q^{-2n-1} (bc)^r) = (q^{-2}\zeta; q^{-2})_r,$$

implies $d^{n-i} a^{n-j} = d^{j-i} (q^{-2}\zeta; q^{-2})_{n-j}$ for $j \geq i$ and $(q^{-2}\zeta; q^{-2})_{n-j} a^{i-j}$ for $i < j$. Thus, for $j \geq i$,

$$\begin{aligned} u^i d^n (u^j d^n)^* &= q^{\binom{i}{2} + \binom{j}{2}} (-q)^j b^i d^{j-i} (q^{-2}\zeta; q^2)_{n-j} c^j \\ &= q^{\binom{i}{2} + \binom{j}{2}} (-q)^j b^i (q^{-2}\zeta; q^2)_{n-j} d^{j-i}, \end{aligned}$$

what is for $j > i$ an element in compl(ζ) hence it vanishes after integration, likewise an expression for $i < j$, and only the terms with $i = j$ survive. Then

$u^i d^n (u^i d^n)^* = -q^{2\binom{i}{2}} \zeta^i (q^{-2}\zeta; q^{-2})_{n-i}$, and using (52′) in Chapter 4 of [15] one derives

$$\int\limits_{SU_q(2)} \zeta^i (q^{-2}\zeta; q^{-2})_{n-i} = \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}} q^n [n+1]_q^{-1}, \tag{9}$$

and the rest of the calculation is immediate.

Now

$$A = -\int\limits_{SU_q(2)} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}} |i\rangle\langle i| \otimes \zeta^i (q^{-2}\zeta; q^{-2})_{n-i}$$

$$\int\limits_{SU_q(2)} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}} |i\rangle\langle i| \otimes \zeta^i (q^{-2}\zeta; q^{-2})_{n-i} = [n+1]_q^{-1} q^n \sum_{i=0}^{n} |i\rangle\langle i|.$$

The sum on RHS is of course the unity. The fact that there was no additional factors depending on i is the nontrivial property of coherent states (Theorem 4). There are many proposals for "$SU_q(2)$-coherent states" in literature (search e.g. MathSciNet) with similar (partly guessed) formulas with wrong q-factors and still having some "resolution of unity" formulas. The wrong factors are compensated by effectively changing the measure as well, for which there is no freedom as $SU_q(2)$ has only one invariant integral up to an *overall* constant!

In other words, $\alpha = q^n [n+1]_q^{-1}$ and the resolution of unity is

$$I = q^{-n} [n+1]_q \int\limits_{SU_q(2)} |C\rangle d\mu(\chi) \langle C|.$$

Formula (9) boils down to an integral representation of Ramanujan's q-beta function (Theorem 10.3.1 in [1]; cf. also [2])

$$\int_0^1 x^\alpha \frac{(qx;q)_\infty}{(q^\beta x;q)_\infty} \mathrm{d}_q x = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

If $\beta \geq 1$ is a positive integer, then the ratio in the integrand equals a polynomial in q and x, namely, $(1-x)(1-qx)\ldots(1-q^{\beta-1}x)$. Like for the ordinary beta function, there is another integral representation involving q-integral from 0 to ∞ with a polynomial in the *denominator*. Namely, instead of the Haar integral over $\zeta = -qbc$ one effectively has a geometric integration over (deformed) two-sphere with real coordinates $u = bd^{-1}$ and $\bar{u}$. However, in the denominator form, new q-factors appear depending on q^i. Jurčo [12] wrote a similar formula without extra q^i factors, but both the "measure" and the coherent states are changed. Hence those "coherent states" do not satisfy the defining factorization property (6) and the measure is not the invariant one.

q-coherent states in selected local coordinates in concrete examples $\mathcal{O}(SU_q(n))$ with $n = 2, 3$, appeared in [9,10,12,25], though without full geometric justification, and sometimes with nongeometric factors. Rudiments of another picture involving quantum group coherent states, related to geometric quantization and orbit method, are discussed in [35]. Finally, a local picture (i.e. calculations in main Bruhat cell) of the coherent states for the case of compact forms of quantum groups of types A,B,C,D, which differs from but is related to ours, is in impressive work by Jurčo and Šťovíček [13,14]. Their family of coherent states, Γ (cf. (5.1) in [14]), live in $V \otimes \mathcal{G}$, i.e. generalize a map $G \to V$ rather than $G/B \to V$. They however calculate some expression in corresponding coordinates on a cell in homogeneous space, working in a big Zariski open cell (without rigorous justification for localization). Proposition 5.11 in [14] is stating the factorization property (our formula (6)) of their quantity $w_\lambda^{-1}\langle\Gamma, u\rangle$ which "belongs to some completion" and basically agrees with our coherent states. Their construction relies on structure properties of quantum groups, while our axiomatics allows a priori treatment of Hopf algebras of more general origin. Furthermore, our construction utilizes the globalization of the geometry on the quantum homogeneous space.

In commutative case, the elements of a family of coherent states form the projective orbit of the highest weight vector. The generators of $\mathcal{G}_\lambda^{\mathrm{co}\mathcal{B}}$ are the analogues of the local coordinates on a big open cell in the coset space, and the coherent vector C_λ may be viewed as a parametrization of an open set in projective orbit by points in a coset space. In similar spirit, in the case of $\mathcal{O}(SL_q(3))$, the reference [25] views $\mathcal{G}_\lambda^{\mathrm{co}\mathcal{B}}$ as an analogue of the (algebra of functions on) unipotent group parametrizes quantum orbit (though they note this algebra is not a bialgebra, unlike the classical case). Here we clarify that, as in the classical case, this should be understood as a parametrization of an open dense subset of orbit, the latter being a noncommutative space.

It remains a very interesting problem to extend the geometrical theory of minimal uncertainty relations for coherent states [36] to quantum groups, cf. [7] and arXiv version v4 of this article.

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