Matematika 2, 12. svibnja 2021.

adicijske formule   dokazali smo prije za sinus i kosinus

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

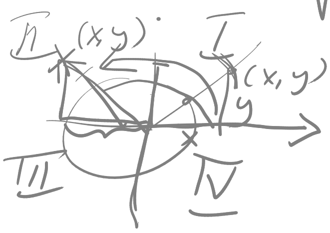
$$\sin 2\alpha = 2\cos\alpha \sin\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\cos\alpha = \cos\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) = 2\cos^2(\alpha/2) - 1$$

$$2\cos^2\alpha/2 = 1 + \cos\alpha$$
$$\cos^2\alpha = \frac{1+\cos\alpha}{2} \Rightarrow \cos\alpha = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

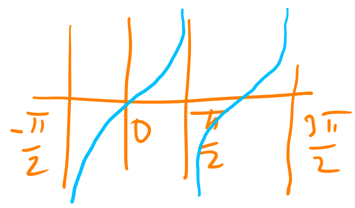
$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

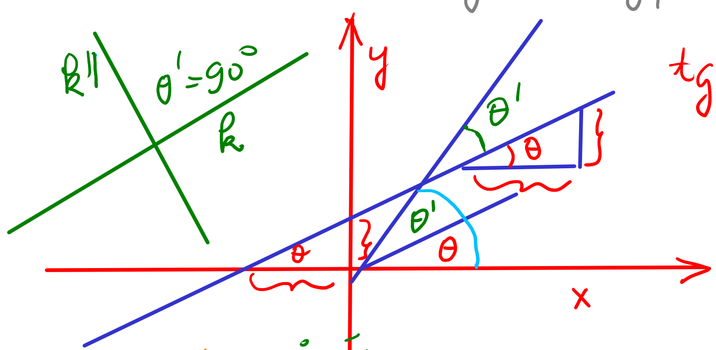
$$\cos 15^\circ = +\sqrt{\frac{1+\sqrt{3}/2}{2}}$$
I kvadrant



$$tg(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \quad \frac{\frac{1}{\cos\alpha\cos\beta}}{\frac{1}{\cos\alpha\cos\beta}}$$
$$= \frac{tg\alpha + tg\beta}{1 - tg\alpha \cdot tg\beta}$$

$$ctg(\alpha+\beta) = \frac{ctg\alpha\, ctg\beta - 1}{ctg\alpha + ctg\beta}$$





$tg\,\theta = k$

naklon pravca (u ravnini) u odnosu na os x

$y = 0$

$\theta' = 90^\circ$

$\cancel{tg\,90^\circ} = \frac{\sin 90^\circ}{\cos 90^\circ}$ nije definiran

$\cancel{k'' = k + ?}$

$$k'' = tg(\theta + \theta') = \frac{tg\theta + tg\theta'}{1 - tg\theta\, tg\theta'} = \frac{k_1 + k_2}{1 - k_1 k_2}$$

$$ctg\,90^\circ = \frac{0 \leftarrow \cos 90^\circ}{1 \leftarrow \sin 90^\circ} = 0$$

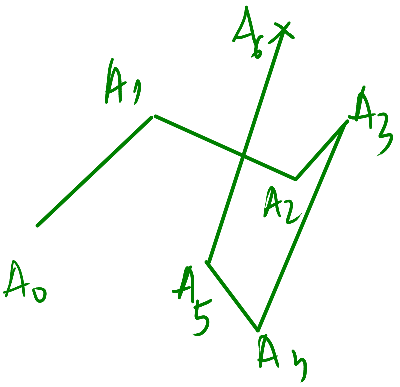
$$k'' = tg(\theta + 90^\circ) = \frac{1}{ctg(\theta+90^\circ)} = \frac{ctg\theta + \cancel{ctg\theta'}^{\,0}}{ctg\theta\,\cancel{ctg\theta'}_{\,0} - 1} = -ctg\theta = -\frac{1}{tg\theta} = -\frac{1}{k}$$

$\theta \neq 0$

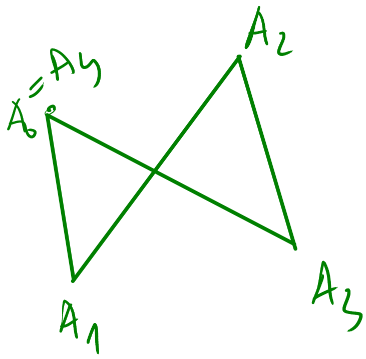
poligonalna linija je unija dužina $\overline{A_0A_1}, \overline{A_1A_2}, \overline{A_2A_3}, \dots, \overline{A_{n-1}A_n}$

za neki $n > 0$ i nekih $(n+1)$ točaka $A_0, \dots, A_n$

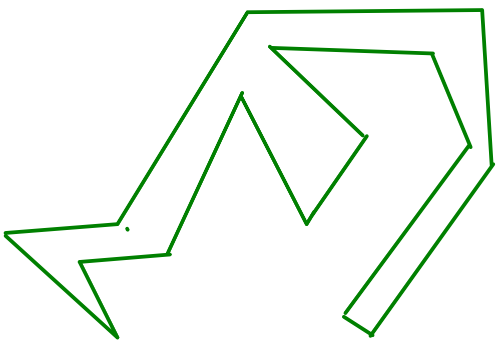
$A_i \neq A_{i+1}$



zatvorena poligonalna linija je
poligonalna linija za koju je n barem 2
i An = A0

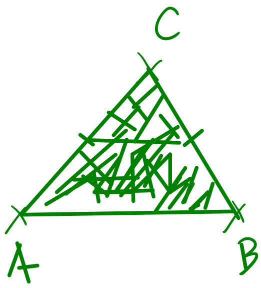


zatvorena poligonalna linija je jednostavna ako nema samopresijecanja
(osim A0 = An) između bilo koje dvije dužine

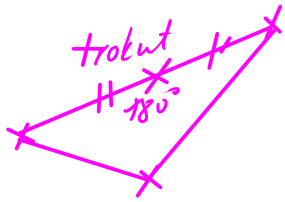


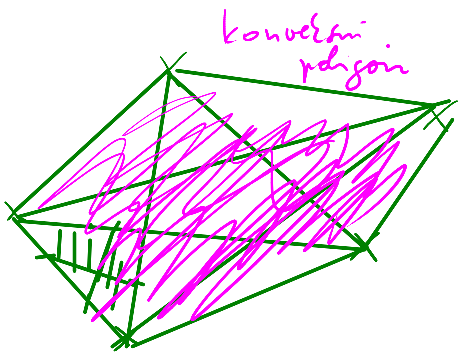
poligonalni Jordanov teorem: svaka jednostavna
zatvorena poligonalna linija dijeli ostatak ravnine
na dva dijela, jedan je ograničeni i zovemo ga
unutrašnjost, a drugi je neograničeni i zovemo ga
vanjština poligona gdje je poligon unija unutrašnjosti
i same poligonalne linije

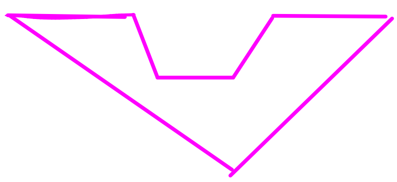
A1,...,An se zovu vrhovi (A0 je isto što i An)
a segmenti poligonalne linije stranice poligona



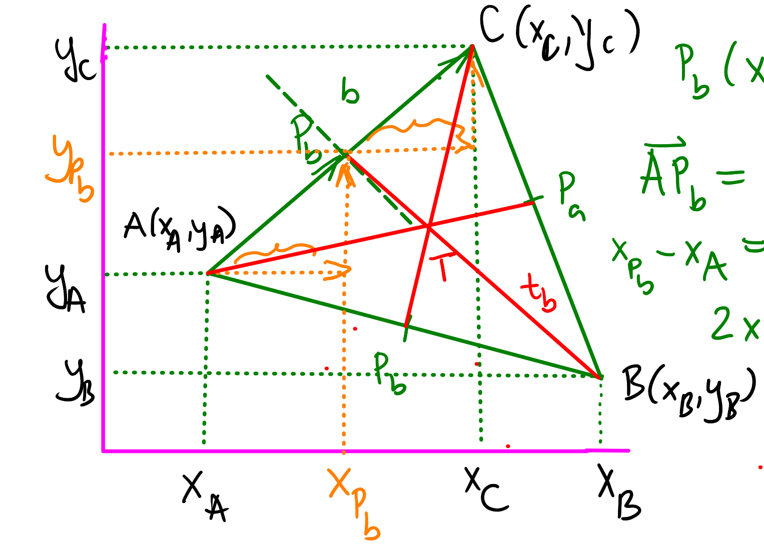
$\mathrm{Conv}(\{A, B, C\}) =: \Delta_{ABC}$







često uzimamo dodatni uvjet da kod poligona dvije susjedne
stranice nikad nisu dijelovi istog pravca



$P_b(x_{P_b}, y_{P_b})$

$\vec{AP_b} = \vec{P_bC}$

$x_{P_b} - x_A = x_C - x_{P_b}$

$2x_{P_b} = x_A + x_C$

$x_{P_b} = \dfrac{x_A + x_C}{2}$

$y_{P_b} = \dfrac{y_A + y_C}{2}$

$\vec{BT} = 2\vec{TP_b}$

teorem o težištu i težišnicama

$x_T - x_B = 2(x_{P_b} - x_T)$

$3x_T = x_B + 2x_{P_b}$

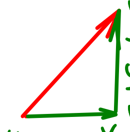
$3x_T = x_B + x_A + x_C$

$x_T = \dfrac{x_A + x_B + x_C}{3}$

$y_T = \dfrac{y_A + y_B + y_C}{3}$

$T\left(\dfrac{x_A + x_B + x_C}{3}, \dfrac{y_A + y_B + y_C}{3}\right)$

$P_b\left(\dfrac{x_A + x_C}{2}, \dfrac{y_A + y_C}{2}\right)$



$d(A,C) = \|\vec{AC}\| = \|(x_C - x_A, y_C - y_A)\|$

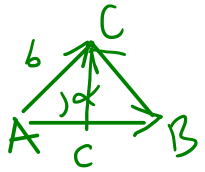
$= +\sqrt{(x_C - x_A)^2 + (y_C - y_A)^2}$

Zad.

$A(2,3), B(3,4), C(0,6)$

Nađi površinu trokuta

$\underset{b}{\vec{AC}} = (-2,3), \underset{a}{\vec{BC}}(-3,2), \underset{c}{\vec{AB}}(1,1)$



$\|\vec{AC}\| = b = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

$\|\vec{BC}\| = a = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

$\|\vec{AB}\| = c = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\underset{c}{\|\vec{AB}\|}\,\underset{b}{\|\vec{AC}\|} \sin\angle(b,c)$

$s = \dfrac{a+b+c}{2} = \dfrac{\sqrt{13} + \sqrt{13} + \sqrt{2}}{2} = \dfrac{2\sqrt{13} + \sqrt{2}}{2} = \sqrt{13} + \dfrac{\sqrt{2}}{2}$

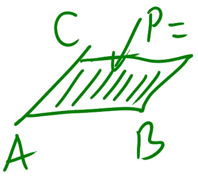
$s - a = \dfrac{b+c-a}{2} = \dfrac{\sqrt{2}}{2}$

$s - b = \dfrac{a+c-b}{2} = \dfrac{\sqrt{2}}{2}$

$P = \sqrt{\underbrace{\left(\sqrt{13} - \frac{\sqrt{2}}{2}\right)\left(\sqrt{13} + \frac{\sqrt{2}}{2}\right)} \frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2}}$

$P = \sqrt{\left(13 - \frac{1}{2}\right)\frac{1}{2}} = \sqrt{\frac{25}{2}\,\frac{1}{2}} = \frac{5}{2}$

$s - c = \dfrac{a+b-c}{2} = \dfrac{\sqrt{13} + \sqrt{13} - \sqrt{2}}{2} = \sqrt{13} - \dfrac{\sqrt{2}}{2}$



$P = \|\vec{AB} \times \vec{AC}\|$

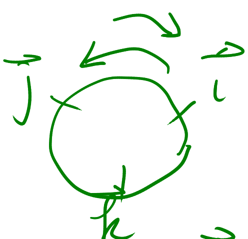
$(-2, 3, 0) \times (-3, 2, 0)$

$(-2\vec{i} + 3\vec{j}) \times (-3\vec{i} + 2\vec{j})$

$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$



$\vec{j} \times \vec{i} = -\vec{k}$

$\vec{AB} \times \vec{AC} = -2 \cdot 2\,\vec{k} - 3 \cdot 3\,(-\vec{k}) = (9-4)\vec{k} = 5\vec{k}$

$\vec{i}^{\times 2} = \vec{j}^{\times 2} = \vec{k}^{\times 2} = 0$

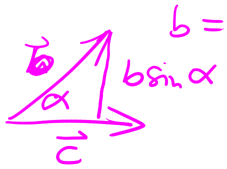
$\vec{i} \times \vec{i}$

$\|\vec{AB} \times \vec{AC}\|^2 = \sqrt{(\underset{0}{\ })^2 + (\underset{0}{\ })^2 + (5)^2} = 5$

$\frac{1}{2} P_{\square} = P_{\triangle} = \frac{1}{2} 5 = \frac{5}{2}$

$(-3,2,0) = -3\vec{i} + 2\vec{j}$

$(2,-3,0) = 2\vec{i} - 3\vec{j}$



$b = \|\vec{b}\|$

$\vec{i} \times \vec{i} = 0 \ \dots \ \vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{i} = -\vec{k}$

$\vec{b} \times \vec{c}$ VEKTORSKI UMNOŽAK → preko komponenti; → sintetički

- duljina $\|\vec{b}\| \, \|\vec{c}\| \sin\sphericalangle(\vec{b},\vec{c}) = P$
- smjer $\perp \vec{b}, \ \perp \vec{c}$
- orijentacija pravilo desne ruke

$$3 \begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} \to 3.\ \text{redak} = \underbrace{(-1)^{3+3}}_{+} \begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} \vec{i} - \vec{j} \begin{vmatrix} -3 & 0 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$$

$$= \cancel{(2\cdot 0 - 0\cdot(-3))\,\vec{i}} - \cancel{\vec{j}\,(-3\cdot 0 - 0\cdot 2)} + \vec{k}\,(-3\cdot(-3) - 2\cdot 2)$$

$$= 5\vec{k}$$

$$(a_x, a_y, a_z) \times (b_x, b_y, b_z)$$

$$= (a_y b_z - a_z b_y, \ a_z b_x - a_x b_z, \ a_x b_y - a_y b_x)$$

skalarni umnožak dva vektora je broj (vektorski je vektor)

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\sphericalangle(\vec{a},\vec{b})$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3dim $(a_x, a_y, a_z) \cdot (b_x, b_y, b_z) = a_x b_x + a_y b_y + a_z b_z$

2dim $(a_x, a_y) \cdot (b_x, b_y) = a_x b_x + a_y b_y$

$\cancel{\|\vec{a}\| \|\vec{b}\|} \cos\sphericalangle$

$\cancel{\|\vec{a}\| \|\vec{b}\|}$

$(-3,2) \cdot (2,-3) = -6 + (-6) = -12$

$$\cos\sphericalangle(\vec{a},\vec{b}) = \frac{-12}{\|\vec{a}\| \, \|\vec{b}\|} = \frac{-12}{\sqrt{9+4}\,\sqrt{4+9}} = \frac{-12}{\sqrt{13}\,\sqrt{13}} = \frac{-12}{13}$$

$(-3,2)$

II. I.

$$\sin\sphericalangle(\vec{a},\vec{b}) = \pm\sqrt{1 - \cos^2\sphericalangle(\vec{a},\vec{b})} = \pm\sqrt{1 - \left(\frac{-12}{13}\right)^2} = \sqrt{\frac{169 - 144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$P = \|\vec{a}\| \, \|\vec{b}\| \, |\sin\sphericalangle(\vec{a},\vec{b})|$$

$$= \cancel{\sqrt{13}}\,\sqrt{13}\,\frac{5}{\cancel{13}} = 5$$

$\sqrt{(-3)^2 + 2^2}$

$$P_\triangle = \frac{1}{2}\, 5 = \frac{5}{2}$$

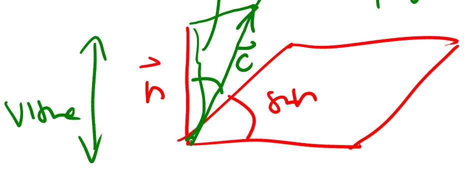
samo u tri dimenzije imamo i mješoviti umnožak vektora, prva dva pomnožimo vektorski i taj vektor pomnožimo skalarno s trećim vektorom (predznak zavisi od poretka)

$$(\vec a \times \vec b)\cdot \vec c = \|\vec a\|\,\|\vec b\|\,\|\vec c\| \sin\sphericalangle(\vec a,\vec b)\cos\sphericalangle(\vec c,\vec n)$$

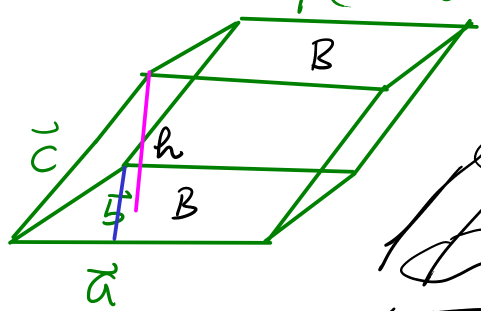
$$\vec n \perp (\vec a, \vec b)$$

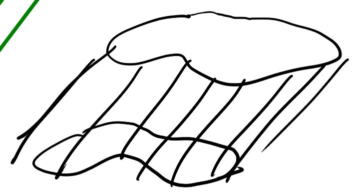
$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ \vec i & \vec j & \vec k \end{vmatrix} \cdot (c_x\vec i + c_y\vec j + c_z\vec k) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$\|\vec c\|\cos\sphericalangle$ = projekcija $\vec c$ na normalu

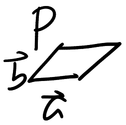


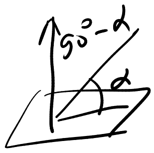
$$|(\vec a \times \vec b)\cdot \vec c| = V_{paralelepiped}$$

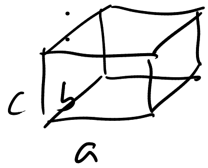




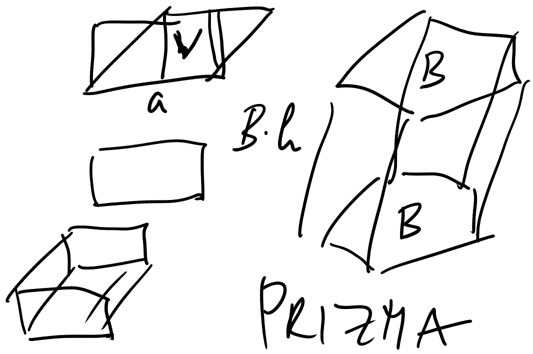
$$\underbrace{\|\vec a\|\,\|\vec b\|\,\sin\sphericalangle}_{V_n} \cdot \underbrace{h}_{\|\vec c\|\cos\sphericalangle}$$







$V = a\cdot b\cdot c$

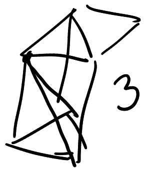


$B\cdot h$

PRIZMA

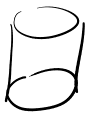
CAVALIERIJEV PRINCIP

ako dva tijela imaju jednake ravninske prereze na svim visinama onda oni imaju jednake volumene

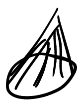


$3V = B\cdot h$

$V = \frac{B\cdot h}{3} = \frac{1}{6}(2B)\cdot h$



$B\cdot h$



$= \frac{1}{3}B\cdot h$