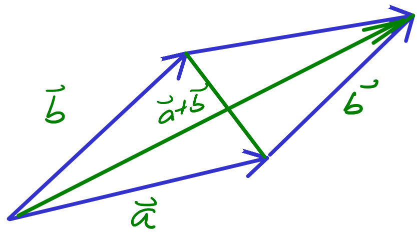
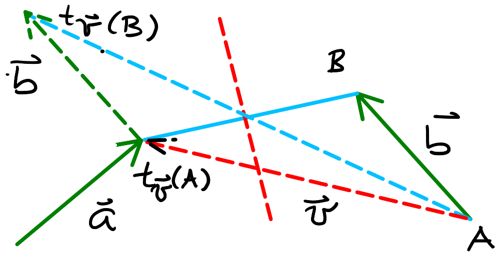
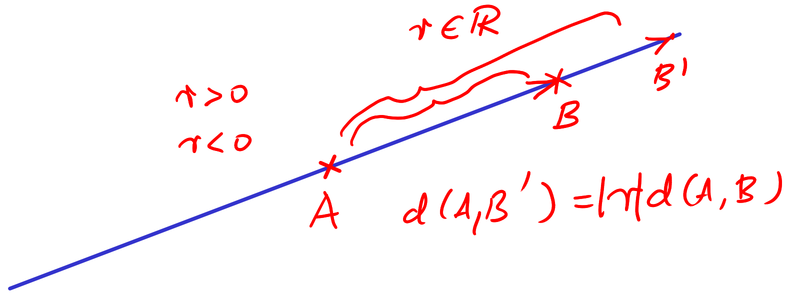
Matematika 4, 21. travnja 2021.



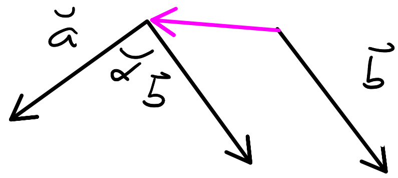




skalarni umnožak vektora

$\vec a, \vec b$ vektori

$$\vec a \cdot \vec b = \|\vec a\| \cdot \|\vec b\| \cdot \cos \measuredangle(\vec a, \vec b)$$



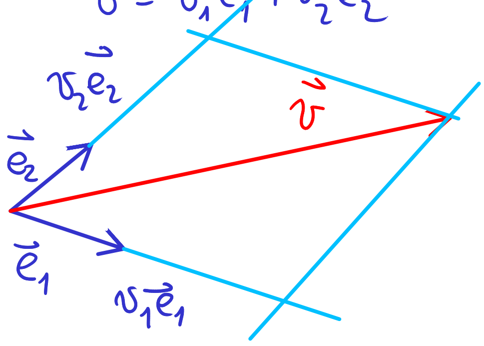
Skalarni umnožak dva vektora je broj (skalar) koji se dobije tako da se duljine ta dva vektora pomnože s kosinusom kuta između ta dva vektora (kuta kojeg zatvaraju predstavnici ta dva vektora s početkom u istoj točki).

→ linearno nezavisni i svaki v. se da napisati preko njih

Sjetimo se komponenti vektora u nekoj bazi u ravnini. Paralelnom projekcijom uzduž dva vektora koji su međusobno nekolinearni možemo svaki drugi vektor u ravnini napisati kao linearnu kombinaciju ta dva vektora

$a_1\vec v_1 + a_2 \vec v_2$ linearna kombinacija od $\vec v_1$ i $\vec v_2$

$$\vec v = v_1 \vec e_1 + v_2 \vec e_2$$



Ako su vektori baze međusobno okomiti i jedinični, kažemo da je to ortonormirana baza.

jedinični vektori -- ortovi

$\vec v = v_x \vec i + v_y \vec j$

komponente vektora

Ako je ortonormirana baza zadana, zajedno s početnom točkom, onda ona zadaje koordinatni sustav u ravnini $(O, \vec e_1, \vec e_2)$

$(O, \vec i, \vec j)$

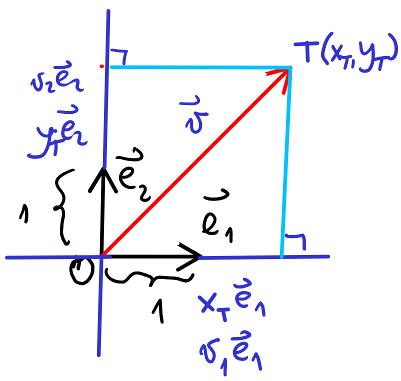
ishodište

$T \mapsto (x_T, y_T) \in \mathbb{R}^2$

ravnina $\longleftrightarrow \mathbb{R}^2$

bijekcija

koordinate točke



Tvrdnja: ako je vektor $\vec{v}$ u ravnini zadan preko komponenti (v_x, v_y) tada je skalarni umnožak dan formulom

$$\vec{v} = v_x\vec{i} + v_y\vec{j}$$
$$\vec{w} = w_x\vec{i} + w_y\vec{j}$$

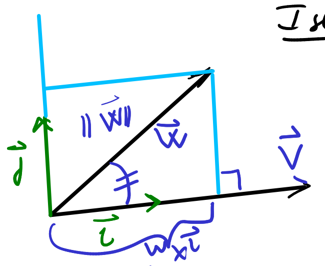
$$\vec{v}\cdot\vec{w} = v_x \cdot w_x + v_y \cdot w_y$$

I slučaj

$\vec{v} \parallel \vec{i}$

$\vec{v} = v_x \vec{i}$

$\vec{w} = w_x\vec{i} + w_y\vec{j}$



$w_x = \|\vec{w}\| \cos\measuredangle$ = ortogonalna projekcija na w

$\|\vec{v}\| = v_x$

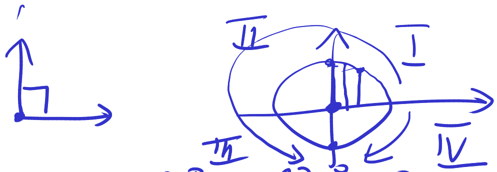
$\|\vec{v}\|\|\vec{w}\| \cos\measuredangle(\vec{v},\vec{w}) =$

$\cos\measuredangle(\vec{v},\vec{w}) = 0 \iff \vec{v} \perp \vec{w}$

$\vec{v}\cdot\vec{w} = 0 \iff \vec{v}\perp\vec{w}$ ili je jedan od njih $\vec{0}$

OPĆI SLUČAJ:

ako rotiramo koordinatni sustav $v_x w_x + v_y w_y$ se ne mijenja pa rotiramo tako da dobijemo poziciju I



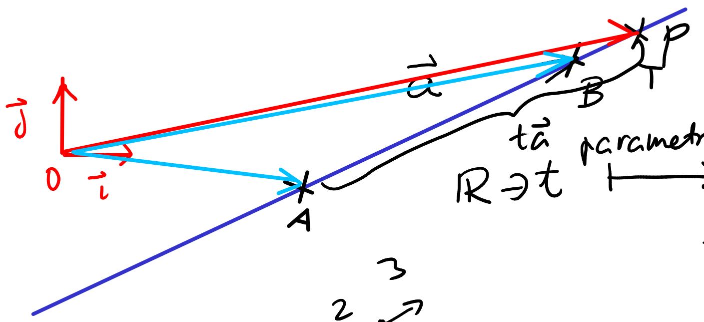
$\cos 90° = \cos 270° = 0$

$\pi/2 \quad 3\pi/2 \quad -\pi/2$

parametarska jednadžba pravca

$\vec{a}$ je na pravcu p

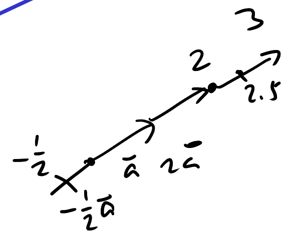
A, B bilo koje točke, $A \neq B$



$\mathbb{R} \ni t \xmapsto{\text{parametrizacija}} T, \quad \vec{AT} = t\vec{a}$

$T = A + t\vec{AB}$ ← translacija

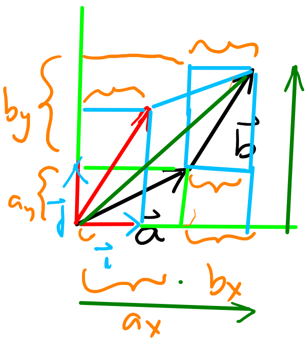
$T = A + t\vec{a}$ $\qquad T = \text{transl}_{t\vec{AB}}(A)$



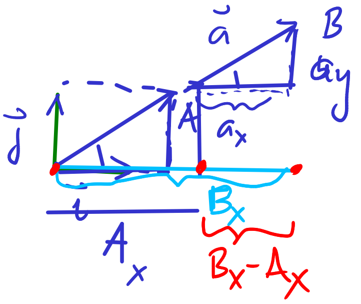
$A(A_x, A_y)$

$\vec{OA} = A_x\vec{i} + A_y\vec{j}$

$\vec{a} = (B_x - A_x)\vec{i} + (B_y - A_y)\vec{j}$



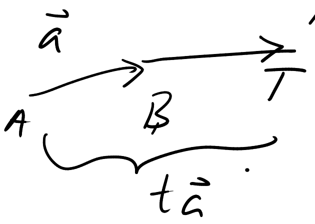
$(\vec{a}+\vec{b}) = (a_x+b_x)\vec{i} + (a_y+b_y)\vec{j}$



$B_y - A_y$

$t \mapsto (A_x + ta_x, A_y + ta_y)$

Neka su A(2,3), B(1,5). Nađi parametarsku jednadžbu pravca kroz A i B i neku implicitnu jednadžbu pravca kroz A i B.



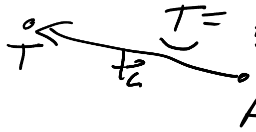
$\vec{AT} = t\vec{AB}$

$$\vec{a} = \vec{AB} = (B_x - A_x)\vec{i} + (B_y - A_y)\vec{j}$$
$$= (1-2)\vec{i} + (5-3)\vec{j}$$
$$\vec{a} = -\vec{i} + 2\vec{j} \quad / \cdot t$$
$$t\vec{a} = -t\vec{i} + 2t\vec{j}$$
$$\boxed{\vec{AT} = t\vec{a}}$$
$$\boxed{T = A + t\vec{a}}$$
$$(x_T, y_T) = (2,3) + t(-1,2)$$

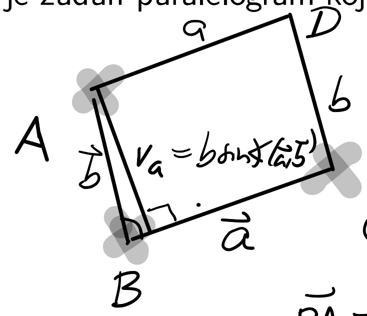
$$-t\vec{i} + 2t\vec{j} = \vec{AT} = (x_T - 2)\vec{i} + (y_T - 3)\vec{j}$$
$$(x_T, y_T) = (2-t,\ 3+2t)$$
$$t \longmapsto (2-t,\ 3+2t)$$
$$(x_T - x_A,\ y_T - y_A) = t\vec{a}$$
$$(x_T, y_T) = (x_A, y_A) + t\vec{a}$$

$\vec{AT} = t\vec{a}$

$T = A + t\vec{a}$



Neka je zadan paralelogram kojem su tri susjedne točke A(2,3), B(1,5), C(3,-3)



Nađi površinu!

$\vec{b} = \vec{BA} = (1,-2)$ $\quad \|\vec{b}\| = \sqrt{5}$

$\vec{a} = \vec{BC} = (2,-8)$ $\quad \|\vec{a}\| = \sqrt{2^2 + (-8)^2}$

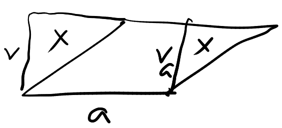
$b = \|\vec{b}\|$ $\quad = \sqrt{4+64}$

$a = \|\vec{a}\|$ $\quad \vec{a}\cdot\vec{b} = 2 + 16 = 18$ $\quad = \sqrt{68}$

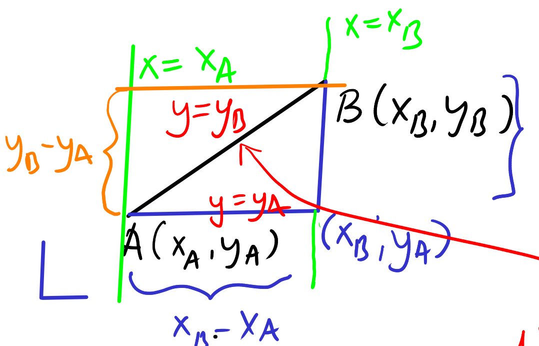
$\vec{BA} = \vec{b}$

$\vec{BC} = \vec{a}$

$$P = a \cdot v_a = a\, b \sin\sphericalangle(\vec{a},\vec{b})$$
$$= \|\vec{a}\|\,\|\vec{b}\|\,\underbrace{\sin\sphericalangle(\vec{a},\vec{b})}_{+} > 0$$



$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\sphericalangle(\vec{a},\vec{b})$$
$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$



PITAGORA

$$\|\vec{AB}\| = d(A,B) = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

$$18 = \vec{a}\cdot\vec{b} = \sqrt{5}\sqrt{68}\cdot\cos\sphericalangle(\vec{a},\vec{b})$$

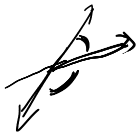
$$\sin\sphericalangle(\vec{a},\vec{b}) = \sqrt{1 - \frac{18^2}{340}} \qquad \cos\sphericalangle(\vec{a},\vec{b}) = \frac{18}{\sqrt{340}}$$
$$= \sqrt{\frac{340-324}{340}} = \sqrt{\frac{16}{340}}$$

$\sin \measuredangle(\vec{a},\vec{b}) = \sqrt{\frac{16}{340}} = \sqrt{\frac{4}{85}}$

$P = \|\vec{a}\| \, \|\vec{b}\| \sin \measuredangle(\vec{a},\vec{b}) = \cancel{\sqrt{5}} \cdot \underbrace{\sqrt{68}}_{\sqrt{\cancel{17}\cdot 4}} \frac{\sqrt{4}}{\sqrt{\cancel{17}\cdot\cancel{5}}} = 4$

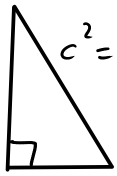
$\vec{a}, \vec{b} \qquad \cos \measuredangle(\vec{a},\vec{b}) \cdot \underset{\sqrt{a_x^2+a_y^2}}{\|\vec{a}\|} \, \underset{\sqrt{b_x^2+b_y^2}}{\|\vec{b}\|} = \vec{a}\cdot\vec{b} = a_x b_x + a_y b_y$

$$-1 \leq \cos \measuredangle(\vec{a},\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} \leq 1$$



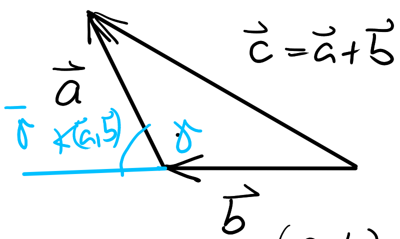
$$= \frac{(a_x b_x + a_y b_y)}{\sqrt{a_x^2+a_y^2} \, \sqrt{b_x^2+b_y^2}}$$

Kosinusov teorem



$c^2 = a^2 + b^2$

skalarni umnožak je distributivan prema zbrajanju $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$



$c = \|\vec{c}\|$

$a = \|\vec{a}\|$

$b = \|\vec{b}\|$

$$\underbrace{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})}_{c^2} = \underbrace{\vec{a}\cdot\vec{a}}_{a^2} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$2ab\cos\measuredangle + \boxed{b^2}$

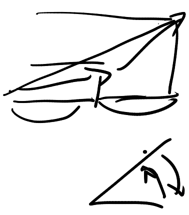
$-\cos\gamma$

$\cos 0 = 1$

$\underset{(a+b)_x}{(a_x+b_x)} c_x = a_x c_x + b_x c_y + a_y c_y + b_y c_y$

$+\underset{(a+b)_y}{(a_y+b_y)} c_y$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$



$(\vec{a}+\vec{b})\cdot\vec{c}$

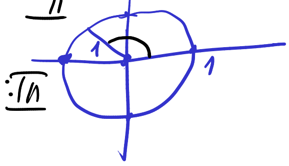
$c^2 = a^2 + b^2 + 2ab\cos\measuredangle(\vec{a},\vec{b})$

$+$ $\bar{\gamma}$ vanjski kut

Kosinusov poučak kaže

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

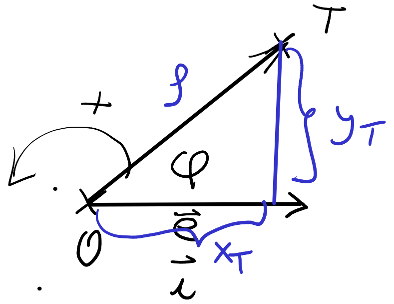
$\cos\pi = -1$



$$\cos(180^\circ - \gamma) = -\cos\gamma$$

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

Polarni koordinatni sustav u ravnini



$$\|\overrightarrow{OT}\| = \rho$$

$$\varphi = \sphericalangle(\vec e, \overrightarrow{OT})$$

$$(\rho, \varphi)$$

$x_T = \rho \cos\varphi$

arccos

inverzna f-ja od f-je cosinus

cos je bijekcija s $[0,\pi] \to [-1,1]$

$\vec j$ ide na $\varphi = \frac{\pi}{2}$ $\quad \rho \in [0, \infty)$

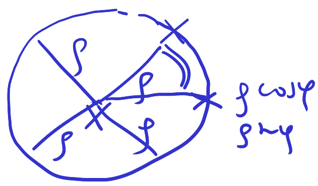
$\varphi \in [0, 2\pi)$

(x_T, y_T)

$$\rho = d(O,T) = \sqrt{(x_T - \underset{0}{\cancel{x_O}})^2 + (y_T - \underset{0}{\cancel{y_O}})^2}$$

$$\rho = \sqrt{x_T^2 + y_T^2}$$

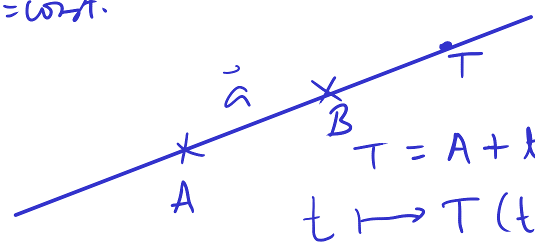
$$\operatorname{tg}\varphi = \frac{y_T}{x_T}$$



$y_T = \rho \sin\varphi$

$x_T = \rho \cos\varphi$

$\rho = \text{const.}$



$T = A + t\vec a$

$t \longmapsto T(t)$

$\varphi \longrightarrow \underbrace{(\rho\cos\varphi, \rho\sin\varphi)}_{\text{pravokutni sustav}}$

$[0, 2\pi]$

kružnica

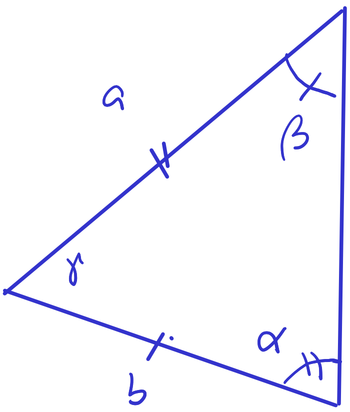
$\varphi \longmapsto (\rho, \varphi)$ POLARNI SUSTAV

$[0, 2\pi] \longmapsto$ ravnina

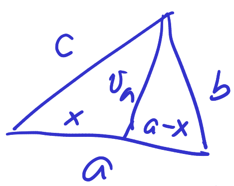
$\rho = \text{const.}$

$\varphi = \varphi$

Sinusov teorem: omjer stranica u trokutu jednak je omjeru sinusa njima nasuprotnih kuteva



$$\frac{\sin\beta}{\sin\alpha} = \frac{b}{a}$$



2x2 sustav jednadžbi

$$\left.\begin{aligned} v_a &= b^2 - (a-x)^2 \\ v_a &= c^2 - x^2 \end{aligned}\right\} \text{nađi } v_a \text{ i } x$$

$$P = \frac{v_a \cdot a}{2}$$

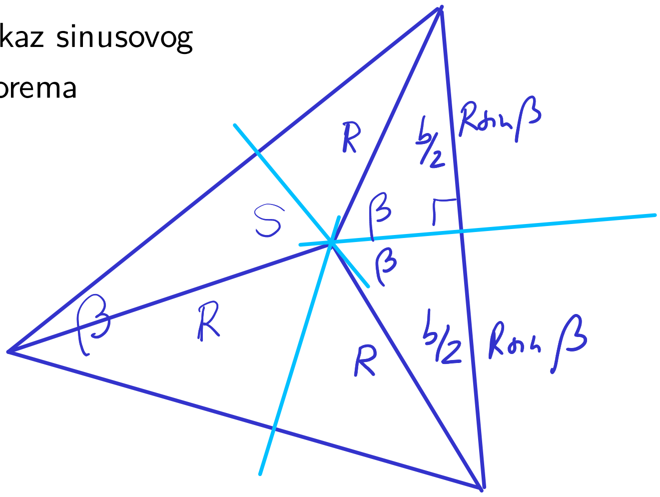
v_a uvrsti

$$s = \frac{a+b+c}{2}$$ poluopseg trokuta

$$P = \sqrt{s(s-a)(s-b)(s-c)}$$

HERONOVA FORMULA

dokaz sinusovog teorema



$$b/2 = R\sin\beta$$

$$\frac{b}{a} = \frac{\cancel{2}R\sin\beta}{\cancel{2}R\sin\alpha}$$

S - sjecište simetrala stranica, ujedno središte trokutu opisane kružnice