Serre A_∞-functors

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3. Serre A_∞-functors

0. Notation

- $\Bbbk$ denotes a (ground) commutative ring
- $\Bbbk$-linear = enriched in $\Bbbk\text{-mod}$
- graded = $\mathbb{Z}$-graded
- $\mathsf{C}_\Bbbk$ = the category of complexes of $\Bbbk$-modules
- A **differential graded (dg) category** is a category enriched in $\mathsf{C}_\Bbbk$.

  In particular, since $\mathsf{C}_\Bbbk$ is a closed monoidal category, it gives rise to a category $\underline{\mathsf{C}}_\Bbbk$ enriched in $\mathsf{C}_\Bbbk$, i.e., to a dg category.
- We use geometric notation for composition:

$$\xrightarrow{f}\xrightarrow{g} = \xrightarrow{fg}$$

1. Preliminaries on A_∞-categories

A_∞-categories

Definition. An A_∞-**category** $\mathcal{A}$ consists of

- a set of objects $\operatorname{Ob}\mathcal{A}$
- for each $X, Y \in \operatorname{Ob}\mathcal{A}$, a graded $\Bbbk$-module $\mathcal{A}(X, Y)$
- for each $n \geqslant 1$ and $X_0, \dots, X_n \in \operatorname{Ob}\mathcal{A}$, a $\Bbbk$-linear map

  $$m_n : \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_{n-1}, X_n) \to \mathcal{A}(X_0, X_n)$$

  of degree $2 - n$,

satisfying the equations

$$\sum_{p+k+q=n} (-1)^{pk+q}(1^{\otimes p} \otimes m_k \otimes 1^{\otimes q})m_{p+1+q} = 0, \quad n \geqslant 1.$$

$(n = 1)$ $\quad m_1^2 = 0$

$(n = 2)$ $\quad m_2 m_1 = (m_1 \otimes 1 + 1 \otimes m_1)m_2$

$(n = 3)$ $\quad (m_2 \otimes 1)m_2 - (1 \otimes m_2)m_2$

$$= m_3 m_1 + (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)m_3$$

Example. A dg category can be viewed as an A_∞-category in which $m_n = 0$ for $n \geqslant 3$.

A_∞-functors

Definition. An A_∞-**functor** $f : \mathcal{A} \to \mathcal{B}$ consists of

- a function $\operatorname{Ob} f : \operatorname{Ob} \mathcal{A} \to \operatorname{Ob} \mathcal{B}$, $X \mapsto Xf$
- for each $n \geqslant 1$ and $X_0, \dots, X_n \in \operatorname{Ob} \mathcal{A}$, a $\Bbbk$-linear map

  $$f_n : \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_{n-1}, X_n) \to \mathcal{B}(X_0 f, X_n f)$$

  of degree $1 - n$,

satisfying the equations

$$\sum_{i_1 + \cdots + i_l = n}^{l>0} (-1)^\sigma (f_{i_1} \otimes \cdots \otimes f_{i_l}) m_l = \sum_{p+k+q=n} (-1)^{pk+q} (1^{\otimes p} \otimes m_k \otimes 1^{\otimes q}) f_{p+1+q}, \quad n \geqslant 1,$$

where $\sigma = (i_2 - 1) + 2(i_3 - 1) + \cdots + (l-1)(i_l - 1)$.

$(n = 1)$ $\qquad f_1 m_1 = m_1 f_1$

$(n = 2)$ $\qquad m_2 f_1 - (f_1 \otimes f_1) m_2 = f_2 m_1 + (m_1 \otimes 1 + 1 \otimes m_1) f_2$

Example. A dg functor can be viewed as an A_∞-functor with $f_n = 0$ for $n \geqslant 2$.

Graded quivers

Definition. A **graded quiver** $\mathcal{A}$ consists of

- a set of objects $\operatorname{Ob}\mathcal{A}$
- for each $X, Y \in \operatorname{Ob}\mathcal{A}$, a graded $\Bbbk$-module $\mathcal{A}(X, Y)$.

A morphism $f : \mathcal{A} \to \mathcal{B}$ of graded quivers consists of

- a function $\operatorname{Ob} f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto Xf$
- for each $X, Y \in \operatorname{Ob}\mathcal{A}$, a $\Bbbk$-linear map

$$f = f_{X,Y} : \mathcal{A}(X, Y) \to \mathcal{B}(Xf, Yf)$$

  of degree 0.

Let $\mathscr{Q}$ denote the category of graded quivers. It is symmetric monoidal with tensor product $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \boxtimes \mathcal{B}$ given by

$$\operatorname{Ob}\mathcal{A} \boxtimes \mathcal{B} = \operatorname{Ob}\mathcal{A} \times \operatorname{Ob}\mathcal{B},$$

$$(\mathcal{A} \boxtimes \mathcal{B})((X, U), (Y, V)) = \mathcal{A}(X, Y) \otimes \mathcal{B}(U, V),$$

($X, Y \in \operatorname{Ob}\mathcal{A}$, $U, V \in \operatorname{Ob}\mathcal{B}$). The unit object is the graded quiver $\mathbb{1}$ with

$$\operatorname{Ob}\mathbb{1} = \{*\}, \qquad \mathbb{1}(*, *) = \Bbbk.$$

Graded quivers with a fixed set of objects

For a set S, denote by $\mathscr{Q}/S$ the fibre of the functor $\mathrm{Ob} : \mathscr{Q} \to \mathbf{Set}$ over S, i.e., the subcategory of $\mathscr{Q}$ whose objects are graded quivers with the set of objects S and whose morphisms are morphisms of graded quivers acting as the identity on objects.

$\mathscr{Q}/S$ is a monoidal category with tensor product $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \otimes \mathcal{B}$ given by

$$(\mathcal{A} \otimes \mathcal{B})(X, Z) = \bigoplus_{Y \in S} \mathcal{A}(X, Y) \otimes \mathcal{B}(Y, Z), \qquad X, Z \in S.$$

The unit object is the **discrete quiver** $\Bbbk S$ given by

$$\mathrm{Ob}\, \Bbbk S = S, \qquad \Bbbk S(X, Y) = \begin{cases} \Bbbk & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases} \qquad X, Y \in S.$$

Cocategories

Definition. An **augmented graded cocategory** is a graded quiver $\mathcal{C}$ equipped with the structure of an augmented counital coassociative coalgebra in the monoidal category $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$. Therefore, $\mathcal{C}$ comes with

- a **comultiplication** $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$
- a **counit** $\varepsilon : \mathcal{C} \to \Bbbk \operatorname{Ob}\mathcal{C}$
- an **augmentation** $\eta : \Bbbk \operatorname{Ob}\mathcal{C} \to \mathcal{C}$

which are morphisms in $\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ satisfying the usual axioms.

A morphism of augmented graded cocategories is a morphism of graded quivers that preserves the comultiplication, counit, and augmentation.

The category of augmented graded cocategories inherits the tensor product $\boxtimes$ from $\mathscr{Q}$.

Main example: tensor cocategory of a quiver

Let $\mathcal{A}$ be a graded quiver. Denote

$$T^n\mathcal{A} = \mathcal{A}^{\otimes n} = \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{n \text{ times}} \text{ (tensor product in } \mathscr{Q}/\operatorname{Ob}\mathcal{A}\text{).}$$

The graded quiver

$$T\mathcal{A} = \bigoplus_{n=0}^{\infty} T^n\mathcal{A}$$

equipped with the **cut comultiplication**

$$\Delta_0 : h_1 \otimes \cdots \otimes h_n \mapsto \sum_{k=0}^{n} h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n,$$

the counit

$$\varepsilon = \mathrm{pr}_0 : T\mathcal{A} \to T^0\mathcal{A} = \Bbbk \operatorname{Ob}\mathcal{A},$$

and the augmentation

$$\eta = \mathrm{in}_0 : \Bbbk \operatorname{Ob}\mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$$

is an augmented graded cocategory.

Cocategory approach to A_∞-categories

For a graded quiver $\mathcal{A}$, denote by $s\mathcal{A}$ its **suspension**:

$$\operatorname{Ob} s\mathcal{A} = \operatorname{Ob}\mathcal{A}, \qquad (s\mathcal{A}(X,Y))^d = \mathcal{A}(X,Y)^{d+1}, \quad X,Y \in \operatorname{Ob}\mathcal{A}.$$

Let $s : \mathcal{A} \to s\mathcal{A}$ denote the 'identity' map of degree -1.

Proposition (folklore). There is a bijection between structures $(m_n)_{n\geqslant 1}$ of an A_∞-category on a graded quiver $\mathcal{A}$ and differentials $b : Ts\mathcal{A} \to Ts\mathcal{A}$ of degree 1 such that $(Ts\mathcal{A}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0, b)$ is an **augmented differential graded cocategory**, i.e.,

$$b^2 = 0, \quad b\Delta_0 = \Delta_0(b\otimes 1 + 1\otimes b), \quad b\,\mathrm{pr}_0 = 0, \quad \mathrm{in}_0\, b = 0.$$

The bijection is given by the formulas

$$m_n = \big[\mathcal{A}^{\otimes n} \xrightarrow{s^{\otimes n}} (s\mathcal{A})^{\otimes n} \xrightarrow{b_n} s\mathcal{A} \xrightarrow{s^{-1}} \mathcal{A}\big],$$
$$b_n = \big[(s\mathcal{A})^{\otimes n} \xhookrightarrow{\mathrm{in}_n} Ts\mathcal{A} \xrightarrow{b} Ts\mathcal{A} \xrightarrow{\mathrm{pr}_1} s\mathcal{A}\big],$$
$$b_{nm} = \sum_{\substack{p+k+q=n\\ p+1+q=m}} 1^{\otimes p}\otimes b_k \otimes 1^{\otimes q} : T^n s\mathcal{A} \to T^m s\mathcal{A}.$$

We may think of A_∞-categories as of augmented dg cocategories of particular form. Then A_∞-functors $f : \mathcal{A} \to \mathcal{B}$ correspond precisely to morphisms

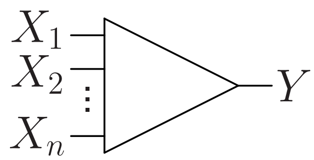
$$(Ts\mathcal{A}, b) \to (Ts\mathcal{B}, b)$$

of augmented dg cocategories.

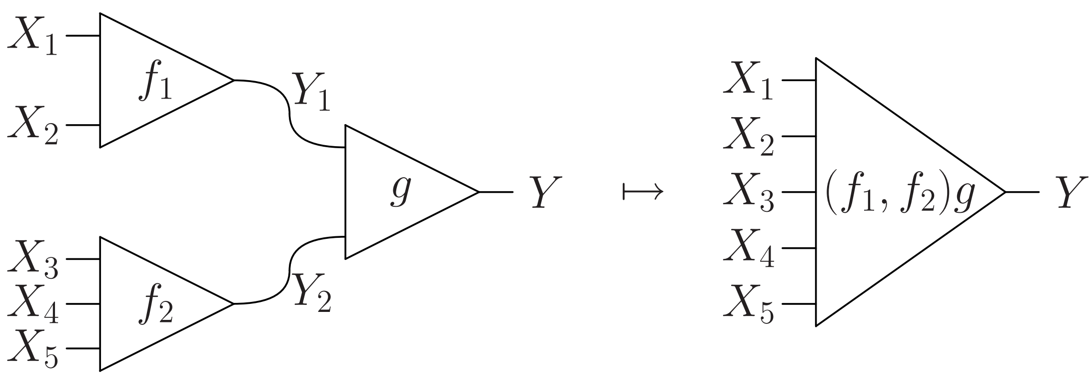
The advantage is that we can easily define A_∞-functors of many arguments!

A short reminder about multicategories

A **multicategory** is just like a category, the only difference being the shape of arrows. An arrow in a multicategory looks like



with a finite family of objects as the source and one object as the target. Composition turns a tree of arrows into a single arrow, e.g.



Example. A one-object multicategory is an operad (multicategories are sometimes called many-object operads, or 'colored operads').

Example. A monoidal category $\mathcal{C}$ gives rise to a multicategory $\widehat{\mathcal{C}}$ with the same set of objects. An arrow

$$X_1, \ldots, X_n \to Y$$

in $\widehat{\mathcal{C}}$ is an arrow

$$X_1 \otimes \cdots \otimes X_n \to Y$$

in $\mathcal{C}$. Composition in $\widehat{\mathcal{C}}$ is derived from composition and tensor product in $\mathcal{C}$.

A_∞-categories constitute a symmetric multicategory

Definition. Let $\mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{B}$ be A_∞-categories. An A_∞-**functor**

$$f : \mathcal{A}_1, \dots, \mathcal{A}_n \to \mathcal{B}$$

is a morphism of augmented dg cocategories

$$Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_n \to Ts\mathcal{B}.$$

Explicitly, an A_∞-functor $f : \mathcal{A}_1, \dots, \mathcal{A}_n \to \mathcal{B}$ consists of

- a function

$$\operatorname{Ob} f : \prod_{i=1}^{n} \operatorname{Ob} \mathcal{A}_i \to \operatorname{Ob} \mathcal{B}, \quad (X_1, \dots, X_n) \mapsto (X_1, \dots, X_n)f$$

- for each $k = (k_1, \dots, k_n) \in \mathbb{N}^n \smallsetminus \{0\}$ and $X_i^j \in \operatorname{Ob} \mathcal{A}_i$, $i = 1, \dots, n$, $j = 1, \dots, k_i$, a $\Bbbk$-linear map

$$\begin{array}{c}[\mathcal{A}_1(X_1^0, X_1^1) \otimes \cdots \otimes \mathcal{A}_1(X_1^{k_1-1}, X_1^{k_1})] \otimes \\ \cdots \otimes [\mathcal{A}_n(X_n^0, X_n^1) \otimes \cdots \otimes \mathcal{A}_n(X_n^{k_n-1}, X_n^{k_n})] \\ \big\downarrow f_k \\ \mathcal{B}((X_1^0, \dots, X_n^0)f, (X_1^{k_1}, \dots, X_n^{k_n})f)\end{array}$$

  of degree $1 - (k_1 + \cdots + k_n)$

subject to equations.

Denote by A_∞ the symmetric multicategory of A_∞-categories and A_∞-functors.

The multicategory A_∞ is closed

For each collection of A_∞-categories $\mathcal{A}_1$, ..., $\mathcal{A}_n$, $\mathcal{B}$, there exists a 'functor' A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$ and an **evaluation** A_∞-functor

$$\mathrm{ev}^{\mathsf{A}_\infty} : \mathcal{A}_1,\dots,\mathcal{A}_n,\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B}) \to \mathcal{B}$$

such that the mapping

$$\mathsf{A}_\infty(\mathcal{B}_1,\dots,\mathcal{B}_m;\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{C})) \to \mathsf{A}_\infty(\mathcal{A}_1,\dots,\mathcal{A}_n,\mathcal{B}_1\dots,\mathcal{B}_m;\mathcal{C}),$$
$$f \mapsto (1_{\mathcal{A}_1},\dots,1_{\mathcal{A}_n},f)\,\mathrm{ev}^{\mathsf{A}_\infty}$$

is a bijection. The objects of the A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$ are A_∞-functors $\mathcal{A}_1,\dots,\mathcal{A}_n \to \mathcal{B}$. For A_∞-functors $f,g : \mathcal{A}_1,\dots,\mathcal{A}_n \to \mathcal{B}$,

$$\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})(f,g) = \{\ A_\infty\text{-transformations } f \to g\}$$
$$= \{\ (f,g)\text{-coderivations } Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_n \to Ts\mathcal{B}\}.$$

The evaluation A_∞-functor acts on objects as expected:

$$\mathcal{A}_1,\dots,\mathcal{A}_n,\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B}) \to \mathcal{B}, \quad (X_1,\dots,X_n,f) \mapsto (X_1,\dots,X_n)f.$$

In the case $n=1$, the A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ has been considered by many authors (Keller, Kontsevich, Lefèvre-Hasegawa, Lyubashenko, Soibelman...).

Unital A_∞-categories

Definition. An A_∞-category $\mathcal{A}$ is called **unital** if, for each $X \in \operatorname{Ob}\mathcal{A}$, there is a cycle $1_X \in \mathcal{A}(X,X)^0$, called the **identity** of X, such that

$$(1_X \otimes \mathrm{id})m_2, (\mathrm{id} \otimes 1_Y)m_2 \sim \mathrm{id} : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y),$$

for each $X, Y \in \operatorname{Ob}\mathcal{A}$.

A unital A_∞-category $\mathcal{A}$ gives rise to a $\Bbbk$-linear category $H^0(\mathcal{A})$:

$$\operatorname{Ob} H^0(\mathcal{A}) = \operatorname{Ob}\mathcal{A}, \quad H^0(\mathcal{A})(X,Y) = H^0(\mathcal{A}(X,Y), m_1), \quad X, Y \in \operatorname{Ob}\mathcal{A}.$$

Composition is induced by m_2, and the identity of an object X is the class $[1_X] \in H^0(\mathcal{A})(X,X)$. The category $H^0(\mathcal{A})$ is called the **homotopy category** of $\mathcal{A}$.

An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is **unital** if it preserves identities modulo boundaries:

$$1_X f_1 - 1_{Xf} \in \operatorname{Im} m_1.$$

A unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ gives rise to a $\Bbbk$-linear functor

$$H^0(f) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$$

such that $\operatorname{Ob} H^0(f) = \operatorname{Ob} f$, and for each $X, Y \in \operatorname{Ob}\mathcal{A}$, the $\Bbbk$-linear map

$$H^0(f) : H^0(\mathcal{A})(X,Y) \to H^0(\mathcal{B})(Xf, Yf)$$

is induced by $f_1 : \mathcal{A}(X,Y) \to \mathcal{B}(Xf, Yf)$.

An A_∞-functor of many argument is **unital** if it is unital in each argument.

The symmetric closed multicategory of unital A_∞-categories

Composition of unital A_∞-functors is unital. Let $\mathsf{A}_\infty^{\mathrm{u}} \subset \mathsf{A}_\infty$ denote the submulticategory of unital A_∞-categories and unital A_∞-functors. It is also closed:

$$\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{B}) \subset \underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{B})$$

is the full A_∞-subcategory whose objects are unital A_∞-functors. It is a unital A_∞-category. The evaluation A_∞-functor $\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{u}}}$ is the restriction of $\mathrm{ev}^{\mathsf{A}_\infty}$. It is a unital A_∞-functor.

Definition. Unital A_∞-functors

$$f, g : \mathcal{A}_1, \dots, \mathcal{A}_n \to \mathcal{B}$$

are called **isomorphic** if they are isomorphic as objects of the category

$$H^0(\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{B})).$$

Opposite A_∞-categories

Definition. Let $\mathcal{A}$ be an A_∞-category. The **opposite** A_∞**-category** $\mathcal{A}^{\mathrm{op}}$ is given by

$$\operatorname{Ob}\mathcal{A}^{\mathrm{op}} = \operatorname{Ob}\mathcal{A}, \qquad \mathcal{A}^{\mathrm{op}}(X,Y) = \mathcal{A}(Y,X), \quad X,Y \in \operatorname{Ob}\mathcal{A},$$

and operations $m_n^{\mathcal{A}^{\mathrm{op}}}$ are given by

$$m_n^{\mathcal{A}^{\mathrm{op}}} = (-1)^{n(n+1)/2+1} \begin{pmatrix} \text{signed permutation} \\ \text{of arguments} \end{pmatrix} \cdot m_n^{\mathcal{A}}.$$

The correspondence $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$ extends to A_∞-functors and yields a symmetric multifunctor $-^{\mathrm{op}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$.

The opposite of a unital A_∞-category (resp. A_∞-functor) is again unital, hence $-^{\mathrm{op}}$ restricts to a symmetric multifunctor $-^{\mathrm{op}} : \mathsf{A}_\infty^{\mathrm{u}} \to \mathsf{A}_\infty^{\mathrm{u}}$.

2. Serre functors

Hereafter, $\Bbbk$ is a **field**.

Definition (Bondal–Kapranov). Let $\mathcal{C}$ be a $\Bbbk$-linear category. A $\Bbbk$-linear functor $S : \mathcal{C} \to \mathcal{C}$ is called a **(right) Serre functor** if there exists an isomorphism

$$\mathcal{C}(X, YS) \cong \mathcal{C}(Y, X)^*$$

natural in $X, Y \in \operatorname{Ob} \mathcal{C}$, where $*$ denotes the dual vector space. A right Serre functor, if it exists, is unique up to isomorphism.

Example. Let X be a smooth projective variety of dimension n over the field $\Bbbk$. Let ω_X denote the canonical sheaf on X. Let $\mathcal{C} = D^b(\mathsf{Coh}_X)$ be the bounded derived category of coherent sheaves on X. Then the functor

$$S = - \otimes \omega_X[n]$$

is a right Serre functor.

3. Serre A_∞-functors

Definition

For an A_∞-category $\mathcal{A}$, there is an A_∞-functor

$$\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}, \qquad (X, Y) \mapsto (\mathcal{A}(X, Y), m_1).$$

It is unital if so is $\mathcal{A}$. The A_∞-functor $\mathcal{A} \to \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}; \underline{\mathsf{C}}_{\Bbbk})$ that corresponds to $\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ by closedness of the multicategory A_∞ is precisely the Yoneda embedding.

Definition (Kontsevich–Soibelman). Let $\mathcal{A}$ be a unital A_∞-category. A unital A_∞-functor $S : \mathcal{A} \to \mathcal{A}$ is called a **(right) Serre A_∞-functor** if the diagram

$$\begin{array}{ccc}
\mathcal{A}^{\mathrm{op}}, \mathcal{A} & \xrightarrow{1,S} & \mathcal{A}^{\mathrm{op}}, \mathcal{A} \\
{\scriptstyle \mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}} \\
\underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}} & \xrightarrow{D} & \underline{\mathsf{C}}_{\Bbbk}
\end{array}$$

commutes up to isomorphism (in $H^0(\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{A}; \underline{\mathsf{C}}_{\Bbbk}))$). Here

$$D : \underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}} \to \underline{\mathsf{C}}_{\Bbbk}, \qquad M \mapsto M^* = \underline{\mathsf{C}}_{\Bbbk}(M, \Bbbk),$$

is the duality dg functor.

Proposition. As in the case of ordinary Serre functors, if a right Serre A_∞-functor exists, then it is unique up to isomorphism.

A_∞-categories closed under shifts

(see also V. Lyubashenko's talk)

Let $\mathcal{A}$ be an A_∞-category. It gives rise to an A_∞-category $\mathcal{A}^{[]}$ obtained from $\mathcal{A}$ by formally adding shifts of objects:

$$\operatorname{Ob}\mathcal{A}^{[]} = \operatorname{Ob}\mathcal{A} \times \mathbb{Z}, \qquad \mathcal{A}^{[]}((X,n),(Y,m)) = \mathcal{A}(X,Y)[m-n].$$

$\mathcal{A}$ embeds as a full A_∞-subcategory into $\mathcal{A}^{[]}$ via

$$u : \mathcal{A} \hookrightarrow \mathcal{A}^{[]}, \qquad X \mapsto (X,0).$$

Definition. A unital A_∞-category $\mathcal{A}$ is called **closed under shifts** if u is an A_∞-equivalence.

Equivalently, each object (X,n) of $\mathcal{A}^{[]}$ is isomorphic in $H^0(\mathcal{A}^{[]})$ to an object of the form $(Y,0)$.

Example. Pretriangulated A_∞-categories (to be defined by V. Lyubashenko) are closed under shifts.

Main theorem

As above, assume that $\Bbbk$ is a field.

Theorem. (1) If $S : \mathcal{A} \to \mathcal{A}$ is a right Serre A_∞-functor, then the induced functor $H^0(S) : H^0(\mathcal{A}) \to H^0(\mathcal{A})$ is an ordinary right Serre functor.

(2) Conversely, suppose that $\mathcal{A}$ is closed under shifts and that $H^0(\mathcal{A})$ admits a right Serre functor $\bar{S} : H^0(\mathcal{A}) \to H^0(\mathcal{A})$. Then there exists a right Serre A_∞-functor $S : \mathcal{A} \to \mathcal{A}$ such that $H^0(S) = \bar{S}$.

Example. By results of Drinfeld, we know that $D^b(\mathsf{Coh}_X)$ is of the form $H^0(\mathcal{A})$, where $\mathcal{A}$ is the dg quotient of the dg category of complexes of coherent sheaves over the full dg subcategory of acyclic complexes. Therefore, the Serre functor $S = - \otimes \omega_X[n]$ lifts to a Serre A_∞-functor $\mathcal{A} \to \mathcal{A}$.