

# Leibniz rules for enveloping algebras and a diagrammatic expansion

Stjepan Meljanac, Zoran Škoda

*Theoretical Physics Division, Institute Rudjer Bošković, Bijenička cesta 54, P.O.Box 180,  
HR-10002 Zagreb, Croatia*

---

## Abstract

Given a finite-dimensional Lie algebra, and a representation by derivations on the completed symmetric algebra of its dual, a number of interesting twisted constructions appear: certain twisted Weyl algebras, deformed Leibniz rules, quantized “star” product. We first illuminate a number of interrelations between these constructions and then proceed to study a special case in certain precise sense corresponding to the symmetric ordering. It has been known earlier that this case is related to the computations with Hausdorff series, for example an expression for the star product is in such terms. For the deformed Leibniz rule, hence a coproduct, we present here a new nonsymmetric expression, which is then expanded into a sum of expressions labelled by a class of planar trees, and for a given tree evaluated by Feynman-like rules. These expressions are graded by a bidegree and we show recursion formulas for the component of fixed bidegree, and compare the recursion to the recursions for Hausdorff series, including a nontrivial comparison of initial conditions. This way we show a direct correspondence between the Hausdorff series and the expression for twisted coproduct. The expansion labelled by planar series in particular gives a new way to reorganize the Hausdorff series.

*Keywords:* universal enveloping algebra, coexponential map, deformed coproduct, star product, Hausdorff series, Weyl algebra, planar tree

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>The data defining the setup</b>	<b>3</b>
<b>3</b>	<b>Deformed derivatives</b>	<b>9</b>
<b>4</b>	<b>Symmetric ordering</b>	<b>14</b>

---

*Email address:* meljanac@irb.hr, zskoda@irb.hr (Stjepan Meljanac, Zoran Škoda)

5	Tree calculus for symmetric ordering	16
6	Some facts on Hausdorff series	23
7	Fourier notation and using exponentials	25
8	The results leading to the proof of the main theorem	25
9	Special cases and other results	27

## 1. Introduction

**1.1.** It is a standard and useful viewpoint to consider the enveloping algebras as deformations of symmetric algebras. Still, there are many structures and formulas coming from exploration of this viewpoint which are not yet demonstrated or known in full generality.

Enveloping algebras appear in the deformation quantization of linear Poisson structures on the dual vector space  $\mathfrak{g}^*$  of a Lie algebra  $\mathfrak{g}$  ([6]). In deformation quantization one deforms given commutative product to a noncommutative product on the same vector space of 'functions' or 'observables'; this tautologically gives a vector space isomorphism between the commutative and the noncommutative algebra. In the case of linear Poisson structure, one considers a vector space isomorphism  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ . One of the common choices is the coexponential map (cf. Section 4). Another important case is the case of so-called Duflo map, when the coexponential map is composed with another natural map. The Kontsevich quantization for linear Poisson structures corresponds to the latter case.

**1.2.** When one interprets  $U(\mathfrak{g})$  as a 'noncommutative space' one would like also to have appropriate notions of noncommutative tangent vectors, or conjugate momenta, and appropriate Leibniz rules for corresponding 'noncommutative derivations'. This way one is not deforming only the 'coordinate space' but the whole 'phase space' (and even the space symmetries). We do not know good proposals how to do this for general  $\xi$ .

However, in some examples of particular  $\mathfrak{g}$  and particular  $\xi$ , physicists introduced (see e.g. [1, 3, 10]) *ad hoc* techniques to determine consistent definitions and formulas for such 'tangent vectors'. Unlike the case of various  $q$ -derivations in quantum group literature, the 'deformed derivations' in these proposals commute by (simplifying) assumption, which surprisingly worked (for the purposes of extending many elaborated algebraic constructions from commutative to well-behaved noncommutative counterparts).

We noticed that in fact a similar story works for general  $\mathfrak{g}$ , but *precisely* for those  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  which are *isomorphisms of coalgebras*, and not only of vector spaces. That class of deformation maps has not been systematically studied in the mathematics literature, except for the coexponential map, which is an example. The map corresponding to the Kontsevich quantization is *not* an example. In physics literature many other such maps are considered, and the

choice is usually determined by other data (the results in Section 2 show that these data are equivalent to giving  $\xi$ ). For some  $\mathfrak{g}$  we know the classification of all such maps  $\xi$  (cf.[8, 9, 10]).

**1.3.** In the present article we define and study the deformed Leibniz rules and related structures for general  $\xi$  in our class (being a *coalgebra* isomorphism and in certain sense close to identity). Then we make elaborate calculations when  $\xi$  is the coexponential map ('symmetric/Weyl ordering', 'PBW quantization'), including the main results on coproducts.

Our trees and Feynman-like rules explored to handle the combinatorics of the main proof, have no apparent connection to now standard graphical calculations in the study of Kontsevich quantization ([6, 7]).

**1.4.** The main result of this paper says that the formula in **5.13**, which relates the deformed coproduct for derivations to the star product holds for symmetric ordering. This result can be extended to some other cases, and we conjecture it to be true in general. The first author has earlier shown and practically used this formula in some special cases. In some cases the deformed coproduct can be explicitly calculated by other means, what makes the whole expression effective. It is an open problem for which  $\mathfrak{g}$  and which  $\xi$  the corresponding star product may be obtained using Drinfeld twist. In few special cases our formula relating deformed coproduct for derivations with the star product, suggested a way to find the formula for the Drinfeld twist.

**1.5. (Hopf algebras)** All bialgebras in the article will be associative, coassociative, with a unit map  $\eta$  and a counit  $\epsilon$ , without gradings. Hopf algebras will be bialgebras with an antipode and the standard Sweedler notation for the coproduct  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$  is few times used, with or without the summation sign. Recall that the elements  $h \in H$  such that  $\Delta(h) = 1 \otimes h + h \otimes 1$  are called primitive.

## 2. The data defining the setup

**2.1.** Fix an  $n$ -dimensional Lie algebra  $\mathfrak{g}$  over a field  $\mathbf{k}$  with basis  $\hat{x}_1, \dots, \hat{x}_n$ , and let  $\partial^1, \dots, \partial^n$  be the dual basis of  $\mathfrak{g}^*$ . The equation  $[\hat{x}_\mu, \hat{x}_\nu] = C_{\mu\nu}^\lambda \hat{x}_\lambda$  defines the structure constants  $C_{\mu\nu}^\lambda$  of  $\mathfrak{g}$ . We will usually write  $x_1, \dots, x_n$  for the basis  $\hat{x}_1, \dots, \hat{x}_n$  considered as generators of the symmetric algebra  $S(\mathfrak{g})$  to distinguish it from elements  $\hat{x}_1, \dots, \hat{x}_n$  viewed as generators of the universal enveloping algebra  $U(\mathfrak{g})$ . Thus  $x_\mu$  mutually commute, while  $\hat{x}_\mu$  don't. By  $\widehat{S(\mathfrak{g}^*)}$  or  $\hat{S}(\mathfrak{g}^*)$  we will denote the completed symmetric algebra on  $\mathfrak{g}^*$ , which may be viewed as a ring of formal power series in  $n$  variables  $x_1, \dots, x_n$ . The main message of this section is that there are correspondences between several kinds of data:

- Homomorphisms of Lie algebras  $\tilde{\phi} : \mathfrak{g} \rightarrow \text{Der}_{\mathbf{k}}(\hat{S}(\mathfrak{g}^*), \hat{S}(\mathfrak{g}^*))$
- Matrices  $(\phi_\beta^\alpha)_{\alpha, \beta=1, \dots, n}$  of elements  $\phi_\beta^\alpha \in \hat{S}(\mathfrak{g}^*)$  satisfying the system of formal differential equations (4).

- $\mathbf{k}$ -linear maps  $\phi : \mathfrak{g} \rightarrow \text{Hom}_{\mathbf{k}}(\mathfrak{g}^*, \hat{S}(\mathfrak{g}^*))$  such that the matrix with components  $\phi(-\hat{x}_\nu)(\partial^\mu)$  satisfies (4).
- Hopf actions of  $U(\mathfrak{g})$  on  $\hat{S}(\mathfrak{g}^*)$ .
- Algebra homomorphisms  $U(\mathfrak{g}) \rightarrow \hat{A}_{n,\mathbf{k}}$  of the form  $\hat{x}_\mu \mapsto \sum_{\alpha=1}^n x_\alpha \phi_\mu^\alpha$  on a basis  $\hat{x}_1, \dots, \hat{x}_n$  of  $\mathfrak{g}$ , with  $\phi_\beta^\alpha \in \hat{S}(\mathfrak{g}^*)$  for  $\alpha, \beta = 1, \dots, n$ . Here  $\hat{A}_{n,\mathbf{k}}$  is the (semi)completed Weyl algebra with generators  $x_1, \dots, x_n, \partial^1, \dots, \partial^n$  which is completed with respect to the powers of  $\partial^i$ -s.
- Coalgebra isomorphisms  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  which are identity on  $\mathfrak{g} \oplus \mathbf{k} = U^1(\mathfrak{g}) \subset U(\mathfrak{g})$ .

These correspondences are not difficult to show. Nevertheless, many special cases of such data are studied in (mainly recent physics) literature, often with confusion about the definitions, nature, the level of generality, the correspondences between these data and related constructions.

**2.1.1.** The correspondences above are bijections under the assumption that the maps  $\phi$  etc. are close to the “unit” case: for example in the language of  $\phi_\beta^\alpha$  the unit case means  $\phi_\beta^\alpha = \delta_\beta^\alpha$  and the nearby is in the sense of the topology on  $\mathbf{k}[[\partial^1, \dots, \partial^n]]$  corresponding to the filtration by powers. Analytic (nonformal) case of the correspondences (variants in which one considers *converging* power series) is interesting as well, but more difficult, and we do not have any closed sufficiently general results for that case.

**2.1.2.** Note that the list can be meaningfully extended. For example, there are popular “ordering prescriptions” which are various concrete ways determining the coalgebra isomorphism  $\xi$  above (as the isomorphism is trivial on generators in  $\mathfrak{g}$  one needs to know what to do with higher polynomials, hence “ordering prescriptions”).

**2.1.3. (Related datum: bijection  $K$ )** There are various interesting functional space extensions of  $S(\mathfrak{g})$  viewed as the space of polynomial functions on  $\mathfrak{g}^*$ . Exponential functions are central to many calculations and one would like to extend the map  $\xi$  to some nice completions  $\hat{S}(\mathfrak{g}) \rightarrow \hat{U}(\mathfrak{g})$ . For algebraic manipulations the natural candidate is the formal power series completion  $\hat{S}(\mathfrak{g})$  as a domain, but unfortunately the appropriate analogue  $\hat{U}(\mathfrak{g})$  for general  $\mathfrak{g}$  does not exist. It is enough to know the restriction of  $\xi$  to some dense subset of  $\hat{S}(\mathfrak{g})$ , and the vector subspace spanned by all exponential power series  $\exp(kx)$  for varying  $k \in \mathbf{k}$  is a good candidate for such subset. For convenience while comparing with the methods in the literature, we will prefer to work with  $\exp(ikx)$ ; of course all our calculations with  $\exp(ikx)$  may be easily redone for  $\exp(kx)$  in the case that  $i = \sqrt{-1} \notin \mathbf{k}$  (this case is less interesting for applications we have in mind). First observation is that  $\xi(\exp(ikx))$  is of the form  $\xi(\exp(iK(kx)))$  where  $K : \mathbf{k}^n \rightarrow \mathbf{k}^n$  is a bijection with  $K(0) = 0$ . The bijection  $K$  is determined by  $\phi$  and in turn determines  $\phi$ . However, we do not know any *general* rule which bijections  $K$  are admissible, though we know the classification results for some very special  $\mathfrak{g}$  (cf. [9], especially formula (27)).

**2.2. (Morphism  $\phi$  and the equation it satisfies)** Suppose we are also given a linear map  $\phi : \mathfrak{g} \rightarrow \text{Hom}_{\mathbf{k}}(\mathfrak{g}^*, \widehat{S(\mathfrak{g}^*)})$ . We want to extend this map to a  $\mathbf{k}$ -linear map into continuous derivations also denoted  $\tilde{\phi} : \mathfrak{g} \rightarrow \text{Der}_{\mathbf{k}}(\widehat{S(\mathfrak{g}^*)}, \widehat{S(\mathfrak{g}^*)})$ . By the commutativity of  $\widehat{S(\mathfrak{g}^*)}$ , it must hold that

$$\tilde{\phi}(\hat{x})(v_1 \cdots v_n) = \sum_{i=1}^n v_1 \cdots v_{i-1} v_{i+1} \cdots v_n \phi(\hat{x})(v_i). \quad (1)$$

This formula is linear in all arguments and symmetric under their permutations, hence by linearity in all arguments it defines a unique extension of  $\phi(\hat{x})$  to a well-defined map  $\tilde{\phi}(x) \in \text{Hom}_{\mathbf{k}}(S(\mathfrak{g}^*), \widehat{S(\mathfrak{g}^*)})$ . It is straightforward to check that  $\tilde{\phi}(\hat{x})$  defined via (1) is indeed a derivation. By abuse of notation, we will henceforth denote the extension  $\tilde{\phi}$  also by  $\phi$ .

Let  $\partial^1, \dots, \partial^n$  be a vector space basis of  $\mathfrak{g}^*$ . Then, in terms of (algebraically defined) partial derivatives  $\frac{\partial}{\partial(\partial^i)}$ , the condition (1) generalizes to the usual chain rule on  $\widehat{S(V)}$

$$\phi(\hat{x})(f) = \sum_{i=1}^m \frac{\partial}{\partial(\partial^i)}(f) \phi(\hat{x})(\partial^i) \quad (2)$$

Finally, we continuously extend  $\phi$  to a  $\mathbf{k}$ -linear map  $\phi : \mathfrak{g} \rightarrow \text{Der}_{\mathbf{k}}(\widehat{S(\mathfrak{g}^*)}, \widehat{S(\mathfrak{g}^*)})$ . It is a known fact that every derivation of the ring of formal power series in  $n$  variables is continuous, hence this procedure gives all  $\mathbf{k}$ -linear maps  $\phi : \mathfrak{g} \rightarrow \text{Der}_{\mathbf{k}}(\hat{S}(\mathfrak{g}), \hat{S}(\mathfrak{g}))$ .

The enveloping algebra  $U(\mathfrak{g})$  is a Hopf algebra with elements of  $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$  being primitive. If the linear map  $\phi : \mathfrak{g} \rightarrow \text{Der}_{\mathbf{k}}(\widehat{S(\mathfrak{g}^*)})$  is a homomorphism of Lie algebras, i.e.

$$\phi(\hat{x})\phi(\hat{y}) - \phi(\hat{y})\phi(\hat{x}) - \phi([\hat{x}, \hat{y}]) = 0, \quad \hat{x}, \hat{y} \in \mathfrak{g}, \quad (3)$$

then  $\phi$  extends multiplicatively to a unique Hopf action of  $U(\mathfrak{g})$ , i.e. to a homomorphism  $\phi : U(\mathfrak{g}) \rightarrow \text{End}_{\mathbf{k}}(\widehat{S(\mathfrak{g}^*)})$  satisfying  $\phi(u)(fg) = m_{\hat{S}(\mathfrak{g}^*)}(\phi \otimes \phi)\Delta(u)(f \otimes g) = \sum \phi(u_{(1)})(f)\phi(u_{(2)})(g)$ , for all  $f, g \in \widehat{S(\mathfrak{g}^*)}$  and  $u \in U(\mathfrak{g})$ , where  $m_{U(\mathfrak{g})}$  is the multiplication map on  $U(\mathfrak{g})$ . Denote

$$\phi_{\beta}^{\alpha} = \phi_{\beta}^{\alpha}(\partial^1, \dots, \partial^n) := \phi(-\hat{x}_{\beta})(\partial^{\alpha}) \in \widehat{S(\mathfrak{g}^*)}.$$

The formal power series  $\phi_{\beta}^{\alpha} = \phi_{\beta}^{\alpha}(\partial^1, \dots, \partial^n)$  has algebraically defined partial derivatives

$$\frac{\partial}{\partial(\partial^i)} \phi_{\beta}^{\alpha} \in \widehat{S(\mathfrak{g}^*)}.$$

Then  $\phi(\hat{x}_i)\phi(\hat{x}_j)(\partial^k) = \phi(\hat{x}_i)(-\phi_j^k) = -\frac{\partial}{\partial(\partial^i)}(\phi_j^k)\phi(\hat{x}_i)(\partial^l) = -\frac{\partial}{\partial(\partial^i)}(\phi_j^k)\phi_i^l$ . Thus the condition (3) reads for  $\hat{x} = \hat{x}_i$  and  $\hat{y} = \hat{x}_j$

$$\phi_j^l \frac{\partial}{\partial(\partial^l)}(\phi_i^k) - \phi_i^l \frac{\partial}{\partial(\partial^l)}(\phi_j^k) = C_{ij}^s \phi_s^k. \quad (4)$$

**2.3. (Weyl algebras)** Consider the usual Weyl algebra  $A_{n,\mathbf{k}}$  with generators  $x_1, \dots, x_n, \partial^1, \dots, \partial^n$ , and relations  $[x_i, x_j] = [\partial^i, \partial^j] = 0$  and  $x_i \partial^j - \partial^j x_i = \delta_j^i$  for all  $i, j = 1, \dots, n$ . As the notation for the generators suggests, the underlying vector space of  $A_{n,\mathbf{k}}$  will be identified with  $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$ . Let also  $\hat{A}_{n,\mathbf{k}}$  be the completion of  $A_{n,\mathbf{k}}$  with respect to the descending filtration by the degree of “differential operator”.

**2.4. Proposition.** *The correspondence  $\hat{x}_i \mapsto \hat{x}_i^\phi := \sum_{j=1}^n x_j \phi_j^i$ , where  $\phi_j^i \in \hat{S}(\mathfrak{g}^*)$  extends to an algebra homomorphism*

$$()^\phi : U(\mathfrak{g}) \rightarrow \hat{A}_{n,\mathbf{k}}$$

iff (4) holds.

**2.5.** A universal formula for a “symmetric” solution to (4) has been found ([4]), for any ring  $\mathbf{k} \supset \mathbb{Q}$ , and  $\mathfrak{g}$  a Lie algebra over  $\mathbf{k}$  which is finite rank free as a module over  $\mathbf{k}$ , and where  $\phi$  is a monomorphism. See also Section 4.

**2.6. (Smash product algebras)** Given any Hopf algebra  $H$  and a, say left, Hopf action of  $H$  on algebra  $\mathcal{S}$ ,  $h \otimes s \mapsto h \triangleright s$ , one forms a crossed product algebra (in Hopf literature “smash product”)  $\mathcal{S} \# H$ . As a vector space, it is simply the tensor product vector space  $S \otimes \mathcal{H}$  and the associative product is given by

$$(s \otimes h)(s' \otimes h') = \sum s(h_{(1)} \triangleright s') \otimes h_{(2)} h'.$$

The canonical embeddings  $S \hookrightarrow \mathcal{S} \# H$  and  $H \hookrightarrow \mathcal{S} \# H$  will be considered identifications, and one usually omits the tensor sign because  $s \otimes h = sh$  with respect to these embeddings and the product in  $\mathcal{S} \# H$ . Then  $h \triangleright s = \sum h_{(1)} s S_H(h_{(2)})$  where  $S_H : H \rightarrow H$  is the antipode. Furthermore, the rule

$$(s \# h) \triangleright s' := s(h \triangleright s') \tag{5}$$

defines an action of  $\mathcal{S} \# H$  on  $\mathcal{S}$ .

Analogously, for any right action of  $H$  on  $\mathcal{S}$  one defines the crossed product denoted by  $H \# \mathcal{S}$ , whose underlying vector space is  $H \otimes \mathcal{S}$ . If the antipode  $S_H : H \rightarrow H^{\text{op}}$  is bijective, there is a bijective correspondence between the left and right actions (namely, composing with  $S_H$ ) and the crossed products for the two corresponding (left and right) actions are canonically isomorphic and we often identify them throughout the article.

**2.7. (( $\mathfrak{g}, \phi$ )-deformed Weyl algebras.)** Regarding that for any  $\mathfrak{g}$  and  $\phi$  such that (4) holds, the representation  $\phi : U(\mathfrak{g}) \rightarrow \widehat{S(\mathfrak{g}^*)}$  corresponds to a left Hopf action, and we may define the smash product algebra

$$A_{\mathfrak{g},\phi} := \widehat{S(\mathfrak{g}^*)} \# U(\mathfrak{g}) = \widehat{S(\mathfrak{g}^*)} \#_\phi U(\mathfrak{g}),$$

where the left action  $u \triangleright s := \phi(u)(s)$  is uniquely determined by the values  $\phi(-\hat{x}_i)(\partial^j) = \phi_j^i$  as explained above. In Lie case, the antipode is invertible, hence the above smash product is isomorphic to  $U(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)}$  where the right

action is  $s \triangleleft u = \phi(Su)(s)$  and where  $Su = -u$  if  $u \in \mathfrak{g} \subset U(\mathfrak{g})$ . The rule (5) specializes to a (dual) "natural" action of  $A_{\mathfrak{g},\phi}$  on  $\widehat{S(\mathfrak{g}^*)}$ . In particular, if  $\mathfrak{g} = \mathfrak{a}$  is an abelian Lie algebra, and  $\phi$  is given by the bilinear pairing  $\phi(-\hat{x}_i)(\partial^j) = \delta_i^j$ , then  $A_{\mathfrak{g},\phi}$  is isomorphic as algebra to the usual (semi)completed Weyl algebra  $\widehat{A}_{n,\mathbf{k}}$  and the action is the usual action of  $S(\mathfrak{a})$  on  $\widehat{S(\mathfrak{a}^*)}$ . For another extreme case, consider a general  $\mathfrak{g}$  but with full degeneration:  $\phi = 0$  identically. Then  $A_{\mathfrak{g},\phi} \cong U(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$  as algebras (elements in  $\mathfrak{g}$  and  $\mathfrak{g}^*$  commute).

**2.8.** From now on we suppose

- (i)  $\phi : \mathfrak{g} \rightarrow \text{Der}(\widehat{S(\mathfrak{g}^*)})$  is a homomorphism of Lie algebras
- (ii) the matrix  $\phi$  (not bold) with entries  $\phi_j^i := \phi(-\hat{x}_i)(\partial^j)$  has the unit matrix as its constant term, i.e.  $\phi_j^i = \delta_j^i + O(\partial)$ .

**2.9.** Under the assumptions from **2.8**,  $\phi$  is invertible as a matrix over the formal power series ring  $\mathbf{k}[[\partial^1, \dots, \partial^n]]$  and the homomorphism  $U(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)} \cong S(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)}$  given on generators by

$$\hat{x}_\alpha \mapsto x_\beta \phi_\alpha^\beta, \quad \partial^\mu \mapsto \partial^\mu$$

is an isomorphism. Hence the (one-sidedly) completed deformed and undeformed Weyl algebras are isomorphic via a nontrivial map and we often identify them when doing calculations.

**2.10.** This isomorphism enables us to consider the homomorphism

$$(\ )^\phi : U(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)} \cong S(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)} \cong \widehat{A}_{n,\mathbf{k}}$$

which agrees with the unique homomorphism  $U(\mathfrak{g}) \rightarrow \widehat{A}_{n,\mathbf{k}}$  extending the rule

$$\hat{x}_\alpha \mapsto \hat{x}_\alpha^\phi := x_\beta \phi_\alpha^\beta \in \widehat{A}_{n,\mathbf{k}}$$

Furthermore, we may identify  $S(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)} \cong \text{Hom}_{\mathbf{k}}(S(\mathfrak{g}), S(\mathfrak{g}))$ . Here  $\phi_\alpha^\beta = \phi_\alpha^\beta(\partial^1, \dots, \partial^n)$  is understood as an element of the completed Weyl algebra  $\widehat{A}_{n,\mathbf{k}} \cong S(\mathfrak{g}) \# \widehat{S(\mathfrak{g}^*)}$  acting in the usual way (as differential operator;  $\phi_\alpha^\beta$  is with constant coefficients) on the polynomial algebra. Therefore we obtained an action, depending on  $\phi$ , of  $U(\mathfrak{g})$  on  $S(\mathfrak{g})$ .

**2.11. Lemma.** *Let  $\chi \in \mathbf{k}[[\partial^1, \dots, \partial^n]]$  and let  $\Delta$  be the standard coproduct making the polynomial algebra  $\mathbf{k}[x_1, \dots, x_n]$  a bialgebra. The natural action of  $\widehat{A}_{n,\mathbf{k}}$  on  $\mathbf{k}[x_1, \dots, x_n]$  makes  $x_\sigma \chi \in \widehat{A}_{n,\mathbf{k}}$  a coderivation of  $\mathbf{k}[x_1, \dots, x_n]$ :*

$$(x_\sigma \chi \otimes \text{id} + \text{id} \otimes x_\sigma \chi)(\Delta(f)) = \Delta(x_\sigma \chi(f)), \quad \forall f \in \mathbf{k}[x_1, \dots, x_n]. \quad (6)$$

*Proof.* By linearity it is enough to prove (6) when  $f$  is a monomial. We prove this by induction on the sum of the polynomial degree of  $f$  and the order of differential operator  $\chi$ . The identity is clearly true if either the degree of  $f$

or order of  $\chi$  is 0. Regarding that  $f$  is monomial it is of the form  $x_\gamma g$  where  $g$  is a monomial of a lower order. We identify  $\text{id}$  with 1 in Weyl algebra and  $x_\mu$  with multiplication with  $x_\mu$ . For step of induction we want to prove that

$$(x_\sigma \chi \otimes 1 + 1 \otimes x_\sigma \chi) \Delta(x_\gamma g) = \Delta(x_\sigma \chi(x_\gamma g))$$

provided this is true for  $\chi$  of lower order or  $x_\gamma g$  replaced by  $g$  what is of lower degree. Using the fact that  $\Delta$  is a homomorphism of algebras and that  $x_\gamma$  is primitive, we rewrite this equality using commutators:

$$\begin{aligned} (x_\sigma[\chi, x_\gamma] \otimes 1 + 1 \otimes x_\sigma[\chi, x_\gamma]) \Delta(g) + \\ + (x_\gamma \otimes 1 + 1 \otimes x_\gamma)(x_\sigma \chi \otimes 1 + 1 \otimes x_\sigma \chi) \Delta(g) \\ = \Delta(x_\sigma[\chi, x_\gamma](g)) + (x_\gamma \otimes 1 + 1 \otimes x_\gamma) \Delta(x_\sigma \chi(g)) \end{aligned}$$

and recall that  $[\chi, x_\gamma]$  is of lower order. This equality holds because it is a sum of two equations which hold by the assumption of the induction. Q.E.D.

**2.12. Corollary.** *The action from 2.10 restricted on  $\mathfrak{g}$  is an action by coderivations with respect to the standard coalgebra structure on  $S(\mathfrak{g})$ :*

$$(x_\beta \phi_\alpha^\beta \otimes \text{id} + \text{id} \otimes x_\beta \phi_\alpha^\beta) (\Delta_{S(\mathfrak{g})}(f)) = \Delta_{S(\mathfrak{g})}(x_\beta \phi_\alpha^\beta(f)), \quad \forall f \in S(\mathfrak{g}). \quad (7)$$

**2.13.** For us it is important to consider the special case of the action of  $U(\mathfrak{g})$  on  $S(\mathfrak{g})$  from 2.10, when  $f$  is  $1_{S(\mathfrak{g})} =: |0\rangle$  (action on ‘‘Fock vacuum’’). In the following proposition, the notation  $(\cdot)^\phi : \hat{u} \mapsto \hat{u}^\phi \in \hat{A}_{n, \mathbf{k}}$  is taken from 2.10, and the evaluation in the expression  $\hat{u}^\phi(1) = \hat{u}^\phi|0\rangle$  is understood in the sense of the natural action of  $\hat{A}_{n, \mathbf{k}}$  on  $\hat{S}(\mathfrak{g})$ .

**Proposition.** *The rule  $\xi_\phi^{-1} : \hat{u} \mapsto \hat{u}^\phi(1) = \hat{u}^\phi|0\rangle$  for  $u \in U(\mathfrak{g})$  is an isomorphism of coalgebras, which restricts to the identity on  $\mathbf{k} \oplus \mathfrak{g}$ .*

*Proof.* It is clear that  $\hat{x}_\mu|0\rangle = x_\mu$ , hence  $\xi_\phi^{-1}$  is indeed tautological on degree 1 terms. Consider the standard ascending filtration of the enveloping algebra  $\mathbf{k} = U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \dots \subset \cup_{n \geq 0} U_n(\mathfrak{g}) = U(\mathfrak{g})$ . We will now prove that  $(\cdot)^\phi|0\rangle$  restricts to a linear isomorphism from  $U_n(\mathfrak{g})$  onto  $S^{\leq n}(\mathfrak{g})$ . Suppose by induction that we have proven that  $(U_n(\mathfrak{g}))^\phi|0\rangle = S^{\leq n}(\mathfrak{g})$  for all  $n \leq n_0$ . By PBW theorem the dimensions of  $U_{n_0+1}(\mathfrak{g})$  and  $S_{n_0+1}(\mathfrak{g})$  are equal and finite, hence it is sufficient to prove that the linear map  $(\cdot)^\phi|0\rangle$  restricted to  $U_{n_0+1}(\mathfrak{g})$  surjects onto  $S^{\leq n_0+1}(\mathfrak{g})$ . Clearly the expressions of the form  $x_\nu P$  where  $P \in S^{\leq n_0}$  span  $S^{\leq n_0+1}(\mathfrak{g})$ . By assumption  $\hat{u}^\phi|0\rangle = P$  for some  $\hat{u} \in U_{n_0}(\mathfrak{g})$ . Now  $(\hat{x}_\nu \hat{u})^\phi|0\rangle = x_\alpha \phi_\nu^\alpha(P) = x_\nu P + x_\alpha (\phi_\nu^\alpha - \delta_\nu^\alpha)(P)$ . By assumption 2.8 (ii) on  $\phi$ ,  $\phi_\nu^\alpha - \delta_\nu^\alpha = O(\partial)$ , so that  $x_\alpha (\phi_\nu^\alpha - \delta_\nu^\alpha)(P)$  is a polynomial of the order at most  $n_0$ , hence by the assumption of the induction it is of the form  $\hat{v}^\phi|0\rangle$  where  $\hat{v} \in U_{n_0}(\mathfrak{g})$ . Therefore  $(\hat{x}_\nu \hat{u} - \hat{v})^\phi|0\rangle = x_\nu P$ , as required.

To show that the isomorphism respects the coalgebra structure we proceed by induction on the degree of monomial in the source  $U(\mathfrak{g})$ . Thus suppose that  $(\Delta_{U(\mathfrak{g})}(\hat{u}))^\phi(|0\rangle \otimes |0\rangle) = \Delta_{S(\mathfrak{g})}(\hat{u}^\phi|0\rangle)$  for each  $\hat{u} \in U_m(\mathfrak{g})$ . Then the expressions of the form  $\hat{x}_\lambda \hat{u}$  span  $U_{m+1}(\mathfrak{g})$  and  $\Delta_{U(\mathfrak{g})}(\hat{x}_\lambda \hat{u}) = (\hat{x}_\lambda \otimes 1 + 1 \otimes \hat{x}_\lambda) \Delta_{U(\mathfrak{g})}(\hat{u})$ . Therefore  $(\Delta_{U(\mathfrak{g})}(\hat{x}_\lambda \hat{u}))^\phi(|0\rangle \otimes |0\rangle) = (x_\beta \phi_\lambda^\beta \otimes 1 + 1 \otimes x_\beta \phi_\lambda^\beta) (\Delta_{U(\mathfrak{g})}(\hat{u}))^\phi|0\rangle$  what is by the assumption of induction equal to  $(x_\beta \phi_\lambda^\beta \otimes 1 + 1 \otimes x_\beta \phi_\lambda^\beta) \Delta_{S(\mathfrak{g})}(\hat{u}^\phi|0\rangle)$ , hence by (7) also to  $\Delta_{S(\mathfrak{g})}(x_\beta \phi_\lambda^\beta \hat{u}^\phi|0\rangle) = \Delta_{S(\mathfrak{g})}((\hat{x}_\lambda \hat{u})^\phi|0\rangle)$ , as required. Q.E.D.

**2.14.** Of course, the inverse of  $\xi_\phi^{-1}$  will be some isomorphism of coalgebras  $\xi = \xi_\phi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ . Conversely, every isomorphism of coalgebras  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  which is identity on  $\mathfrak{k} \oplus \mathfrak{g}$ , defines a map  $D^T : \mathfrak{g} \rightarrow \text{Coder}(S(\mathfrak{g}))$  into coderivations by  $D_x^T(f) = D^T(x)(f) = \xi^{-1}(\xi(x) \cdot_{U(\mathfrak{g})} \xi(f))$ . The dual map  $D_x : \widehat{S(\mathfrak{g}^*)} \rightarrow \widehat{S(\mathfrak{g}^*)}$  is a continuous derivation, and one has  $D_x^T(f) = -\sum_\alpha x_\alpha D_x(\partial^\alpha)(f)$  where the action on the left is the usual action as differential operator. Here  $\sum_\alpha x_\alpha \otimes \partial^\alpha \in \mathfrak{g} \otimes \mathfrak{g}^*$  is the “canonical element” (the image of  $\text{id}_{\mathfrak{g}}$  under the isomorphism  $\widehat{\text{Hom}_{\mathfrak{k}}(\mathfrak{g}, \mathfrak{g})} \rightarrow \mathfrak{g} \otimes \mathfrak{g}^*$ ). Thus one defines a Lie homomorphism  $\phi : \mathfrak{g} \rightarrow \text{Der}(\widehat{S(\mathfrak{g}^*)}, \widehat{S(\mathfrak{g}^*)})$  by  $x \mapsto D_x$  such that  $\phi_j^i = D_{x_j}(\partial^i)$  and  $\phi_j^i = \delta_j^i + O(\partial)$ .

**2.15. (Star product)** We saw in **2.14** that giving the Lie homomorphism  $\phi$  for which the matrix  $\phi(-\hat{x}_i)(\partial^j) = \delta_j^i + O(\partial)$  is equivalent to giving a coalgebra isomorphism  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  which is identity when restricted to  $\mathfrak{k} \oplus \mathfrak{g}$ . This isomorphism helps us define the star product

$$\star : S(\mathfrak{g}) \otimes S(\mathfrak{g}) \rightarrow S(\mathfrak{g}), \quad f \star g := \xi^{-1}(\xi(f) \cdot_{U(\mathfrak{g})} \xi(g)). \quad (8)$$

**2.15.1.** One should note that in literature related to the representation theory ([13]) and the deformation quantization ([7]) usually some other vector space isomorphisms  $S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  are important, which do not respect the coalgebra structure, but do have some other favorable properties. Our constructions below, however, essentially use the compatibility with the coalgebra structure.

### 3. Deformed derivatives

**3.1.** We now define the deformed derivatives in several ways and show that they are equivalent. Physicist who view  $U(\mathfrak{g})$  as some sort of the algebra of functions on a “Lie type noncommutative space” consider the deformed derivatives (which mutually commute) as a sensible choice of a basis of the tangent space to this noncommutative space ([3, 5, 8, 9, 10]).

**3.2. Proposition.** *Consider the topological algebra freely generated by the discrete  $U(\mathfrak{g})$  and the topological algebra  $\hat{S}(\mathfrak{g}^*)$  with the topology of the formal power series ring and quotient it by the closed ideal generated by the relations  $[\partial^i, \hat{x}_j] = \phi_j^i$  where  $(\phi_j^i)$  is a matrix with elements in  $\hat{S}(\mathfrak{g}^*)$ . The kernel of the quotient map restricted to  $\hat{S}(\mathfrak{g}^*)$  is zero iff the formal system of differential equations 4 holds.*

*Proof.* In the quotient, the Jacobi identities gives

$$[\partial^l, [\hat{x}_i, \hat{x}_j]] = [[\partial^k, \hat{x}_i], \hat{x}_j] + [\hat{x}_i, [\partial^k, \hat{x}_j]]$$

therefore

$$\sum_s C_{ij}^s \phi_s^k = [\phi_i^k, \hat{x}_j] + [\hat{x}_i, \phi_j^k].$$

This is the identity in the image of  $\hat{S}(\mathfrak{g}^*)$  in the quotient algebra and the commutators on the right hand side can be evaluated by the chain rule. The equations

identical to the system 4 follow, but interpreted in the quotient. Now if the kernel is zero, this condition is non-degenerate, hence the differential equations are satisfied. Conversely, if the equations follow then the above algebra is the smash product algebra.

**3.3. (An abstract action of the Fock space type)** Given a Hopf algebra  $H$ , a right Hopf action  $H$  on an algebra  $\mathcal{S}$ , and a homomorphism of unital algebras  $\epsilon^{\mathcal{S}} : \mathcal{S} \rightarrow \mathbf{k}$ , in addition to the smash product algebra  $H\sharp\mathcal{S}$  (cf. 2.6) one can also define a  $\mathbf{k}$ -linear left action of the smash product algebra on  $H$ . This action  $(H\sharp\mathcal{S}) \otimes H \xrightarrow{\blacktriangleright} H$  is the composition

$$(H\sharp\mathcal{S}) \otimes H \hookrightarrow (H\sharp\mathcal{S}) \otimes (H\sharp\mathcal{S}) \xrightarrow{m_{H\sharp\mathcal{S}}} H\sharp\mathcal{S} \xrightarrow{H\sharp\epsilon^{\mathcal{S}}} H \otimes \mathbf{k} \cong H.$$

This action restricts along the algebra embedding  $\mathcal{S} \hookrightarrow H\sharp\mathcal{S}$ ,  $s \mapsto 1 \otimes s$  to a left action  $\mathcal{S} \otimes H \rightarrow H$ . If the antipode  $S_H : H \rightarrow H^{\text{op}}$  is an isomorphism, the corresponding representation  $\rho : \mathcal{S} \rightarrow \text{End}_{\mathbf{k}}(H)$  is faithful. Using the definition 2.6 of the smash product algebra, we may write the restricted action  $\blacktriangleright|_{\mathcal{S} \otimes H} : \mathcal{S} \otimes H \rightarrow H$  in terms of Hopf action  $\triangleleft : \mathcal{S} \otimes H \rightarrow \mathcal{S}$  only:

$$s \otimes h \mapsto \sum h_{(1)}\sharp(s \triangleleft h_{(2)}) \mapsto \sum h_{(1)}\epsilon^{\mathcal{S}}(s \triangleleft h_{(2)}) = s \blacktriangleright h. \quad (9)$$

In particular,  $s \blacktriangleright 1_H = \epsilon^{\mathcal{S}}(s)1_H$ , and, if  $u \in H$  is primitive and  $h \in H$  then

$$\begin{aligned} s \blacktriangleright (uh) &= \sum (uh)_{(1)}(s \triangleleft (uh)_{(2)}) \blacktriangleright 1_H \\ &= \sum uh_{(1)}(s \triangleleft h_{(2)}) + h_{(1)}((s \triangleleft u) \triangleleft h_{(2)}) \blacktriangleright 1_H \\ &= us \blacktriangleright h + (s \triangleleft u) \blacktriangleright h. \end{aligned} \quad (10)$$

Similar formula holds for skew-primitive elements. The symbol for the action  $\blacktriangleright$  is often omitted below, unless when it is useful for clarity.

**3.3.1. (Interpretation)** This construction has a spirit of Fock space construction: think of derivatives as primitive elements generating  $\mathcal{S}$ . The smash product  $H\sharp\mathcal{S}$  has a multiplication which involves rearrangement of factors in  $H$  and factors in  $\mathcal{S}$ . Once we put derivatives to the right we act with them on vacuum, what amounts to the map  $\epsilon^{\mathcal{S}}$ , while the polynomial factor in  $\mathcal{S}$  stays intact.

**3.4.** We now specialize 3.3 to the case where  $\mathcal{S} := \widehat{S(\mathfrak{g}^*)}$ ,  $H := U(\mathfrak{g})$  and the Hopf action is induced by  $\phi : U(\mathfrak{g}) \rightarrow \text{Der}(\widehat{S(\mathfrak{g}^*)}, \widehat{S(\mathfrak{g}^*)})$ , and  $\epsilon^{\mathcal{S}} : P \mapsto P(1)$  is the application of a constant coefficient differential operator on 1 (derivatives act in the undeformed way on 1; nevertheless we view the unit 1 as *deformed vacuum*  $1_{U(\mathfrak{g})}$ ). Recall that in that case  $H\sharp\mathcal{S} = \hat{A}_{\phi, \mathbf{k}}$ . The corresponding representation  $\hat{A}_{\mathfrak{g}, \phi} \rightarrow \text{End}_{\mathbf{k}}(U(\mathfrak{g}))$  is called  **$\phi$ -deformed Fock space**. Then the restricted action  $\hat{S}(\mathfrak{g}^*) \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is given by  $s \blacktriangleright h = \sum h_{(1)}\epsilon^{\mathcal{S}}(\phi(S_{U(\mathfrak{g})}h_{(2)})(s))$  and if  $h = u \in \mathfrak{g}$  and  $s = \partial \in \mathfrak{g}^* \subset \hat{S}(\mathfrak{g}^*)$ , this gives

$$\partial \blacktriangleright u = \phi(-u)(\partial) \blacktriangleright 1_{U(\mathfrak{g})} \quad (11)$$

as from  $\partial 1 = 0$  the summand  $u\epsilon^S(\phi(1_H)(\partial))$  vanishes. In other words, if we restrict the left action  $\hat{S}(\mathfrak{g}^*) \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  to  $\mathfrak{g}^* \otimes \mathfrak{g}$  it coincides with the restriction of right Hopf action  $\hat{S}(\mathfrak{g}^*) \otimes U(\mathfrak{g}) \rightarrow S(\mathfrak{g}^*)$  to  $\mathfrak{g}^* \otimes \mathfrak{g}$ , followed by the evaluation at deformed vacuum  $1_{U(\mathfrak{g})}$ .

**3.5.** The restricted representation  $\rho \equiv \rho_\phi : \mathcal{S} \rightarrow \text{End}_{\mathbf{k}}(H)$  from **3.3** may be alternatively described in terms of its values  $\hat{\partial}^\mu := \rho_\phi(\partial^\mu) \equiv (\partial^\mu \blacktriangleright) \in \text{End}_{\mathbf{k}}(U(\mathfrak{g}))$  on the standard algebra generators  $\partial^1, \dots, \partial^n$  of  $S(\mathfrak{g}^*)$ . By (11), (10) and the fact that  $\hat{x}_\nu$  is primitive for each  $\hat{f} \in U(\mathfrak{g})$ , the identity

$$\hat{\partial}^\mu(\hat{x}_\nu \hat{f}) = \hat{x}_\nu \hat{\partial}^\mu(\hat{f}) + \phi(-\hat{x}_\nu)(\partial^\mu) \blacktriangleright \hat{f} = \hat{x}_\nu \hat{\partial}^\mu(\hat{f}) + \phi_\nu^\mu(\hat{\partial})(\hat{f}) \quad (12)$$

holds in  $U(\mathfrak{g})$ . Here  $\phi_\nu^\mu(\hat{\partial}) := \rho_\phi(\phi_\nu^\mu)$  is obtained from  $\phi_\nu^\mu$  by replacing  $\partial^\tau$  by their commuting images  $\hat{\partial}^\tau \in \text{End}_{\mathbf{k}}(U(\mathfrak{g}))$ .

**3.6.** We describe the action of  $\hat{\partial}^\mu$  on  $U(\mathfrak{g})$  alternatively taking (12) as the step of an inductive definition. First of all,  $\hat{\partial}^\mu(1) = 0$  and  $\hat{\partial}^\mu(\hat{x}_\nu) = \delta_\nu^\mu$ . Suppose  $\hat{\partial}^\mu$  is already defined on monomials of order up to  $n$ . Then any monomial of order  $n+1$  is of the form  $\hat{x}_\nu \hat{f}$  where  $\hat{\partial}(\hat{f})$  is already defined. We set

$$\hat{\partial}^\mu(\hat{x}_\nu \hat{f}) := [\hat{\partial}^\mu, \hat{x}_\nu](\hat{f}) + \hat{x}_\nu \hat{\partial}^\mu(\hat{f}) := \phi_\nu^\mu(\hat{f}) + \hat{x}_\nu \hat{\partial}^\mu(\hat{f}),$$

where  $\phi_\nu^\mu = \phi_\nu^\mu(\hat{\partial})$  (we can substitute  $\hat{\partial}$  because  $S(\mathfrak{g}^*)$  is a free commutative algebra and  $\hat{\partial}^\mu$  mutually commute as it may be shown a posteriori).  $\hat{\partial}$  is well defined on  $S(\mathfrak{g}^*)$  (hence by continuity on  $\widehat{S(\mathfrak{g}^*)}$ ), namely it is obviously well defined linear operator from the free algebra on abstract variables  $\hat{x}_\alpha$  to  $U(\mathfrak{g})$ , and if one takes the generators of the defining ideal of the enveloping algebra  $i_{\nu_1 \nu_2} = \hat{x}_{\nu_1} \hat{x}_{\nu_2} - \hat{x}_{\nu_2} \hat{x}_{\nu_1} - C_{\nu_1 \nu_2}^\alpha \hat{x}_\alpha$  then, applying our inductive rules for every of the three monomials on the right-hand side, we conclude that for every  $\hat{f} \in U(\mathfrak{g})$ ,

$$\begin{aligned} \hat{\partial}^\gamma(i_{\nu_1 \nu_2} \hat{f}) &= \phi_{\nu_1}^\gamma(\hat{x}_{\nu_2} \hat{f}) + \hat{x}_{\nu_1} \hat{\partial}^\gamma \hat{x}_{\nu_2}(\hat{f}) - \phi_{\nu_2}^\gamma(\hat{x}_{\nu_1} \hat{f}) - \hat{x}_{\nu_2} \hat{\partial}^\gamma \hat{x}_{\nu_1}(\hat{f}) - \\ &\quad - C_{\nu_1 \nu_2}^\alpha \phi_\alpha^\gamma(\hat{f}) - C_{\nu_1 \nu_2}^\alpha \hat{x}_\alpha \hat{\partial}^\gamma(\hat{f}) \\ &= \frac{\partial}{\partial(\partial^{\nu_2})}(\phi_{\nu_1}^\gamma)(\hat{f}) + \hat{x}_{\nu_2} \phi_{\nu_1}^\gamma(\hat{f}) + \hat{x}_{\nu_1} \phi_{\nu_2}^\gamma(\hat{f}) + \hat{x}_{\nu_1} \hat{x}_{\nu_2} \hat{\partial}^\gamma(\hat{f}) \\ &\quad - \left( \frac{\partial}{\partial(\partial^{\nu_1})}(\phi_{\nu_2}^\gamma)(\hat{f}) + \hat{x}_{\nu_1} \phi_{\nu_2}^\gamma(\hat{f}) + \hat{x}_{\nu_2} \phi_{\nu_1}^\gamma(\hat{f}) + \hat{x}_{\nu_1} \hat{x}_{\nu_2} \hat{\partial}^\gamma(\hat{f}) \right) \\ &\quad - C_{\nu_1 \nu_2}^\alpha \phi_\alpha^\gamma(\hat{f}) - C_{\nu_1 \nu_2}^\alpha \hat{x}_\alpha \hat{\partial}^\gamma(\hat{f}) \\ &= \left( \frac{\partial}{\partial(\partial^{\nu_2})}(\phi_{\nu_1}^\gamma) - \frac{\partial}{\partial(\partial^{\nu_1})}(\phi_{\nu_2}^\gamma) - C_{\nu_1 \nu_2}^\alpha \phi_\alpha^\gamma \right)(\hat{f}) \end{aligned}$$

The injectivity of  $\rho$  implies that  $\hat{\partial}^\gamma(i_{\mu\nu} \hat{f}) = 0$  for every  $\hat{f}$  iff the operator in the brackets on the right-hand side vanishes, what amount to our main assumption (4). It is trivial that  $\hat{\partial}(\hat{f} i_{\mu\nu}) = 0$  as well, namely this is sufficient to check for monomial  $\hat{f}$ , but this is  $\hat{f} \hat{\partial}(i_{\mu\nu}) + [\hat{\partial}, \hat{f}](i_{\mu\nu})$ . We already know that the first summand is zero. The commutator in the second summand is some polynomial in  $\hat{\partial}$ -s, hence it is clearly zero modulo  $i_{\mu\nu}$  by induction on the degree of monomials and linearity.

Notice for the classical case of the abelian Lie algebra, that  $[\hat{\partial}, \hat{f}] = \hat{\partial}(\hat{f})$ , while this is not true in general (the equality always makes sense: the left-hand side is the bracket  $\hat{\partial}\hat{f} - \hat{f}\hat{\partial}$  in the smash product  $\widehat{S(\mathfrak{g}^*)\sharp U(\mathfrak{g})}$ , while the right-hand side is in  $U(\mathfrak{g}) \hookrightarrow \widehat{S(\mathfrak{g}^*)\sharp U(\mathfrak{g})}$ ).

**3.7.** Given any operator  $P \in \text{End}_{\mathbf{k}} S(\mathfrak{g})$  we can transport it via the vector space isomorphism  $\xi = \xi_\phi$  from **2.14** to an operator  $P_U := \xi P \xi^{-1} \in \text{End}_{\mathbf{k}} U(\mathfrak{g})$ . It is particularly important for us to transport  $P \in \hat{A}_{n,\mathbf{k}}$  understood as operators  $S(\mathfrak{g}) \rightarrow S(\mathfrak{g})$  via the usual Fock representation. Now we give a convenient invariant description of  $\hat{\partial}^\mu$ .

**Proposition.**  $\partial_U^\mu = \hat{\partial}^\mu : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ .

In other words, the operators  $\hat{\partial}^\mu = \rho_\phi(\partial^\mu)$  from **3.4** are characterized by the formula

$$\hat{\partial}^\mu(\xi(f)) = \xi(\partial^\mu(f)), \quad f \in S(\mathfrak{g}),$$

where  $\xi = \xi_\phi$  is described in **2.14**. Therefore also  $\xi^{-1}\hat{\partial}^\mu = \partial^\mu\xi^{-1}$ .

*Proof.* It is clear that the values of  $\partial_U^\mu$  and  $\hat{\partial}^\mu$  agree on  $\mathbf{k} \oplus \mathfrak{g}$  giving a basis of induction on the standard filtration of  $U(\mathfrak{g})$ . Suppose now  $\partial_U^\mu \hat{u} = \hat{\partial}^\mu \hat{u}$  for every  $\hat{u}$  in  $U_n(\mathfrak{g})$ , and for all  $\mu$ . Regarding that both operators,  $\partial_U^\mu$  and  $\hat{\partial}^\mu$ , decrease the filtration, the equality extends to all formal power series  $P$  in  $\partial_U$ -s versus in  $\hat{\partial}$ -s:  $P(\partial_U)(\hat{u}) = P(\hat{\partial})(\hat{u})$ , for all  $\hat{u} \in U_n(\mathfrak{g})$ . For step of the induction, it suffices to show that  $\partial_U^\mu(\hat{x}_\nu \hat{u}) = \hat{\partial}^\mu(\hat{x}_\nu \hat{u})$  for every such  $\hat{u}$  and every  $\nu$ ; by the definition this reads  $\partial^\mu((\hat{x}_\nu \hat{u})|0) = (\partial^\mu \blacktriangleright (\hat{x}_\nu \hat{u}))|0$ . Now  $\partial^\mu \blacktriangleright (\hat{x}_\nu \hat{u}) = (\partial^\mu \hat{x}_\nu \hat{u}) \blacktriangleright 1_{U(\mathfrak{g})} = (\phi_\nu^\mu \hat{u}) \blacktriangleright 1_{U(\mathfrak{g})} + (\hat{x}_\nu [\partial^\mu, \hat{u}]) \blacktriangleright 1_{U(\mathfrak{g})} + (\hat{x}_\nu \hat{u} \partial^\mu) \blacktriangleright 1_{U(\mathfrak{g})}$ , and the rightmost summand is zero. The element  $[\partial^\mu, \hat{u}]$  is not only in  $\hat{A}_{\mathfrak{g},\phi}$  but actually in  $U(\mathfrak{g}) \subset \hat{A}_{\mathfrak{g},\phi}$ , therefore  $(\hat{x}_\nu [\partial^\mu, \hat{u}]) \blacktriangleright 1_{U(\mathfrak{g})} = \hat{x}_\nu \blacktriangleright [\partial^\mu, \hat{u}] = \hat{x}_\nu \blacktriangleright (\partial^\mu \blacktriangleright \hat{u}) = \hat{x}_\nu \blacktriangleright \partial_U^\mu(\hat{u})$  by the assumption of the induction.  $\partial_U^\mu(\hat{u})$  is in  $U(\mathfrak{g})$  hence we may write  $\hat{x}_\nu \blacktriangleright \partial_U^\mu(\hat{u}) = \hat{x}_\nu \partial_U^\mu(\hat{u})$ . On the other hand,  $(\phi_\nu^\mu \hat{u}) \blacktriangleright 1_{U(\mathfrak{g})} = \phi_\nu^\mu \blacktriangleright \hat{u} = (\phi_\nu^\mu)_U(\hat{u})$  by the assumption of induction (multiplicative extension to polynomials in  $\partial$ -s, discussed above). The conclusion is  $\partial^\mu \blacktriangleright (\hat{x}_\nu \hat{u}) = \hat{x}_\nu \blacktriangleright \partial_U^\mu(\hat{u}) + (\phi_\nu^\mu)_U(\hat{u})$ . Therefore,  $(\partial^\mu \blacktriangleright (\hat{x}_\nu \hat{u}))|0 = (\hat{x}_\nu \partial_U^\mu(\hat{u}) + (\phi_\nu^\mu)_U(\hat{u}))|0 = \hat{x}_\nu^\phi \partial^\mu(\hat{u}|0) + \phi_\nu^\mu(\hat{u}|0) = (x_\lambda \phi_\nu^\lambda \partial^\mu + \phi_\nu^\mu)(\hat{u}|0)$ . By the usual Leibniz rule this equals  $\partial^\mu(x_\lambda \phi_\nu^\lambda(\hat{u}|0)) = \partial^\mu((\hat{x}_\nu \hat{u})|0) \equiv (\partial_U^\mu(\hat{x}_\nu \hat{u}))|0$ . Q.E.D.

**3.8. Corollary.** *The composition  $S(\mathfrak{g}) \hookrightarrow \hat{A}_{n,\mathbf{k}} \cong \hat{A}_{\mathfrak{g},\phi} \xrightarrow{\blacktriangleright 1_{U(\mathfrak{g})}} U(\mathfrak{g})$  is  $\xi_\phi$ .*

*Proof.* By the definition of  $\xi_\phi^{-1}$ , this says  $(\hat{u}^\phi|0) \blacktriangleright 1_{U(\mathfrak{g})} = \hat{u}$  for any  $\hat{u} \in U(\mathfrak{g})$ . Again we proceed by induction by filtration.  $((\hat{x}_\nu \hat{u})^\phi|0) \blacktriangleright 1_{U(\mathfrak{g})} = (\hat{x}_\nu^\phi \hat{u}^\phi|0) \blacktriangleright 1_{U(\mathfrak{g})} = (x_\lambda \phi_\nu^\lambda \hat{u}^\phi|0) \blacktriangleright 1_{U(\mathfrak{g})}$ . Because  $x_\lambda \in S(\mathfrak{g})$ , we may write  $(x_\lambda (\phi_\nu^\lambda \hat{u}^\phi|0)) \blacktriangleright 1_{U(\mathfrak{g})} = x_\lambda \blacktriangleright ((\phi_\nu^\lambda \hat{u}^\phi) \blacktriangleright 1_{U(\mathfrak{g})})$ . By **3.7**,  $\phi_\nu^\lambda \hat{u}^\phi|0 = \phi_\nu^\lambda(\hat{\partial})(\hat{u}^\phi)|0$  and, regarding that  $\phi_\nu^\lambda(\hat{\partial})(\hat{u}^\phi)$  is of degree not bigger than the degree of  $\hat{u}^\phi$ , we may use the assumption of induction, in the form  $\phi_\nu^\lambda(\hat{\partial})(\hat{u}^\phi)|0 \blacktriangleright 1_{U(\mathfrak{g})} = \phi_\nu^\lambda(\hat{\partial})(\hat{u}^\phi)$ . Therefore,  $(\hat{x}_\nu^\phi \hat{u}^\phi|0) \blacktriangleright 1_{U(\mathfrak{g})} = x_\lambda \blacktriangleright \phi_\nu^\lambda(\hat{\partial}) \blacktriangleright \hat{u}^\phi = \hat{x}_\nu \hat{u}^\phi$ . Q.E.D.

**3.9. Corollary.** *The following (external) square of linear maps commutes:*

$$\begin{array}{ccc}
\hat{A}_{\mathfrak{g},\phi} & \xrightarrow{(-)\blacktriangleright 1_{U(\mathfrak{g})}} & U(\mathfrak{g}) \\
\hat{x} \rightarrow \hat{x}^\phi, \partial \rightarrow \partial \downarrow & & \downarrow (-)^\phi |0\rangle \\
\hat{A}_{n,\mathbf{k}} & \xrightarrow{(-)|0\rangle} & S(\mathfrak{g}).
\end{array} \tag{13}$$

3 inner triangles also commute in the sense of those morphism compositions which start either at  $U(\mathfrak{g})$  or  $S(\mathfrak{g})$  (hence 3 identities total). The diagonal  $S(\mathfrak{g}) \rightarrow \hat{A}_{\mathfrak{g},\phi}$  is the homomorphism of algebras  $x_\nu \mapsto \hat{x}_\lambda(\phi^{-1})_\nu^\lambda$  and the diagonal  $U(\mathfrak{g}) \rightarrow \hat{A}_{n,\mathbf{k}}$  is the homomorphism of algebras  $\hat{x}_\nu \mapsto x_\lambda \phi_\nu^\lambda$ . The left vertical arrow is an isomorphism of algebras and the right vertical arrow an isomorphism of coalgebras; the horizontal arrows are surjective linear maps, not respecting the multiplication in general. However, the horizontal arrows are homomorphisms of algebras when restricted to either the first or second tensor factor in  $\hat{A}_{\mathfrak{g},\phi} \cong U(\mathfrak{g}) \sharp_\phi \hat{S}(\mathfrak{g}^*)$  and in  $\hat{A}_{n,\mathbf{k}} \cong S(\mathfrak{g}) \sharp_\delta \hat{S}(\mathfrak{g}^*)$  respectively.

**3.10. Corollary.** *The deformed Fock space is a faithful module over  $(\mathfrak{g}, \phi)$ -deformed (completed) Weyl algebra  $\hat{A}_{\mathfrak{g},\phi}$  if  $\phi_\nu^\mu = \delta_\nu^\mu + O(\partial)$ .*

*Proof.* Suppose the opposite: the representation is not faithful. Then there is an element  $P \in \hat{A}_{\mathfrak{g},\phi}$  such that  $P \blacktriangleright \hat{u} = 0$  for all  $\hat{u} \in U(\mathfrak{g})$ . Thus  $(P\hat{u}) \blacktriangleright 1_{U(\mathfrak{g})} = 0$ , hence  $0 = ((P\hat{u}) \blacktriangleright 1_{U(\mathfrak{g})})|0\rangle = (P^\phi \hat{u}^\phi |0\rangle) \blacktriangleright 1_{U(\mathfrak{g})}$ . As  $P^\phi \hat{u}^\phi |0\rangle$  is by definition in  $S(\mathfrak{g})$  (not only in  $\hat{A}_{\mathfrak{g},\phi}$ ) then this means a fortiori  $P^\phi \hat{u}^\phi |0\rangle = 0$ . But as  $\hat{u}$  runs through whole  $U(\mathfrak{g})$ ,  $\hat{u}^\phi |0\rangle$  runs through whole  $S(\mathfrak{g})$ . Therefore this means that  $P^\phi$  is zero, because the classical Fock representation is faithful. But  $()^\phi$  is an isomorphism on the whole twisted Weyl algebra, therefore  $P$  vanishes as well. Q.E.D.

**3.11. Definition.** *The deformed coproduct  $\Delta(\hat{\partial}^\mu) = \sum \hat{\partial}_{(1)}^\mu \otimes \hat{\partial}_{(2)}^\mu$  is defined by*

$$\hat{\partial}(u \cdot_{U(\mathfrak{g})} v) = \sum \hat{\partial}_{(1)}^\mu(u) \cdot_{U(\mathfrak{g})} \hat{\partial}_{(2)}^\mu(v) \text{ for } u, v \in U(\mathfrak{g}).$$

**3.11.1.** This coproduct is equivalent to the “deformed Leibniz rule”, popular in some physics works:

$$\partial^\mu(f \star g) = \sum_i \partial_{(1)}^\mu f \star \partial_{(2)}^\mu g \equiv m_\star \Delta(\partial)(f \otimes g), \quad f, g \in S(\mathfrak{g}),$$

as the following calculation shows:  $\partial^\mu(f \star g) = \partial^\mu(\xi^{-1}(\xi(f) \cdot_{U(\mathfrak{g})} \xi(g))) = \xi^{-1}(\hat{\partial}^\mu(\xi(f) \cdot_{U(\mathfrak{g})} \xi(g))) = \xi^{-1}(\hat{\partial}_{(1)}^\mu \xi(f) \cdot_{U(\mathfrak{g})} \hat{\partial}_{(2)}^\mu \xi(g)) = \xi^{-1}(\xi \hat{\partial}_{(1)}^\mu(f) \cdot_{U(\mathfrak{g})} \xi \hat{\partial}_{(2)}^\mu(g)) = \hat{\partial}_{(1)}^\mu(f) \star \hat{\partial}_{(2)}^\mu(g)$ .

**3.12.** This coproduct is related to but different from the dual coproduct  $(S(\mathfrak{g}))^* \cong \widehat{S(\mathfrak{g}^*)} \xrightarrow{\Delta'} \widehat{S(\mathfrak{g}^*)} \widehat{\otimes} \widehat{S(\mathfrak{g}^*)}$  to the star product (8). The defining property of  $\Delta'$  is  $\langle u_{1'}, f \rangle \langle u_{2'}, g \rangle \equiv \langle \Delta'(u), f \otimes g \rangle = \langle u, f \star g \rangle$  for  $u \in S(\mathfrak{g})^* \cong \widehat{S(\mathfrak{g}^*)}$ ,  $f, g \in S(\mathfrak{g})$ .

The correspondence  $P \mapsto (f \mapsto P(f)(0))$  is the linear isomorphism from the space of derivations of  $S(\mathfrak{g})$  to the space of linear functionals  $S(\mathfrak{g})^*$ . Evaluating at zero the  $n$ -th partial derivative is the same as evaluating the product of first partial derivatives except that one has to adjust the factor of  $n!$  what amounts to a different pairing between the graded components  $S^n(\mathfrak{g})$  and  $S^n(\mathfrak{g}^*)$  (i.e. a different identification  $S^n(\mathfrak{g}^*) \cong S^n(\mathfrak{g})^*$ ).

In [15] it is shown that in fact the coproducts  $\Delta$  and  $\Delta'$  may be directly related; and that the dual coproduct  $\Delta'$  may be used to identify  $A_{\mathfrak{g},\phi}$  with a topological Heisenberg double of  $U(\mathfrak{g})$ .

**3.13. Lemma.** *If  $\hat{a} = a^\alpha \hat{x}_\alpha$  and  $\hat{f} \in U(\mathfrak{g})$  then*

$$\hat{\partial}^\mu(\hat{a}^p \hat{f}) = \sum_{k=0}^{p-1} \binom{n}{k} a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_k} \hat{a}^{p-k} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{f}) \quad (14)$$

*Proof.* This is a tautology for  $p = 0$ . Suppose it holds for all  $p$  up to some  $p_0$ , and for all  $\hat{f}$ . Then set  $\hat{g} = \hat{a}\hat{f} = a^\alpha \hat{x}_\alpha$ . Then  $\hat{\partial}^\mu(\hat{a}^{p_0+1}\hat{f}) = \hat{\partial}^\mu(\hat{a}^{p_0}\hat{g})$  and we can apply (14) to  $\hat{\partial}^\mu(\hat{a}^{p_0}\hat{g})$ . Now

$$\begin{aligned} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{g}) &= a^{\alpha_k} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{x}_{\alpha_{k+1}}\hat{g}) \\ &= \hat{a} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{f}) + \\ &\quad + a^{\alpha_{k+1}} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}], \hat{x}_{\alpha_{k+1}}]](\hat{f}). \end{aligned}$$

Collecting the terms and the Pascal triangle identity complete the induction step.

## 4. Symmetric ordering

**4.1.** Given a basis  $\hat{x}_1, \dots, \hat{x}_n$  in a Lie algebra  $\mathfrak{g}$ , and structure constants defined by  $[\hat{x}_i, \hat{x}_j] = C_{ij}^k \hat{x}_k$ , denote by  $\mathcal{C}$  the matrix with entries in  $A_{n,\mathbf{k}}$  whose  $(i, j)$ -th entry is

$$C_j^i = C_{jk}^i \partial^k$$

In [4] we have shown that if  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is the coexponential map then the corresponding  $\phi$  is determined by

$$\phi(-\hat{x}_\beta)(\partial^\alpha) = \phi_\beta^\alpha = \sum_{N=s}^{\infty} (-1)^N \frac{B_N}{N!} (C^N)_\beta^\alpha$$

where  $B_N$  are the Bernoulli numbers. *For the reason of structure of the coexponential map, from now on we will say that this is the case of **symmetric ordering**.* It has the property that  $\xi^{-1}(\exp(a^\alpha \hat{x}_\alpha)) = \exp(a^\alpha x_\alpha)$ .

**4.2.** For general  $\phi$ ,  $[\hat{\partial}^\mu, \hat{x}_\alpha] = \phi_\alpha^\mu$ ,  $[\hat{\partial}^\mu, \hat{x}_\alpha](1) = \delta_\alpha^\mu$ , and

$$[[[\hat{\partial}^\mu, \hat{x}_\alpha], \hat{x}_\beta]] = \frac{\partial}{\partial(\partial^\rho)}(\phi_\alpha^\mu)\phi_\beta^\rho = \phi_{\alpha,\rho}^\mu \phi_\beta^\rho$$

holds. In the case of the symmetric ordering (cf. **4.1**), that is when  $\xi$  is the coexponential map ([4]), also

$$\phi_{\alpha,\rho}^\mu \phi_\beta^\rho(1) = \frac{1}{2} C_{\alpha\beta}^\mu$$

and the higher order terms are not so easy to evaluate at 1 in general in a closed form (this involves identities between different tensors in  $C$ -s, what is combinatorially involved, hence one should probably handled it using tree calculus).

**4.3.** (Notation: subscripts after comma for derivatives.) Given  $\phi_\beta^\alpha \in \widehat{S(\mathfrak{g}^*)}$  as above, denote

$$\phi_{\beta,\rho_1\rho_2\dots\rho_k}^\alpha := \frac{\partial}{\partial(\partial_{\rho_k})} \cdots \frac{\partial}{\partial(\partial_{\rho_2})} \frac{\partial}{\partial(\partial_{\rho_1})} \phi_\beta^\alpha$$

and we use the extension of this notation to more complicated expressions, e.g.  $(ab)_{,\rho} = a_{,\rho}b + ab_{,\rho}$  is the derivative of the product  $ab$  with respect to  $\partial_\rho$ .

**4.4. Lemma.** *Let  $\hat{x}_1, \dots, \hat{x}_n$  be a basis of  $\mathfrak{g}$ . For any  $\phi$  as above,*

$$[\dots [[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \hat{x}_{\alpha_2}], \dots, \hat{x}_{\alpha_k}] = (\dots ((\phi_{\alpha_1,\rho_1}^\mu \phi_{\alpha_2}^{\rho_1}, \rho_2 \phi_{\alpha_3}^{\rho_2}, \rho_3 \dots)_{,\rho_{k-1}} \phi_{\alpha_k}^{\rho_{k-1}} \dots) \quad (15)$$

The proof is an obvious induction, using the chain rule.

**4.4.1.** Using the Leibniz rule we can rewrite the formula (15) as a sum of terms for which every derivative operator  $\frac{\partial}{\partial(\partial_\rho)}$  is applied only to a single  $\phi$ -series, rather than to products. Indeed, it is clear that  $\frac{\partial}{\partial(\partial_{\rho_1})}$  applies only to  $\phi_{\alpha_1}^\mu$ , then  $\frac{\partial}{\partial(\partial_{\rho_2})}$  applies either to  $\phi_{\alpha_1}^\mu$  or  $\phi_{\alpha_2}^{\rho_1}$ , and in general,  $\frac{\partial}{\partial(\partial_{\rho_s})}$  applies to  $\phi_{\alpha_p}^{\rho_{p-1}}$  where  $1 \leq p \leq s$  and  $\rho_0 := \mu$ . This means that we have  $(k-1)!$  summands. For example for  $k=4$  we have 6 summands:

$$\begin{aligned} & \phi_{\alpha_1,\rho_1}^\mu \phi_{\alpha_2,\rho_2}^{\rho_1} \phi_{\alpha_3,\rho_3}^{\rho_2} \phi_{\alpha_4}^{\rho_3} + \phi_{\alpha_1,\rho_1}^\mu \phi_{\alpha_2,\rho_2\rho_3}^{\rho_1} \phi_{\alpha_3}^{\rho_2} \phi_{\alpha_4}^{\rho_3} + \phi_{\alpha_1,\rho_1\rho_3}^\mu \phi_{\alpha_2,\rho_2}^{\rho_1} \phi_{\alpha_3}^{\rho_2} \phi_{\alpha_4}^{\rho_3} \\ & + \phi_{\alpha_1,\rho_1\rho_2}^\mu \phi_{\alpha_2}^{\rho_1} \phi_{\alpha_3,\rho_3}^{\rho_2} \phi_{\alpha_4}^{\rho_3} + \phi_{\alpha_1,\rho_1\rho_2}^\mu \phi_{\alpha_2,\rho_3}^{\rho_1} \phi_{\alpha_3}^{\rho_2} \phi_{\alpha_4}^{\rho_3} + \phi_{\alpha_1,\rho_1\rho_2\rho_3}^\mu \phi_{\alpha_2}^{\rho_1} \phi_{\alpha_3}^{\rho_2} \phi_{\alpha_4}^{\rho_3} \end{aligned}$$

I will call this expansion ‘‘expansion 1’’.

**4.5.** We now specialize to the case of the series corresponding to the symmetric ordering

$$\phi_{\beta,\rho_1,\dots,\rho_s}^\alpha = \sum_{N=s}^{\infty} (-1)^N \frac{B_N}{N!} (\mathcal{C}^N)_{\beta,\rho_1,\dots,\rho_s}^\alpha$$

The sum over  $N \geq k$  for each  $\phi$  in the form of expansion 1, will be called expansion 2. By applying the Leibniz rule again, we notice that  $(\mathcal{C}^N)_{,\rho_1,\dots,\rho_s}$  is a sum of  $N!/(N-s)!$  summands, each of which is monomial which is a product of  $N-s$   $\mathcal{C}$ -s and  $s$   $C$ -s. This is the expansion 3. Performing consequently expansions 1,2 and 3, the commutator in (15) becomes a multiple sum of terms which are labelled by certain class of attributed planar trees and each summand is certain contraction of several  $\mathcal{C}$ -tensors and several  $C$ -tensors with  $k+1$  external indices  $\mu, \alpha_1, \dots, \alpha_k$ , and with some pre-factor involving (products of) Bernoulli numbers and factorials. To describe the details, we introduce several ‘‘classes’’ of planar rooted trees and their ‘‘semantics’’.

## 5. Tree calculus for symmetric ordering

**5.1.** *Class  $\mathcal{T}$  consists of all planar rooted trees with two kinds of nodes, white and black, where black nodes may only be leaves.*

We will draw the trees in  $\mathcal{T}$  with the root on the top. 'Planar' implies that the (left to right) order of child branches of every node matters. If  $t \in \mathcal{T}$ , then  $w(t) \geq 0$  and  $b(t) \geq 0$  are the number of white and black nodes in  $t$  respectively. Class  $\mathcal{T}$  is graded in obvious way  $\mathcal{T} = \coprod_{P=1}^{\infty} \mathcal{T}_P$  by the total number of nodes  $P$ , and bigraded by the numbers  $b$  and  $w$  of black and white nodes:  $\mathcal{T} = \coprod_{w+b>0} \mathcal{T}_{w,b}$ . Clearly  $\mathcal{T}_P = \coprod_{w+b=P} \mathcal{T}_{w,b}$ .

Class  $\mathcal{T}^{ord}$  consists of pairs  $(t, l)$  where  $t \in \mathcal{T}$  and  $l$  is a numeration (with values  $1, \dots, w$ ) on the set of white nodes of  $t$  which is descending in the sense that white children nodes are always assigned greater values than their parent nodes. Let  $\mathcal{T}_P^{ord}$  and  $\mathcal{T}_{w,b}^{ord}$  be the sets of all pairs  $(t, l) \in \mathcal{T}^{ord}$  such that  $t \in \mathcal{T}_P$  and  $t \in \mathcal{T}_{w,b}$  respectively. Given  $s \in \mathcal{T}^{ord}$  and  $t \in \mathcal{T}$  we say  $s \in t$  if  $s = (t, l)$  for some numeration  $l$ . This means that we identify  $t$  with the set of all pairs of the form  $(t, l)$ .

**5.2.** (Example: counting trees in  $\mathcal{T}^{ord}$ ) Let  $s_w$  be the cardinality of  $\mathcal{T}_{w,0}^{ord}$ , that is the number of distinct numerated planar rooted trees with descending numeration and only white nodes. We suggest reader to check that  $s_1 = s_2 = 1$ ,  $s_3 = 3$  and  $s_5 = 15$ . It is easy to derive a recursion for  $s_w$ . The trees in  $\mathcal{T}_{w+1,0}^{ord}$  have a root node with at most  $w$  numerated branches which are themselves planar rooted trees with labels. The exact labelling is determined by first choosing the set of labels of each branch, and then choosing a descending numeration on the labels within each branch. For the whole process  $w$  labels are available, regarding that the root branch is mandatory labelled with 1. Thus we obtain the recursion

$$s_{w+1} = \sum_{k=1}^w \sum_{w_1+w_2+\dots+w_k=w} \frac{w!}{w_1!w_2!\dots w_k!} s_{w_1} s_{w_2} \dots s_{w_k}, \quad w \geq 1.$$

The solution of this recursion is  $s_w = (2w - 3)!! = 1 \cdot 3 \cdot 5 \dots (2w - 3)$ .

Cardinality of  $\mathcal{T}_{b,w}^{ord}$  may be determined similarly: for  $w \geq 0$ ,

$$s_{w+1,b} = \sum_{k=1}^w \sum_{\substack{w_1 + \dots + w_k = w \\ b_1 + \dots + b_k = b}} \frac{w!}{w_1!w_2!\dots w_k!} s_{w_1,b_1} s_{w_2,b_2} \dots s_{w_k,b_k}.$$

**5.3.** Suppose now  $\mathfrak{g}$  and its basis  $\hat{x}_1, \dots, \hat{x}_n$  are fixed; and hence the dual basis  $\partial^1, \dots, \partial^n$  and the structure constants  $C_{jk}^i$ .

Given a tree  $t \in \mathcal{T}_{w,b}^{ord}$  and labels  $1 \leq \mu, \alpha_1, \dots, \alpha_w \leq n$ , **define**

$$\text{ev}(t)_{\alpha_1, \dots, \alpha_w}^{\mu} \in S(\mathfrak{g}^*)$$

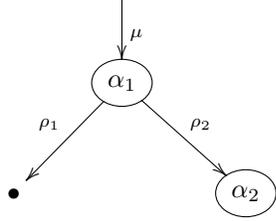
as follows. First replace the numeration labels  $1, \dots, w$  on white nodes with  $\alpha_1, \dots, \alpha_w$ . Then label arbitrarily the inner lines by distinct new variables

$\rho_1, \dots, \rho_{w+b-1}$ , and attach a new *external* incoming line to the root node and label it with label  $\mu$ .

To form an expression  $\text{ev}(t)_{\alpha_1, \dots, \alpha_w}^\mu$  apply the *Feynman-like rules*:

- To each **white node** with label  $\alpha_k$ , incoming node  $\rho_l$  and outgoing nodes  $\rho_{v_1}, \dots, \rho_{v_s}$  assign value  $(-1)^s \frac{B_s}{s!} \sum_{k_1, \dots, k_{s-1}=1}^n C_{\alpha_k \rho_{v_1}}^{k_1} C_{k_1 \rho_{v_2}}^{k_2} \dots C_{k_{s-1} \rho_{v_s}}^{\rho_l}$ . If  $s = 0$  (the white node is a white leaf), the value is Kronecker delta  $\delta_\alpha^{\rho_l}$ .
- To each **black leaf** assign  $\partial^{\rho_l} \in \mathfrak{g}^* \subset S(\mathfrak{g}^*)$ .
- Multiply so assigned values of all nodes and **sum over all values from 1 to  $n$  of labels of all internal lines**  $\rho_1, \dots, \rho_{w+b-1}$ .

5.3.1. Example:



$$\frac{B_2}{2!} \sum_k (C_{\alpha_1 \rho_1}^k \partial^{\rho_1}) C_{k \rho_2}^\mu \delta_{\alpha_2}^{\rho_2} = \frac{1}{12} \sum_k C_{\alpha_1}^k C_{k \alpha_2}^\mu \quad (16)$$

5.3.2. Clearly  $\text{ev}(t)_{\alpha_1, \dots, \alpha_w}^\mu$  are components of some tensor which will be of course denoted  $\text{ev}(t) \in \mathfrak{g} \otimes T^n(\mathfrak{g}^*)$ . In this notation,

$$[\dots [[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \hat{x}_{\alpha_2}], \dots, \hat{x}_{\alpha_w}] = \sum_{b=0}^{\infty} \sum_{t \in \mathcal{T}_{w,b}^{\text{ord}}} \text{ev}(t)_{\alpha_1, \dots, \alpha_w}^\mu \quad (17)$$

5.4. For a tree  $t \in \mathcal{T}_{w,b}^{\text{ord}}$  one defines its **full evaluation**

$$\text{fev}(t)^\mu := \frac{1}{w!} \partial^{\alpha_1} \dots \partial^{\alpha_w} \otimes \text{ev}(t)_{\alpha_1, \dots, \alpha_w}^\mu,$$

and for  $s \in \mathcal{T}$  one defines

$$\text{fev}(s)^\mu := \sum_{t \in s, t \in \mathcal{T}^{\text{ord}}} \text{fev}(t)^\mu.$$

5.5. (**Basic selection rule**) Suppose a tree  $t \in \mathcal{T}$  has at least one white node  $y$  such that its most left child branch is a white leaf. Then for all  $\mu$ ,

$$\text{fev}(t)^\mu = 0.$$

*Proof.* Once the Feynman rules are applied the fact is rather obvious. Namely, suppose that white node has  $s$  child branches, its label is  $k$  and of its most left child branch is  $l$  (then  $l > k$ ). Then the Feynman rules for  $\text{ev}(t)_{\alpha_1, \dots, \alpha_k, \dots, \alpha_l, \dots, \alpha_w}^\mu$

assign to the white node  $y$  the factor  $(-1)^s \frac{B_s}{s!} C_{\alpha_1 \rho_1}^* C_{*\rho_2}^* \dots C_{*\rho_s}^{\rho_0}$  if the incoming line to  $y$  is labelled by  $\rho_0$  and outgoing from left to right by  $\rho_1, \dots, \rho_s$ . The white leaf contributes by a factor  $\delta_{\alpha_2}^{\rho_1}$ . Thus we get a factor of the type  $C_{\alpha_k \rho_1}^* \delta_{\alpha_l}^{\rho_1} = C_{\alpha_k \alpha_l}^*$  which is antisymmetric in lower indices. To obtain  $\text{fev}(t)^\mu$  contract  $1 \otimes \text{ev}(t)_{\alpha_1, \dots, \alpha_k, \dots, \alpha_l, \dots, \alpha_w}^\mu$  with the symmetric tensor  $\frac{1}{w!} \partial^{\alpha_1} \dots \partial^{\alpha_w} \otimes 1$  what vanishes by symmetry reasons. Q.E.D.

**5.5.1.** Notice that this selection rule holds for fev but not for ev (the latter does not involve symmetrization). The subset of trees which are not excluded in calculation of fev by the basic selection rules are called (fev)-**contributing trees** and the corresponding subclasses are distinguished with superscript  $c$ , e.g.  $\mathcal{T}_{w,b}^c \subset \mathcal{T}_{w,b}$ .

By similar symmetry reasons, the following result holds:

**5.6. Lemma.** *Let  $\hat{x}_1, \dots, \hat{x}_n$  be a basis of  $\mathfrak{g}$ . If  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is the coexponential map, then for  $w \geq 2$ ,*

$$\sum_{\sigma \in \Sigma(w)} [\dots [[\hat{\partial}^\mu, \hat{x}_{\sigma \alpha_1}], \hat{x}_{\sigma \alpha_2}], \dots, \hat{x}_{\sigma \alpha_w}](1) = 0,$$

where on the left hand side the evaluation at unit element (“vacuum”) is in the sense of the action of the Weyl algebra on the usual symmetric algebra  $S(\mathfrak{g})$ .

The evaluation at vacuum simply kills all the strictly positive powers of  $\partial$ -s, hence only the terms coming from trees in  $\mathcal{T}_{w,0}^{\text{ord}}$  survive. Thus the lemma may be restated as

$$\sum_{\sigma \in \Sigma(k)} \sum_{t \in \mathcal{T}_{w,0}^{\text{ord}}} \text{ev}(t)_{\sigma \alpha_1 \dots \sigma \alpha_w}^\mu = 0.$$

The proof in the latter form is obvious: applying the Feynman rules to a graph with  $w$  nodes and  $w - 1$  internal lines produces a tensor which is proportional to some contracted product of  $w - 1$  copies of the structure constants tensor  $C$ ,  $w - 1$  contractions,  $w$  lower external labels and one upper external label  $\mu$ . In particular at least one pair of labels  $\alpha_i, \alpha_j$  will be attached as lower labels of the same  $C$ -tensor. By the antisymmetry in subscripts of  $C$ , after symmetrization of  $\alpha_1, \dots, \alpha_w$  we obtain zero.

**5.7. Corollary.** *In the symmetric ordering (if  $\xi$  is the coexponential map), the formula for the derivatives of  $(\hat{a})^p = (a^\beta \hat{x}_\beta)^p$  is of the classical (undeformed) shape, i.e.*

$$\frac{1}{s!} \hat{\partial}^{\alpha_1} \hat{\partial}^{\alpha_2} \dots \hat{\partial}^{\alpha_s} (\hat{a}^p) = \binom{p}{s} a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_s} \hat{a}^{p-s}, \quad p \geq s.$$

This follows by an induction on  $k$ ; the induction step involves applying the case  $k = 1$ . For  $k = 1$ , the formula follows from (14) for  $\hat{f} = 1$  after noticing that  $a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_k}$  in (14) is symmetric under permutations of  $\alpha_1, \dots, \alpha_k$ , hence by **5.6** the only term which survives is the top degree term which is of classical shape.

**5.8.** Up to the fourth order in total derivative, or equivalently, third order in  $C$ -s one gets the following

$$\begin{aligned}\Delta\hat{\partial}^\mu &= 1 \otimes \hat{\partial}^\mu + \hat{\partial}^\mu \otimes 1 + \frac{1}{2}C_{\alpha\beta}^\mu \hat{\partial}^\alpha \otimes \hat{\partial}^\beta + \frac{1}{12}C_{\alpha\beta}^* C_{*\gamma}^\mu (\hat{\partial}^\alpha \otimes \hat{\partial}^\beta \hat{\partial}^\gamma + \hat{\partial}^\beta \hat{\partial}^\gamma \otimes \hat{\partial}^\alpha) \\ &\quad - \frac{1}{24}C_{\alpha\beta}^* C_{*\gamma}^* C_{*\delta}^\mu \hat{\partial}^\alpha \hat{\partial}^\gamma \otimes \hat{\partial}^\beta \hat{\partial}^\delta + O(C^4)\end{aligned}$$

where we sum on pairs of repeated indices (including  $*$ , where on two consecutive ones).

**5.9. Theorem.** *If  $\xi$  is the coexponential map, the coproduct is given by*

$$\Delta\hat{\partial}^\mu = 1 \otimes \hat{\partial}^\mu + \hat{\partial}^\alpha \otimes [\hat{\partial}^\mu, \hat{x}_\alpha] + \frac{1}{2}\hat{\partial}^\alpha \hat{\partial}^\beta \otimes [[\hat{\partial}^\mu, \hat{x}_\alpha], \hat{x}_\beta] + \dots$$

or, in symbolic form,

$$\Delta\hat{\partial}^\mu = \exp(\hat{\partial}^\alpha \otimes \text{ad}(-\hat{x}_\alpha))(1 \otimes \hat{\partial}^\mu) \quad (18)$$

and in the tree expansion form, using the notation from **5.4**,

$$\Delta\hat{\partial}^\mu = \sum_{t \in \mathcal{T}^{\text{ord}}} \text{fev}(t)^\mu. \quad (19)$$

Of course, each  $\text{ad}(-\hat{x}_\alpha)$  in (18) has to be applied to  $\hat{\partial}^\mu$  before applying the whole expression on the elements in  $S(\mathfrak{g}) \otimes S(\mathfrak{g})$  (for the Leibniz rule for the star product) or on the elements in  $U(\mathfrak{g}) \otimes U(\mathfrak{g})$  (for the Leibniz rule for the usual noncommutative product).

*Proof.* It is well known that the expressions of the form  $(\hat{a})^p$  where  $\hat{a} = \sum_\alpha a^\alpha \hat{x}_\alpha$  with varying  $a = (a^\alpha)$  span  $U(\mathfrak{g})$ . Thus it is sufficient to show that for all  $a$ , all  $\hat{f} \in U(\mathfrak{g})$  and all  $p$  the twisted Leibniz rule

$$\hat{\partial}^\mu(\hat{a}^p \hat{f}) = \sum_{w=0}^p \frac{1}{w!} \sum_{\alpha_1, \dots, \alpha_w} \hat{\partial}^{\alpha_1} \dots \hat{\partial}^{\alpha_w}(\hat{a}^p) [[\dots [\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots], \hat{x}_{\alpha_w}](\hat{f}).$$

holds. This follows by comparing the Corollary **5.7** which holds for symmetric ordering only with the formula (14) which holds for general ordering.

**5.10.** Let  $\partial^{abc} = \partial^a \partial^b \partial^c$  and so on. Recall  $\phi_\nu^\mu = \phi_\nu^\mu(\partial) = [\hat{\partial}^\mu, \hat{x}_\nu]$ .

**Corollary.** *In symmetric ordering, for any  $\hat{f}, \hat{g}$  in  $U(\mathfrak{g})$ ,*

$$\phi_\nu^\mu(\partial)(\hat{f}\hat{g}) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{i_1, \dots, i_N} \sum_{k=1}^N \partial^{i_1 \dots i_{k-1} i_{k+1} \dots i_N} \phi_\nu^{i_k}(\partial)(\hat{f}) \cdot [[\dots [\hat{\partial}^\mu, \hat{x}_{i_1}], \dots], \hat{x}_{i_N}](\hat{g})$$

Notice that the last sum is from 1, not 0. Summation over repeated indices understood. This formula is equivalent to giving the formula deformed coproduct for the argument  $\Delta([\hat{\partial}^\mu, \hat{x}_\nu]) = \Delta(\phi_\nu^\mu)$ . For the proof, calculate  $\hat{\partial}^\mu((\hat{x}_\nu \hat{f})\hat{g})$  using the twisted Leibniz rule from the theorem **5.9**, and subtract similarly  $\hat{x}_\nu \hat{\partial}^\mu(\hat{f}\hat{g})$  and group the terms and commutators appropriately.

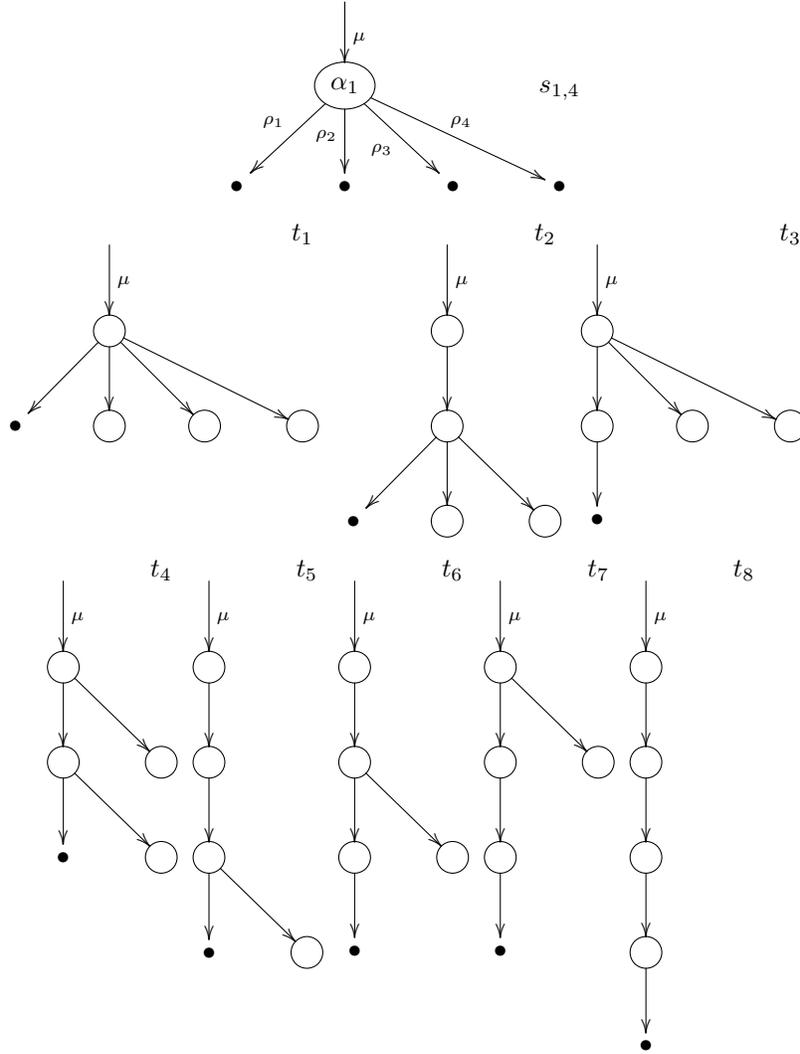
**5.11.** Let  $\tau : S(\mathfrak{g}^*) \hat{\otimes} S(\mathfrak{g}^*) \rightarrow S(\mathfrak{g}^*) \hat{\otimes} S(\mathfrak{g}^*)$  be the standard flip interchanging the tensor factors (in the completed tensor product).

**Theorem.** Let  $s_{1,p}$  be the unique tree in  $\mathcal{T}_{1,p}^{ord}$ . Then for all  $\mu$ ,

$$\tau(\text{fev}(s_{1,p})^\mu) = (-1)^{p+1} \sum_{(t,l) \in \mathcal{T}_{p,1}^{ord}} \text{fev}(t)^\mu \quad (20)$$

or more explicitly

$$\text{ev}(s_{1,p})^\mu_\beta \otimes \partial^\beta = (-1)^{p+1} \sum_{t \in \mathcal{T}_{p,1}^{ord}} \frac{1}{p!} \partial^{\alpha_1} \dots \partial^{\alpha_p} \otimes \text{ev}(t)^\mu_{\alpha_1, \dots, \alpha_p} \quad (21)$$



The diagrams above show  $s_{1,4}$  and the 8 diagrams  $t_1, \dots, t_8 \in \mathcal{T}_{4,1}^c$ .

*Proof.* For  $p = 1$  the assertion is a tautology. Let us prove the assertion for  $p > 1$ .

By the Feynman rules, the left-hand side of (21) equals

$$(-1)^p \frac{B_p}{p!} \sum_{k_1, \dots, k_{p-1}} C_{\beta \rho_1}^{k_1} C_{k_1 \rho_2}^{k_2} \dots C_{k_{p-1} \rho_p}^\mu \partial^{\rho_1} \partial^{\rho_2} \dots \partial^{\rho_p} \otimes \partial^\beta = (-1)^p \frac{B_p}{p!} (\mathcal{C}^p)_\beta^\mu \otimes \partial^\beta.$$

Therefore it is sufficient and we will show by induction that

$$\sum_{t \in \mathcal{T}_{p,1}^{\text{ord}}} \frac{1}{p!} \partial^{\alpha_1} \dots \partial^{\alpha_p} \otimes \text{ev}(t)_{\alpha_1, \dots, \alpha_p}^\mu = (-1)^{p+1} \frac{B_p}{p!} (\mathcal{C}^p)_\beta^\mu \otimes \partial^\beta.$$

Bournulli numbers and hence this expression are zero for odd  $p > 1$  and nonzero for even  $p > 1$ .

By the basic selection rule **5.5**, the only trees  $t \in \mathcal{T}_{p,1}$  (no labelling) which may give a nonzero contribution are those who have no leftmost white leaves, and regarding that there is only one black node in our case, only one white node may have a leftmost leaf (which is black). That means that every contributing tree in  $\mathcal{T}_{p,1}^c$  is composed as follows: start with a vertical chain made of  $r+1 \leq p$  white nodes ending with a black node on the bottom and on this white chain there are attached  $(p-r-1) \geq 0$  right-hand side leaves (to some among the white nodes of the vertical chain), but no branches of length  $\geq 2$  are attached.

Notice that each  $t \in \mathcal{T}_{p,1}^c$  for  $p > 1$  may be also composed alternatively starting with the top white node, attaching the left-most branch  $t' \in \mathcal{T}_{r,1}^c$  and  $p-r-1$  leaves,  $r \geq 0$ . We group the trees by the number  $0 \leq r < p$ . Let us now consider the ordered trees  $t \in \mathcal{T}_{p,1}^{c, \text{ord}}$ . To the top node we must assign label 1, then we may choose any  $r$  remaining numbers  $\beta_1, \dots, \beta_r$  to distribute them within  $t'$  branch according to the usual ordering rules within  $t'$  and distribute the remaining  $p-r-1$  labels  $\gamma_1, \dots, \gamma_{p-r-1}$  to the white leaves in any order. Other way around, given  $t$  with labels, if  $t'$  as a branch of  $t$ , then its labels are renumerated as 1 to  $r$  in the same order. For example, labels 2, 5, 7, 8, 3 of white nodes in  $t'$  as a branch will be replaced by the position labels 1, 3, 4, 5, 2 in  $t'$  as an independent tree. Thus for a given ordering

$$\text{ev}_{1, \dots, r}^\mu(t) = (-1)^p \frac{B_p}{p!} \sum_{\rho, k_1, \dots, k_{p-r-1}} C_{1, \rho}^{k_1} C_{k_1 \gamma_1}^{k_2} \dots C_{k_{p-r-1} \gamma_{p-r-1}}^\mu \text{ev}_{\beta_1, \dots, \beta_r}^\rho(t')$$

(of course each  $i$  has to be replaced by  $\alpha_i$ ). Now we need to count all ordering and combine into fev. The ordering constraints described above give some combinatorial factors, as well as  $1/n!$  in the definition of fev. We obtain

$$\sum_{t \in \mathcal{T}_{p,1}^c} \text{fev}(t)^\mu = \frac{1}{p!} \sum_r (-1)^{p-r+1} \frac{B_{p-r}}{(p-r)!} \binom{p-1}{r} r! \text{fev}(t')^\rho (p-r-1)! ((\mathcal{C}^{p-r})_\rho^\mu \otimes 1).$$

Notice here an additional sign from the first  $C$ -factor (by antisymmetry of lower indices):  $C_{\alpha_1 \rho}^* \partial^{\alpha_1} = -C_\rho^*$ .

By the induction hypothesis,  $\text{fev}(t')^\rho = (-1)^r \frac{B_r}{r!} (\mathcal{C}^r)_\beta^\rho \otimes \partial^\beta$ , hence,

$$\sum_{t \in \mathcal{T}_{1,p}^c} \text{fev}(t)^\mu = \frac{1}{p} (-1)^p \sum_r \frac{B_{p-r}}{(p-r)!} \frac{B_r}{r!} ((\mathcal{C}^p)_\beta^\mu \otimes \partial^\beta)$$

Regarding that, for  $p > 1$ ,  $B_{p-r}$  and  $B_r$  on the right are simultaneously nonzero if and only if  $r$  and  $p-r$  are both even, the proof finishes by applying the well known identity for Bernoulli numbers

$$\sum_{s=1}^l \frac{B_{2s}}{(2s)!} \frac{B_{2l-2s}}{(2l-2s)!} = \frac{-B_{2l}}{(2l-1)!} + \frac{1}{4} \delta_{l,1}, \quad l > 0.$$

**5.12.** In these terms we state the following conjecture on the star product

In our notation we will often not distinguish any more  $\partial$  from  $\hat{\partial}$ ; with the convention that when we write  $[\partial, \hat{x}]$  where  $\hat{x} \in U(\mathfrak{g})$  we mean  $\hat{\partial}$ ; as well as when we apply  $\hat{\partial}(\hat{f})$  with  $\hat{f} \in U(\mathfrak{g})$ ; however when we apply  $\partial(f)$  with  $f \in S(\mathfrak{g})$  we mean the usual (undeformed) Fock representation. In any case  $\Delta$  is deformed and  $\Delta_0$  undeformed coproduct:  $\Delta_0(\partial^\mu) = 1 \otimes \partial^\mu + \partial^\mu \otimes 1$ .

**5.13. General conjecture.** (for all  $\phi$ )

$$f \star g = \sum_{i_1, i_2, \dots, i_n \geq 0} \frac{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{i_1! \cdots i_n!} m \left( \left( \prod_{l=1}^n (\Delta - \Delta_0)((\partial^l)^{i_l}) \right) (f \otimes g) \right), \quad (22)$$

where  $f, g \in S(\mathfrak{g})$  and  $m$  is the commutative multiplication of polynomials  $S(\mathfrak{g}) \otimes S(\mathfrak{g}) \rightarrow S(\mathfrak{g})$ . Notice that for any concrete  $f$  and  $g$ , the summation on the right has only finitely many nonzero terms. This formula is proved in some special cases ([10]) and in this article for general  $\mathfrak{g}$  and symmetric ordering. For general  $\phi$ , if  $f$  is a first order monomial and  $g$  arbitrary, this formula boils down to our main formula of article [4].

Formula (22) can be expressed via normal ordered exponential  $:\exp(\cdot):$  (here  $x$ -s to the left,  $\partial$ -s to the right)

$$f \star g = m : \exp(x_\alpha (\Delta - \Delta_0)(\partial^\alpha)) : (f \otimes g)$$

and  $m$  is the usual product. Notice that  $:\exp(x_\alpha (\Delta - \Delta_0)(\partial^\alpha)):$  is not an element of the tensor product  $H \otimes H$  where  $H$  is the algebra of formal vector fields; namely the position of  $x$ -variables is to the left from the  $\partial$ -s, but the tensor factor is not chosen, and it does not matter as we use  $m$  after application of the derivatives to  $f \otimes g$ . But we believe there is a correct alternative form where the positions of  $x$ -s in tensor factor is chosen and cocycle conditions for a Drinfeld twist are satisfied (cf. **9.1**).

**5.14.** In articles [8, 10] for a particular Lie algebra, the case of “kappa-deformed Euclidean space” in dimension  $n$  for which the commutation relations are of the type  $[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)$  for some vector  $\mathbf{a} = (a_1, \dots, a_m)$ , the conjecture has been verified for general  $\phi$ .

**5.15. Main theorem.** For symmetric ordering the conjecture 5.13 holds for all  $\mathfrak{g}$ .

In fact we can prove the conjecture in more general case, for those  $\phi$  which are obtained using certain procedure of twisting basis by a wide class inner automorphisms of semicompleted Weyl algebra.

## 6. Some facts on Hausdorff series

**6.1. (The recursive form of Hausdorff series)** Given  $X, Y \in \mathfrak{g}$  where  $\mathfrak{g}$  is finite-dimensional with a norm inducing the standard topology. The series  $H(X, Y)$  is uniquely defined by

$$\exp(X)\exp(Y) = \exp(H(X, Y))$$

and it converges in such norm. Then  $H(X, Y) = \sum_{N=0}^{\infty} H_N(X, Y)$  where “**Dynkin’s Lie polynomials**”  $H_N = H_N(X, Y)$  are defined recursively by  $H_1 = X + Y$  and

$$(N+1)H_{N+1} = \frac{1}{2}[X-Y, H_N] + \sum_{r=0}^{\lfloor N/2-1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_s [H_{s_1}, [H_{s_2}, [\dots, [H_{s_{2r}}, X+Y] \dots]]]$$

where the sum over  $s$  is the sum over all  $2r$ -tuples  $s = (s_1, \dots, s_{2r})$  of strictly positive integers whose sum  $s_1 + \dots + s_{2r} = N$ . This identity is well-known and we do not reprove it here.

**6.2. (Linear parts in either  $X$  or  $Y$ )** The linear part in  $X$  of the Hausdorff series is  $H_{1,*}(X, Y) = \sum_{N=0}^{\infty} (-1)^N \frac{B_N}{N!} [Y, [\dots, [Y, X]]]$  where  $N$  is the degree of  $Y$  in the Lie polynomial involved. Similarly, the linear part in  $Y$  is  $H_{*,1}(X, Y) = \sum_{N=0}^{\infty} \frac{B_N}{N!} [X, [\dots, [X, Y]]]$  where  $N$  is the degree of  $Y$  in the Lie polynomial involved.

**6.3. (Symmetries of Hausdorff series)** Identity  $e^X e^Y = (e^{-Y} e^{-X})^{-1}$  implies  $H(-Y, -X) = -H(X, Y)$ . Dynkin’s polynomials are of fixed total degree, hence the change  $(X, Y) \mapsto (-Y, -X)$  does not mix them and  $H_P(-Y, -X) = -H_P(X, Y)$  for all  $P > 0$ . We refine the degree grading on a free Lie algebra on two generators by a bigrading which induces a decomposition  $H_P(X, Y) = \sum_{w+b=P} H_{w,b}(X, Y)$  where  $H_{w,b}$  is the sum of all Lie polynomials in  $H_P(X, Y)$  of degree  $w$  in  $X$  and degree  $b$  in  $Y$ . Clearly, knowing  $H_P$  determines  $H_{w,b}$  for all  $w, b$  with  $w + b = P$ .

**6.4. Proposition.** The following  $w$ -recursion and  $b$ -recursion hold

$$(w+1)H_{w+1,b} = \frac{1}{2}[X, H_{w,b}] + \sum_{r=0}^{\lfloor w/2-1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_{w_i, b_i} [H_{w_1, b_1}, [\dots, [H_{w_{2r}, b_{2r}}, X] \dots]]$$

$$bH_{w+1,b} = -\frac{1}{2}[Y, H_{w,b}] + \sum_{r=0}^{\lfloor b/2-1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_{w_i, b_i} [H_{w_1, b_1}, [\dots, [H_{w_{2r}, b_{2r}}, Y] \dots]]$$

where in the sum on the right-hand side  $\sum_i w_i = w$  and  $\sum_i b_i = b$  for the  $w$ -recursion and  $\sum_i w_i = w + 1$  and  $\sum_i b_i = b - 1$  for the  $b$ -recursion.

*Proof.* For the purpose of the proof we introduce two new sets of Lie polynomials. The first set will have members  $H_{w,b}^W$  and the latter  $H_{w,b}^B$  where  $w \geq 0, b \geq 0, w + b > 0$ . For  $w = 0$  we set  $H_{w,b}^W = H(w, b)$  what is 0 unless  $b = 1$  when  $H_{0,1}^W = X$ ; similarly for  $b = 0$  we set  $H^B(w, b) = H(w, b)$ . Also set  $H_{1,0}^W = Y$  and  $H_{0,1}^B = X$ , regarding that  $(0, 0)$  point is undefined. By definition,  $w$ -recursion is used to define  $H_{w,b}^W$  at all other pairs  $(w, b)$  and similarly the  $b$ -recursion is used to define  $H_{w,b}^B$ . E.g. for  $w$ -recursion we first use the recursion at the line  $b = 0$ , increasing from  $w = 1$  on, then at the line  $b = 1$ , increasing from  $w = 1$ , and so on. Clearly each recursion relation is used exactly once to determine one new value and all instances of relations are used. Notice that on the line  $b = 0$ , the  $w + 1 = w + b + 1 = P + 1$ , hence the  $w$ -recursion gives the same values on this line as the standard recursion for  $H_{w,b}$ . In that manner we notice that the initial values (line  $b = 0$  and  $(0, 1)$ ) given to  $H^B$  agree with the value of  $H^W$  and  $H$  obtained by  $w$ -recursion and the standard recursion. The initial values hence also satisfy the symmetries  $H_{w,b}(X, Y) = -H_{b,w}(-Y, -X)$  in both cases. We want to prove that the values within the quadrant agree as well, not only the conditions on the boundary. But, the  $b$ -recursion may be obtained from  $w$ -recursion also by the same symmetry operation! Regarding that the symmetry holds for initial values and also for the recursion, than this is true for each pair of new points to which the two recursions assign the values. Conclusion:  $H^B = H^W$ . Therefore we can now safely combine two recursions without being afraid of nonconsistency. But adding up the  $w$ -recursion and  $b$ -recursion we clearly get the standard recursion. Regarding that the initial value  $w + b = P = 1$  for standard recursion is checked and that the standard recursion is the consequence, and also that the values  $H_P$  determine  $H_{w,b}$ , we conclude  $H = H^B = H^W$ .

**6.5. (Recursive formula for  $D = D(k, q)$ )** Let  $\hat{x}_1, \dots, \hat{x}_n$  be a basis of  $\mathfrak{g}$ ,  $i = \sqrt{-1}$ ,  $X = ik^a \hat{x}_a$ ,  $Y = iq^a \hat{x}_a$  and  $H(X, Y) = iD^a(k, q) \hat{x}_a$ , where  $k = (k^1, \dots, k^n)$ ,  $q = (q^1, \dots, q^n)$ ; let also  $D = D(k, q) = (D^1(k, q), \dots, D^n(k, q))$ . Then  $D^\mu(k, q) = \sum_{N=0}^{\infty} D_n^\mu(k, q)$  where  $D_1^\mu(k, q) = k^\mu + q^\mu$  and the recursion

$$(N+1)D_{N+1}^\mu = \frac{1}{2}(k^a - q^a)(E_N)_a^\mu + \sum_{r=1}^{\lfloor N/2-1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_s (k^a + q^a)(E_{s_1} \cdots E_{s_{2r}})_a^\mu \quad (23)$$

holds where

$$(E_P)_\nu^\mu := \sum_{\sigma} iC_{\nu\sigma}^\mu D_P^\sigma, \quad P \geq 1,$$

are the components of a matrix  $E_P$ , and the product of matrices on the right is via the convention that the superscript is the *row* index. The sum over  $a$  on the right is understood and the sum over  $s$  is again over  $2r$ -tuples of positive integers adding up to  $N$ .

## 7. Fourier notation and using exponentials

**7.1.** If some linear isomorphism  $S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  preserves the degree filtration, then it clearly extends by continuity to a linear map among the corresponding completions  $\widehat{S(\mathfrak{g})} \rightarrow \widehat{U(\mathfrak{g})}$ . If the isomorphism is a coalgebra map, then the extension respects the completed coproducts  $\Delta : \hat{H} \rightarrow \hat{H} \hat{\otimes} \hat{H}$  ( $H = S(\mathfrak{g})$  or  $U(\mathfrak{g})$ ). Thus, it makes sense to consider the behaviour of exponential series (as a formal series) under coalgebra isomorphism  $\xi$  as above. It is also useful to extend the field by  $\sqrt{-1}$  if it is not present and consider formal series of the type  $\exp(ik^\alpha x_\alpha)$ . If the field is  $\mathbb{C}$  then such series are specially important because of Fourier integral methods. However, Fourier integral is defined only for some formal series, so the formulas, though useful for other spaces of functions (one can extend our coproducts etc. to various functional spaces, but we will avoid this here) the formulas involving Fourier integrals in this paper will be understood just in the following sense: every abstract series involved is a finite sum of formal power series of the form  $\exp(ia^\alpha x_\alpha)$ . The linear space of such such finite sums (of exponentials),  $S_e(\mathfrak{g}) \subset \widehat{S(\mathfrak{g})}$  is dense in the space of all formal power series. Thus if we prove that some identity between functionals continuous with respect to the power series filtration, holds when restricted to this space, the identity holds in general. Even when the identity is proved for finite sums of exponentials we heuristically write integrals, instead of sums. The imaginary unit is just for suggestiveness of applications in physics, one can correct the  $\sqrt{-1}$  factors and prove the formulas just for the sums of functions of the form  $\exp(ia^\alpha x_\alpha)$  but we will not spend time on these niceties.

**7.2.** Coalgebra isomorphisms  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  which are identity on  $\mathfrak{k} \oplus \mathfrak{g}$ , and which are extended to the completions have the property

$$\xi(\exp(ik^\alpha x_\alpha)) = \exp(iK(\vec{k})^\beta \hat{x}_\beta) \quad (24)$$

for some bijection  $K : \mathfrak{k}^n \rightarrow \mathfrak{k}^n$ . (Proof: All group like elements both in  $\hat{S}(\mathfrak{g})$  and in  $\hat{U}(\mathfrak{g})$  are of such exponential form.  $\xi$  is a bijection and preserves the group like elements because it is a coalgebra map.) We would like to know which  $K$  in turn come from such a coalgebra isomorphism  $\xi = \xi_K$ . Main example, if  $K$  is the identity map, is the case of symmetric ordering:  $\xi_K$  is the coexponential map (when considered defined on  $S(\mathfrak{g})$  only).

## 8. The results leading to the proof of the main theorem

**8.1.** For the coexponential map  $\xi$ , the equality  $\xi(\exp(ik^a x_a)) = \exp(ik^a \hat{x}_a)$  holds. Therefore the star product  $f \star g = \xi^{-1}(\xi(f) \cdot \xi(g))$  reduces to calculations with Hausdorff series. Namely if  $f(x) = \exp(ik^a x_a)$ ,  $g(x) = \exp(iq^a x_a)$ , then  $(f \star g)(x) = \exp(iD^a(k, q)x_a)$ . For general  $f$  and  $g$ , it is convenient to expand  $f$  and  $g$  in Fourier components (reasoning understood in the sense of **7.1**)  $f(x) = \int \frac{d^n k}{(2\pi)^n} (Ff)(k) \exp(ik^a x_a)$  and, by bilinearity, we obtain

$$(f \star g)(x) = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} (Ff)(k) (Fg)(q) \exp(iD^a(k, q)x),$$

or alternatively,

$$(f \star g)(x) = m \exp(iz_a(D^a(-i\partial \otimes 1, -i \otimes \partial) + i\partial^a \otimes 1 + i \otimes \partial^a))(f \otimes g)(x)|_{z_a=x_a}$$

where  $\partial = (\partial^1, \dots, \partial^n)$ . Now notice that  $D_1^a(-i\partial \otimes 1, -i \otimes \partial) = -i\partial^a \otimes 1 - i \otimes \partial^a$ , hence

$$(f \star g)(x) = m \exp(iz_a(D^a - D_1^a)(-i\partial \otimes 1, -i \otimes \partial))(f \otimes g)(x)|_{z_a=x_a}$$

Notice that  $iD_1^a(-i\partial \otimes 1, -i \otimes \partial) = \Delta_0(\partial^a)$ . In fact, using the filtration by the total degree, we now see that the **main theorem 5.15 is equivalent to**

**8.2. Theorem.** *Let  $\Delta_P(\partial^a)$  be the summand in  $\Delta(\partial^a)$  consisting of terms of total homogeneity  $P \geq 1$ . Then for every  $P \geq 1$ ,*

$$iD_P^a(-i\partial \otimes 1, -i \otimes \partial) = \Delta_P(\partial^a)$$

The theorem will be proved by induction on  $P$ . In other words, we have to prove the corresponding recursion for  $\Delta_P$ . We use two tools: 1. Fourier transform (this is only heuristic term here, strictly speaking we use the denseness of the linear span of all exponential series  $\exp(a^\alpha x_\alpha)$  in  $\widehat{S(\mathfrak{g})}$  and do not require the existence of the imaginary unit, as explained in 7.1) and 2. the combinatorics of the trees whose Feynman rule contribution is involved here. Namely expression (19) can be recursively computed after being filtered by the bidegree

$$\Delta \hat{\partial}^\mu = \sum_{t \in \mathcal{T}^{ord}} \text{fev}(t)^\mu = \sum_{b+w=1}^{\infty} \sum_{t \in \mathcal{T}_{b,w}^{ord}} \text{fev}(t)^\mu = \sum_{b+w=1}^{\infty} \Delta_{b,w} \hat{\partial}^\mu.$$

After evaluating,  $b$  and  $w$  correspond to the power of  $\partial$ -s in the left and right tensor product factor respectively. In other words, exactly Every degree in homogeneity corresponds to a node (white nodes for right tensor product factor and black leaves for left factor) as it is easily seen from the expression for fev and Feynman rules for ev. The new node in induction procedure can always be assumed to be the top node, and, in particular white. Then one uses the two crucial lemmas which use the expansions encoded in our Feynman-rule calculus:

**8.3. Lemma.** *w-recursion formula holds (in Fourier transformed form) for calculating  $\Delta \hat{\partial}^\mu$ , where increasing w-degree by 1 corresponds to one white node added and this w-recursion formula is the same as for Hausdorff series in Fourier transformed form.*

The proof is obtained using our Feynman rules and accounting for correct combinatorial factors, in the same way as the counting of trees in 5.2, but with weights. We leave this to the reader.

**8.4. Lemma.** *The initial conditions for w-recursion are the same as for the w-recursion of the Hausdorff series.*

This lemma follows from 5.11.

Therefore the theorem 8.2 follows and hence the main theorem 5.15.

## 9. Special cases and other results

**9.1. (Classical cases and twists)** The classical case of Moyal noncommutative space, where the deformation is given by an antisymmetric matrix  $\theta_{\mu,\nu}$  and the commutation relations are given by  $[x_\mu, x_\nu] = \theta_{\mu,\nu}$  can be treated as special case of this framework by multiplying  $\theta_{\mu,\nu}$  by a central element  $c$ . Then one calculates the star product and obtains the classical formula, after setting back  $c$  to 1. In the classical case also one has the formula  $f \star g = mF(f \otimes g)$  where  $F \in H \otimes H$  is a Drinfeld twist where  $H$  is the universal enveloping algebra of Lie algebra of formal vector fields. One would like to have this property in general. Our “normally ordered exponential” formula for the star product should be rewritten in form  $f \star g = mF(f \otimes g)$  where  $F$  is indeed in  $H \otimes H$ . In one case of interest (“kappa-deformed space” [8, 9, 10]) the answer is known for all orderings. The unique choice of an element in  $H \otimes H$  is made there by trying to write our normally ordered exponential using in addition to  $\partial_\mu \in H$  which may be considered as momenta operators also some special operators which have the role of angular momenta (defined in [10]). We hope some similar principles will enable us to find Drinfeld twists which yield our star products in many new cases.

**9.2. (Formal arguments)** By the Hausdorff formula, using the notation from (24),

$$\begin{aligned} \xi(\exp(ikx))\xi(\exp(irq)) &= \exp(iK(k)\hat{x}) \exp(iK(q)\hat{x}) \\ &= \exp(iD(K(k), K(q))\hat{x}) \\ &= \xi(\exp(iK^{-1}(D(K(k), K(q)))x)) \end{aligned}$$

where we wrote the contractions with suppressed indices. If we denote

$$D_\phi(k, q) := K^{-1}(D(K(k), K(q))), \quad K = K_\phi,$$

then we write this as  $\xi(\exp(ikx))\xi(\exp(irq)) = \xi(\exp(iD_\phi(k, q)x))$  or equivalently

$$\exp(ikx) \star_\phi \exp(irq) = \exp(iD_\phi(k, q)x).$$

In physics papers (e.g. [9, 10])  $\xi(\exp(ikx))$  is usually written as  $\phi$ -ordered exponential :  $\exp(ik\hat{x}) :_\phi$ . Similar expressions one can write for the deformed coproducts (in Fourier harmonics picture).

$$\begin{aligned} iD_\phi^\mu(k, q) \exp(iD_\phi(k, q)x) &= \partial^\mu(\exp(iD_\phi(k, q)x)) \\ &= \partial^\mu(\exp(ikx) \star_\phi \exp(irq)) \\ &= m_\phi(\Delta_\phi(\partial^\mu)(\exp(ikx) \otimes \exp(irq))) \\ &= \Delta_\phi^\mu(ik, iq)(\exp(ikx) \star_\phi \exp(irq)) \\ &= \Delta_\phi^\mu(ik, iq) \exp(iD_\phi(k, q)x) \end{aligned}$$

where  $\Delta_\phi^\mu(ik, iq)$  is obtained from  $\Delta_\phi(\partial_\mu)$  by substituting  $\partial^\alpha \mapsto k^\alpha$  or  $q^\alpha$  depending on the tensor factor and multiplying. Thus  $iD_\phi^\mu(k, q) = \Delta_\phi^\mu(ik, iq)$ .

- [1] G. AMELINO-CAMELIA, M. ARZANO, *Coproduct and star product in field theories on Lie-algebra non-commutative space-times*, Phys. Rev. D65:084044 (2002) [hep-th/0105120](#).
- [2] A. BOROWIEC, A. PACHOŁ,  *$\kappa$ -Minkowski spacetimes and DSR algebras: fresh look and old problems*, SIGMA 6:086,2010, [arxiv/1005.4429](#)
- [3] M. DIMITRIJEVIC, F. MEYER, L. MÖLLER, J. WESS, *Gauge theories on the kappa-Minkowski spacetime*, Eur.Phys.J. C36 (2004) 117–126; [hep-th/0310116](#).
- [4] N. DUROV, S. MELJANAC, A. SAMSAROV, Z. ŠKODA, *A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra*, Journal of Algebra 309:1, pp.318–359 (2007) [math.RT/0604096](#)
- [5] S. HALLIDAY, R. J. SZABO, *Noncommutative field theory on homogeneous gravitational waves*, J. Phys. A39 (2006) 5189–5226, [hep-th/0602036](#)
- [6] V. KATHOTIA, *Kontsevich’s universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula*, Internat. J. Math. 11 (2000), no. 4, 523–551; [math.QA/9811174](#)
- [7] M. KONTSEVICH, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), no. 3, 157–216; [math.QA/9709040](#)
- [8] S. KREŠIĆ-JURIĆ, S. MELJANAC, M. STOJIC, *Covariant realizations of kappa-deformed space*, Eur. Phys. J. C **51**, (2007), 1, 229-240, [hep-th/0702215](#).
- [9] S. MELJANAC, A. SAMSAROV, M. STOJIC, K. S. GUPTA, *Kappa-Minkowski space-time and the star product realizations*, Eur.Phys.J. C51 (2007) 229–240 [arXiv:0705.2471](#).
- [10] S. MELJANAC, M. STOJIC, *New realizations of Lie algebra kappa-deformed Euclidean space*, Eur.Phys.J. C47 (2006) 531–539; [hep-th/0605133](#).
- [11] S. MELJANAC, D. SVRTAN, Z. ŠKODA, *Exponential formulas and Lie algebra type star products*, SIGMA **8** (2012) 013; 1–15, [arxiv/1006.0478](#)
- [12] S. MELJANAC, Z. ŠKODA, *Lie algebra type noncommutative phase spaces are Hopf algebroids*, preprint (2013)
- [13] G. OLSHANSKI, *Generalized symmetrization in enveloping algebras*, Transformation groups 2 (1997), no. 2, 197–213.
- [14] E. PETRACCI, *Universal representations of Lie algebras by coderivations*, Bull. Sci. Math. 127 (2003), no. 5, 439–465; [math.RT/0303020](#)
- [15] Z. ŠKODA, *Heisenberg double versus deformed derivatives*, Int. J. of Modern Physics A 26, Nos. 27 & 28 (2011) 4845–4854; [arXiv:0909.3769](#).