A simple algorithm for extending the identities for quantum minors to the multiparametric case

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Abstract. *For any homogeneous identity between q-minors, we provide an identity between P,Q-minors. We also exhibit a 1-1 correspondence between the bihomogeneous Ore sets in 1-parametric quantum matrix bialgebra with such Ore sets in its multiparametric twist.*

1. ([1, 3, 9]) Suppose we are given $n \times n$ matrices $P = (p_{ij})$ and $Q = (q_{ij})$ with invertible entries in the ground field $\mathbf{k}$, for which there exist q such that

$$p_{ij}q_{ij} = q^2, q_{ij} = q_{ji}^{-1}, \quad i < j, \text{ and } \quad q_{ii} = p_{ii}, \text{ for all } i. \tag{1}$$

Define an associative algebra $\mathcal{M}(P, Q; \mathbf{k}, n) := \mathbf{k}\langle T^i_j, i, j = 1, \ldots, n\rangle/I$, where I is the ideal spanned by the relations

$$\begin{array}{ll} T^k_i T^k_j = q_{ij} T^k_j T^k_i, & i < j \\ T^k_i T^l_i = p_{kl} T^l_i T^k_i, & k < l \\ q_{ij} T^k_j T^l_i = p_{kl} T^l_i T^k_j, & i < j, \;\; k < l \\ T^k_i T^l_j - q_{ij} q^{-1}_{kl} T^l_j T^k_i = (q_{ij} - p^{-1}_{ij}) T^k_j T^l_i, & i < j, \;\; k < l \end{array}$$

$\mathcal{M} := \mathcal{M}(P, Q; \mathbf{k}, n)$ is a bialgebra with respect to the "matrix" comultiplication which is the unique algebra homomorphism $\Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ extending the formulas $\Delta T^i_j = \sum T^i_k \otimes T^k_j$ with counit $\epsilon T^i_j = \delta^i_j$ (Kronecker delta). The bialgebra $\mathcal{M}$ is called the **multiparametric quantum linear semigroup**.

Our conventions differ a bit from the cited references: we treat p-s and q-s symmetrically in the sense that if we interchange rows and columns of matrix T and if we simultaneously interchange P and Q, we obtain an isomorphic algebra. If $P = Q$ and $q_{ij} = q$ for $i < j$ and $q_{ij} = q^{-1}$ for $i > j$, then $\mathcal{M} = \mathcal{M}_q(\mathbf{k})$ (1-parametric quantized matrix bialgebra). In this paper, we will denote by t^i_j the generators for 1-parametric case.

2. (*Labels.*) It is convenient to consider that the row and column labels belong to some totally ordered set of labels, not necessarily the set $\{1, \ldots, n\}$. The main reason is that one often needs to treat some subsets of the set of labels and the corresponding submatrices of the matrix $T = (T^i_j)$.

3. The quantum Q-space $\mathcal{O}(\mathbf{k}^n_Q)$, quantum P-space $\mathcal{O}(\mathbf{k}^n_P)$, q- normalized Q-space $S_r(q,Q)$, dual q- normalized Q- space $S_l(q,Q)$, right P-exterior algebra Λ_P, and left Q-exterior algebra Λ_Q, are the algebras *defined* by generators and relations as follows:

$$
\begin{array}{l}
\mathcal{O}(\mathbf{k}^n_Q) := \mathbf{k}\langle x^i, i=1,\ldots,n\rangle / \langle x^i x^j - q_{ij} x^j x^i, i<j\rangle \\
\mathcal{O}(\mathbf{k}^n_P) := \mathbf{k}\langle y_i, i=1,\ldots,n\rangle / \langle y_i y_j - p_{ij} y_j y_i, i<j\rangle\rangle \\
S_r(q,Q) := \mathbf{k}\langle r_1,\ldots,r_n\rangle / \langle r_i r_j - q q_{ij}^{-1} r_j r_i,\ \ i<j\rangle \\
S_l(q,Q) := \mathbf{k}\langle l^1,\ldots,l^n\rangle / \langle l^i l^j - q_{ij} q^{-1} l^j l^i,\ \ i<j\rangle \\
\Lambda_P := \mathbf{k}\langle e^1,\ldots,e^n\rangle / \langle e^i e^j + p_{ij}^{-1} e^j e^i,\ i<j,\ (e^i)^2\rangle \\
\Lambda_Q := \mathbf{k}\langle e_1,\ldots,e_n\rangle / \langle f_i f_j + q_{ij}^{-1} f_j f_i,\ i<j,\ (f_i)^2\rangle
\end{array}
\tag{2}
$$

Of course, we could have written $S_l(q,Q)$ more symmetrically in terms of P, namely $q_{ij}q^{-1} = q^2 p_{ij} q^{-1} = q p_{ij}^{-1}$. We note after MANIN (e.g. [8, 9]) the simple fact that $\mathcal{O}(\mathbf{k}^n_Q), \Lambda_Q$ are right $\mathcal{M}$-comodule algebras and $\mathcal{O}(\mathbf{k}^n_P), \Lambda_P$ are left comodule algebras via the unique coactions $\rho, \rho_\Lambda, \rho', \rho'_\Lambda$ which are the algebra maps and extend the formulas $\rho(x^i) = \sum_j x^j \otimes T^i_j$, $\rho_\Lambda(e^i) = \sum_j e^j \otimes T^i_j$, $\rho'(y_j) = \sum_i T^i_j \otimes y_i$ and $\rho'_\Lambda(f_j) = \sum_i T^i_j \otimes f_i$.

4. Tuples of labels will be called *multilabels*. Let $L = (l_1,\ldots,l_r)$, $K = (k_1,\ldots,k_s)$. The concatenation will be denoted by juxtaposition: $LK = (l_1,\ldots,l_r,k_1,\ldots,k_s)$. Usually the multilabels will be (ascendingly) ordered to start with and $\hat{L}$ denotes the ordered complement of a submultilabel L (usually in $\{1,\ldots,n\}$). By placing the multilabel within the colons, we will denote its ascendingly ordered version. For example, if K and L are ordered, then KL is not necessarily ordered, because some labels in L may be smaller than some labels in K. However $:KL:$ is the multilabel obtained from KL by permuting the labels until they are ascendingly ordered. We identify the notation for a single label j and the multilabel (j), and in this vein $\hat{j} = \widehat{(j)}$ is the same as multilabel $(1,2,\ldots,j-1,j+1,\ldots,n)$.

We use an obvious exponent notation: $r^J := r^{j_1}\cdots r^{j_n}$ and alike.

5. Algebras (2) satisfy the obvious normal basis ("PBW type") theorems: fix an order on generators, then the monomials ordered compatibly with this order form a vector space basis ("PBW basis"), for example $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in the case of $\mathcal{O}(\mathbf{k}^n_Q)$; however, no higher exponents than 1 appear in the bases for Λ_P and Λ_Q because $e_i^2 = 0$.

Passing from arbitrary words in generators to the basis elements clearly reduces to reordering the generators, and accumulating the proportionality constants. Let us introduce the notation for those rearrangement factors which will be needed below. The following are the defining properties of coefficient functions $\epsilon_P, \epsilon_Q, \zeta_r, \zeta_l$ from the set of multilabels to $\mathbf{k}$:

$$e_J := \epsilon_P(J) e_{:J:}, \quad r_J := \zeta_r(J) r_{:J:},$$
$$f_J := \epsilon_Q(J) f_{:J:}, \quad l_J := \zeta_l(J) l_{:J:},$$

where $\epsilon_P(J), \epsilon_Q(J)$ are defined only when J has no repeted labels inside, but ζ_r and ζ_l are defined even for multilabels with repetition. The following is obvious:

$$\zeta_r(k_1, \ldots, k_s) = \prod_{i<j, k_i>k_j} q^{-1} q_{k_i k_j}$$

$$\zeta_l(k_1, \ldots, k_s) = \prod_{i<j, k_i>k_j} q q^{-1}_{k_i k_j} = \zeta_r^{-1}(k_1, \ldots, k_s)$$

Clearly, if J is ascendingly ordered multilabel with out repetitions, then

$$\begin{array}{lllll} \epsilon_P(\sigma J) & = & (-q)^{-l(\sigma)} \zeta_r(\sigma J) & = & \prod_{k<l, \sigma(k)>\sigma(l)} (-p_{j_{\sigma(k)} j_{\sigma(l)}}), \\ \epsilon_Q(\sigma J) & = & (-q)^{l(\sigma)} \zeta_l(\sigma J) & = & \prod_{k<l, \sigma(k)>\sigma(l)} (-q_{j_{\sigma(k)} j_{\sigma(l)}}), \end{array}$$

(recall that $(q_{ij})^{-1} = q_{ji}$).

6. (*Gradings.*) Let $\mathcal{I}$ be the set of labels of generators of $S_r = S_r(q, Q)$ (they label rows!). Let $\mathcal{J}$ be the set of labels of generators of $S_l = S_l(q, Q)$ (they label columns!). Both sets are bijective to $\{1, \ldots, n\}$. Thus the free Abelian group $\mathbf{Z}[\mathcal{I}]$ is isomorphic to $\mathbf{Z}^n$ (and could be naturally identified with the weight lattice for SL_n). Now we assign $\mathbf{Z}[\mathcal{I}]$–$\mathbf{Z}[\mathcal{J}]$-bigrading to the algebras $S_l(q, Q)$, $S_r(q, Q)$ and $\mathcal{M}_q(\mathbf{k})$. If $i_1, \ldots, i_n, j_1, \ldots, j_n$ are the elements of $\mathcal{I}$ and $\mathcal{J}$, then a bidegree is a formal sum of the form $a_1 i_1 + \ldots + a_n i_n + b_1 j_1 + \ldots + b_n j_n$, e.g. $-i_3 + 2i_4 + j_3$, and we may separate the $\mathcal{I}$ and $\mathcal{J}$ grading with comma for clarity, e.g. $(-i_3 + 2i_4,\ j_3) \equiv -i_3 + 2i_4 + j_3$. We assign the bidegree $(-i, 0)$ to the generator r_i of S_l (notice the negative sign!) and the analogous *dual* prescription $(0, -j)$ to l^j of S_r. We also assign the bidegree $(+i, +j)$ to each generator t^i_j of the 1-parametric algebra $\mathcal{M}_q(\mathbf{k})$. The defining ideals are bihomogeneous with respect to this bigrading, hence we extend this prescription multiplicatively to a bigrading on the algebras $S_l(q, Q)$, $S_r(q, Q)$ and $\mathcal{M}_q(\mathbf{k})$.

7. Notation. Consider the tensor products of bigraded algebras

$$\widehat{\mathcal{M}} = \widehat{\mathcal{M}}(P, Q; \mathbf{k}) := S_l(q, Q) \otimes \mathcal{M}_q(\mathbf{k}) \otimes S_r(q, Q), \tag{3}$$
$$\widehat{\mathcal{M}}^- = \widehat{\mathcal{M}}^-(P, Q; \mathbf{k}) := S_l(q, Q)^{\mathrm{op}} \otimes \mathcal{M}(P, Q, \mathbf{k}) \otimes S_r(q, Q)^{\mathrm{op}}$$

where op denotes the opposite algebra.

8. Lemma. *Any (bi)homogenous element in a tensor product of (bi)graded algebras is a sum of tensor products of (bi)homogenous elements in tensor factors. If one chooses a set of homogeneous generators in each tensor factor than the summands can be chosen as tensor products of monomials in those generators.*

9. Lemma. *$l^j \otimes t^i_j \otimes r_i$ generate the subalgebra $\widehat{\mathcal{M}}_{(0,0)}$ of all elements of bidegree $(0,0)$ in $\widehat{\mathcal{M}}$. Similarly, $(l^i)^{\rm op} \otimes T^i_j \otimes (r_j)^{\rm op}$ generate the subalgebra $\widehat{\mathcal{M}}_{(0,0)}$ of all elements of bidegree $(0,0)$ in $\widehat{\mathcal{M}}^-$.*

Proof. If $J = (j_1, \ldots, j_s)$ is some ordered s-tuple of labels (repetitions of labels possible), we denote $l^J = l^{j_1} \cdots l^{j_s}$ and adopt the obvious extension of this multilabel notation to r-s and t-s. It is clear that any tensor product of monomials which is of bidegree $(0,0)$ is of the form $l^{\tau I} \otimes t^J_I \otimes r_{\sigma J}$ where σ and τ are permutations on $|I| = |J|$ letters. Then by lemma **8** it is enough to show that any such tensor product $l^{\tau I} \otimes t^J_I \otimes r_{\sigma J}$ may be written as sum of products of the form $l^j \otimes t^i_j \otimes r_i$. But l^J and $l^{\tau J}$ are proportional (differing by a scalar in $\mathbf{k}$) in S_l, and similarly, r_I and $r_{\sigma I}$ are proportional in S_r. Hence, up to accounting for a scalar factor, we may assume that σ and τ are trivial. The expression $l^{\tau I} \otimes t^J_I \otimes r_{\sigma J}$ is thus manifestly the product of elements of the required form.

10. Theorem. *Suppose (1) holds. Let t^i_j and T^i_j denote the generators of q-deformed and P,Q-deformed quantum matrix algebras respectively. Then*

(i) the rule

$$\iota = \iota_{q,Q} : T^i_j \mapsto l^j \otimes t^i_j \otimes r_i \tag{4}$$

extends to a unique algebra homomorphism $\iota_{q,Q} : \mathcal{M}(P,Q;\mathbf{k}) \to \widehat{\mathcal{M}}$.

(i)' Similarly, the rule

$$\upsilon = \upsilon_{q,Q} : t^i_j \mapsto (l^j)^{op} \otimes T^i_j \otimes (r_i)^{op}$$

extends to a unique algebra homomorphism $\upsilon : \mathcal{M}_q(\mathbf{k}) \to \widehat{\mathcal{M}}^-$.

(ii) The homomorphism ι is injective and its image is the subalgebra $\widehat{\mathcal{M}}_{(0,0)}$ of all elements of $(0,0)$-bidegree in $\widehat{\mathcal{M}}$. Similarly, υ is injective and its image is the subalgebra $\widehat{\mathcal{M}}^-_{(0,0)}$ of all elements of $(0,0)$-bidegree in $\widehat{\mathcal{M}}^-$.

(iii) Similarly, rescaling e^j by l^j produces the relations in Λ_P from the relations in Λ_Q.

Proof. (i) One needs to show that $\iota_{q,Q}$ sends the ideal of relations (in free algebra on T-s) to zero. For example, omitting the tensor product notation, we calculate, for $i < j$ and $k < l$,

$$\begin{array}{rcl}
\iota(q_{kl}T^i_lT^j_k) & = & q_{kl}l^lt^i_lr_i \cdot l^kt^j_kr_j \\
& = & q_{kl}(l^ll^k)(t^i_lt^j_k)(r_ir_j) \\
& = & q_{kl}(qq_{kl}^{-1}l^kl^l)(t^j_kt^i_l)(qq_{ij}^{-1}r_jr_i) \\
& = & q^2q_{ij}^{-1}l^kl^lt^j_kt^i_lr_jr_i \\
& = & \iota(p_{ij}T^j_kT^i_l),
\end{array}$$

$$\begin{array}{rcl}
\iota(T^k_iT^l_j - q_{ij}q_{kl}^{-1}T^l_jT^k_i) & = & (l^il^j)(t^k_it^l_j)(r_kr_l) - q_{ij}q_{kl}^{-1}(l^jl^i)(t^l_jt^k_i)()r_lr_k) \\
& = & (q_{ij}q^{-1}l^jl^i)(t^k_it^l_j)(r_kr_l)- \\
& & \quad -q_{ij}q_{kl}^{-1}(l^jl^i)[t^k_it^l_j - (q-q^{-1})t^k_jt^l_i](q_{kl}q^{-1}r_kr_l) \\
& = & q_{ij}q^{-1}(q-q^{-1})(l^jl^i)(t^k_jt^l_i)(r_kr_l) \\
& = & \iota((q_{ij} - p_{ij}^{-1})T^k_jT^l_i).
\end{array}$$

The remaining two cases are equally straightforward and left to the reader.

(i)' In complete analogy with (i).

(ii) For injectivity one can use e.g. the normal basis theorem for the quantum matrix algebras: monomials of the form $(T^1_1)^{\alpha_{11}}(T^1_1)^{\alpha_{12}}\cdots(T^1_1)^{\alpha_{nn}}$ make a basis of $\mathcal{M}(P,Q;\mathbf{k})$. It is clear that the images are linearly independent because the middle tensor factors of the images are such (by the normal basis theorem for 1-parametric case) and the other two tensor factors are nonzero. The description of the image of $\iota_{q,Q}$ follows from **9**.

(iii) Easy.

11. (Comparison with Artin-Schelter-Tate). The monomorphism $\iota = \iota_{q,Q}$ will be very useful for our purpose. Above proposition introduces a mechanism which is essentially equivalent to the cocycle-twisting of [1]. Namely, in both approaches, the difference between the algebra relations for t^i_j-s and for T^i_j-s is reflected in rescaling factors for each monomial, which may be expressed in terms of a bicharacter of [1], and depends only on the bidegree of the monomial.

However, there is an important difference in using our monomorphism $\iota : \mathcal{M}(P,Q;\mathbf{k}) \to \widehat{\mathcal{M}}(P,Q;\mathbf{k})$ from the usage of twisting in [1] which relates $\mathcal{M}_q(k)$ with $\mathcal{M}(P,Q;k)$. Namely, it is shown in [1] that the correspondence $t^i_j \mapsto T^i_j$ which they use, extends multiplicatively on monomials to an isomorphisms of vector spaces, and even of **coalgebras** $\mathcal{M}_q \to \mathcal{M}(P,Q;\mathbf{k})$; whereas it does *not* respect the algebra structure. On the other hand, our map ι is a monomorphism of **algebras**, as stated in the theorem, but it does *not* respect the coalgebra structure!

12. (*Quantum minors.*) If J and K are ascendingly ordered row and column multilabels of the same cardinality m without repetitions, then the corresponding quantum minor D^K_L is the element of $\mathcal{M}$ satisfying any of the four equivalent expressions

$$\begin{aligned} D^K_L = \sum_{\sigma\in\Sigma(m)} \epsilon_P(\sigma K) T^{k_{\sigma 1}}_{l_1}\cdots T^{k_{\sigma m}}_{l_m} &= \sum_{\sigma\in\Sigma(m)} \epsilon_P(\sigma K)\epsilon_P^{-1}(\tau L) T^{k_{\sigma 1}}_{l_{\tau 1}}\cdots T^{k_{\sigma m}}_{l_{\tau m}} \\ = \sum_{\sigma\in\Sigma(m)} \epsilon_Q(\sigma L) T^{k_1}_{l_{\sigma 1}}\cdots T^{k_m}_{l_{\sigma m}} &= \sum_{\sigma\in\Sigma(m)} \epsilon_Q(\sigma L)\epsilon_Q^{-1}(\tau K) T^{k_{\tau 1}}_{l_{\sigma 1}}\cdots T^{k_{\tau m}}_{l_{\sigma m}} \end{aligned}$$

where $\tau \in \Sigma(m)$ is a fixed permutation.

13. Proposition. *Let ι be the monomorphism (4). If K and J are ascendingly ordered, then $\iota(D^K_J) = l^J d^K_J r_K$ where d^K_J denotes the quantum minor in 1-parametric case.*

Proof.

$$\begin{aligned} \iota(D^K_J) &= \textstyle\sum_{\sigma\in\Sigma(m)}(-q)^{l(\sigma)}\zeta_r(\sigma K)\iota(T^{k\sigma(1)}_{j_1})\cdots\iota(T^{k\sigma(m)}_{j_m}) \\ &= \textstyle\sum_{\sigma\in\Sigma(m)}(-q)^{l(\sigma)} l^{j_1}l^{j_2}\cdots l^{j_m}\otimes t^{k\sigma(1)}_{j_1}\cdots t^{k\sigma(m)}_{j_m}\otimes\zeta_r(\sigma K) r_{k_1}\cdots r_{k_m} \\ &= \textstyle\sum_{\sigma\in\Sigma(m)}(-q)^{l(\sigma)} l^{j_1}l^{j_2}\cdots l^{j_m}\otimes t^{k\sigma(1)}_{j_1}\cdots t^{k\sigma(m)}_{j_m}\otimes r_{k\sigma(1)}\cdots r_{k\sigma(m)} \\ &= l^J D^K_J r_K. \end{aligned}$$

This proposition reminds to, but is different to the statement of Lemma 5 in [1] which asserts that the twisting considered as an identity map but changing its algebra structure, interchanges the quantum determinants. Notice that the D^K_L and d^K_L on the two sides are given by different formulas in terms of generators T^k_l and t^k_l respectively.

14. Now one needs to see what happens when one considers monomials in D-s, for example $\iota(D^K_L D^M_N)$. By the same, method, one gets $(l^L l^M)(d^K_L d^M_N)(r_K r_M)$. One knows that the relations in $\mathcal{M}$ are bihomogeneous in the sense that the total row multilabel and column multilabel are the same up to the ordering. Every relation is a sum of bihomogeneous. Now if we take different monomials $\prod d^*_* = d^{K_1}_{L_1}\cdots d^{K_m}_{L_m}$ in usual quantum minors, then in order to make them manifestly in the image of ι on some monomial in multiparametric quantum minors, we need to homogenize expression by multiplying it by l^S and r_V where S and V are the ascendingly ordered column and row total multilabel of $\prod d^*_*$, that is $S =: K_1\cdots K_m :$ and $V =: L_1\cdots L_m :$. *Thus the S and V are the same for all monomials in the identity (this is more or less the* *definition of a bihomogeneous identity).* Then we reorder the multilabels in l and in r separately to get the same ordering, but this involves introducing inverse of ζ_l and ζ_r corresponding to the ordering on the column and row multilabels seperately. For a bihomogeneous identity the multiplier l^V and r_S will be the same for all summands, however the reordering factors will be clearly different. Thus we get, in terms of D-s the same identity up to different bihomogeneous factors in front of different monomials will be $\zeta_l^{-1}(L_1L_2\cdots L_m)\zeta_r^{-1}(K_1K_2\cdots K_m)$ where K_i-s is the row multilabel of i-th row and L_i of i-th column. We can express this in terms of the formula

$$\begin{aligned}&\iota(\textstyle\sum_\alpha a_\alpha\zeta_l(L_{1,\alpha}L_{2,\alpha}\cdots L_{r,\alpha})\zeta_r(K_{1,\alpha}K_{2,\alpha}\cdots K_{r,\alpha})D^{L_{1,\alpha}}_{K_{1,\alpha}}D^{L_{2,\alpha}}_{K_{2,\alpha}}\ldots D^{L_{r,\alpha}}_{K_{r,\alpha}}) =\\ &= l^V\otimes(\textstyle\sum_\alpha a_\alpha d^{L_{1,\alpha}}_{K_{1,\alpha}}d^{L_{2,\alpha}}_{K_{2,\alpha}}\ldots d^{L_{r,\alpha}}_{K_{r,\alpha}})\otimes r_S\end{aligned}$$

which holds for bihomogeneous identities for which $V =: L_{r,\alpha}L_{r,\alpha}\ldots L_{r,\alpha}$: and $S =: K_{r,\alpha}K_{r,\alpha}\ldots K_{r,\alpha}$: for all α.

15. Theorem. *The above procedure induces the 1-1 correspondence between the quantum minor identities for the 1-parametric and the minor identities for multiparametric minors.*

16. Proposition. *The correspondence which to a bihomogeneous element $\sum_\alpha r^{L_\alpha}_{K_\alpha}$ in $\mathcal{M}_q(\mathbf{k})$ assigns an element in $\mathcal{M}(P,Q,\mathbf{k})$ which all quantum determinants are interchanged by their multiparametric counterparts and each such new monomial with multilabels L and K is multiplied by*

$$\zeta_l(L_{1,\alpha}L_{2,\alpha}\cdots L_{r,\alpha})\zeta_r(K_{1,\alpha}K_{2,\alpha}\cdots K_{r,\alpha})$$

is multiplicative up to a constant factor, i.e. if $r_1\mapsto R_1$ and $r_2\mapsto R_2$ then $r_1r_2\mapsto e(r_1,r_2)R_1R_2$ for some $e(r_1,r_2)\in\mathbf{k}$.

Proof. While $\zeta_l(L_{1,\alpha}L_{2,\alpha}\cdots L_{r,\alpha})\zeta_r(K_{1,\alpha}K_{2,\alpha}\cdots K_{r,\alpha})$ depends on α, the homomorphism $\iota(r_1) = l^S\otimes R_1\otimes r_V$ where S and V do not depend on α. When doing the same procedure for r_1r_2 we need to rearrange the factors $l^{S_1}l^{S_2}$ and $r_{V_1}r_{V_2}$ what gives an additional *overall* factor $e(r_1,r_2)$.

Remark. Notice that despite the fact that the map $r\mapsto R$ is not an algebra homomorphism, essentially ι and extracting the proprotionality constants from reorderings of r-s and l-s do the job. If one would use the original twisting of [1] one has over there a global *coalgebra map*, which therefore sends identities to something what are not, hence it is not clear how to directly use it for the same result. Our map is however not global, in the sense that it is defined not on the whole matrix bialgebra, but only for the bihomogeneous elements.

17. It is well-known that $\mathcal{M}_q(\mathbf{k})$ is an iterated Ore extension of $\mathbf{k}$, hence a domain. Recall also that a subset S in a ring is multiplicative if $1 \in S$ and $s_1, s_2 \in S$ implies that the product $s_1 s_2 \in S$. A multiplicative subset S of a domain Ω is left Ore [13] if it satisfied the left Ore condition: for all $s \in S$, for all $a \in \Omega$, there exist $s' \in S$ and $a' \in \Omega$ so that $s'a = a's$. Let us consider left Ore subsets in $\Omega = \mathcal{M}_q$ whose all elements are bihomogeneous and such that the subgroup $\mathbf{k}^\times$ of units in $\mathbf{k}$ is in S.

18. Theorem. *There is a 1-1 correspondence between the bihomogeneous left Ore subsets in $\mathcal{M}_q$ which contain $\mathbf{k}^\times$ and such subsets in $\mathcal{M}(P, Q, \mathbf{k})$.*

Proof. To $s \in S$ we assign $\tilde{s}$ by the correspondence in **16**. Regarding that it is multiplicative up to a constant in $\mathbf{k}^\times$ and such constants are in the multiplicative set S, then the set S maps to some multiplicative set $\tilde{S}$. (Notice that s is possibly not a monomial, but the overall constant correction is ensured by **16**.) Let now $\tilde{s} \in \tilde{S}$ and $\tilde{a} \in \mathcal{M}(P, Q, \mathbf{k})$. As they correspond by **16** to some $s \in S$ and $a \in \mathcal{M}_q(\mathbf{k})$ we can find $s' \in S$ and $a' \in \mathcal{M}_q(\mathbf{k})$ so that $s'a = a's$. The correspondence is by **16** multiplicative up to a scalar which can for this quadratic identity then be pushed to the right hand side of the equation, yielding $\tilde{s}'\tilde{a} = \lambda\tilde{a}'\tilde{s}$ for some $\lambda \in \mathbf{k}^\times$. Hence we found $\tilde{s}' \in S$ and $\lambda\tilde{a}' \in \mathcal{M}(P, Q, \mathbf{k})$ completing the left Ore condition.

19. Corollary. *Every set of quantum minors in $\mathcal{M}(P, Q, \mathbf{k})$ multiplicatively generates a left and right Ore subset.*

Proof. I have proved in [11] that every set of quantum minors in $\mathcal{M}_q(\mathbf{k})$ multiplicatively generates an Ore set. By the theorem above and the concrete shape of correspondence **16** (up to scalars quantum minors go to quantum minors) we get the same in $\mathcal{M}(P, Q, \mathbf{k})$. The additional scalar factors are not needed as we have the explicit description of the Ore set as a well-defined multiplicative set.

20. Theorem **18** of course passes to some interesting quotients by ideals and to the localizations of $\mathcal{M}(P, Q, \mathbf{k})$ (e.g. quantum linear groups [1, 2, 3, 5, 8, 12], big Bruhat cell [12]) and corresponding quantum homogeneous spaces [7, 12]. Ore sets yield Ore localizations which give certain analogues of open subsets important in the study of noncommutative algebraic geometry of quantum groups, their homogeneous spaces and principal fibrations over them [7, 12, 13]. In some localizations and in the quotient ring of $\mathcal{M}(P, Q, \mathbf{k})$ we can consider also quasideterminants of submatrices $T^I_J = (T^i_j)_{i \in I, j \in J}$. It is known that up to a scalar factor, such a quasideterminant is a ratio of two quantum minors (of sizes differing by one). Thus, by a straightforward extension of our methods, the (homogeneous in appropriate sense) relations between quasiminors in $\mathcal{M}_q(\mathbf{k})$ also lead to the relations between quasiminors in $\mathcal{M}(P, Q, \mathbf{k})$.

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