Search for Leibniz groups

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0.1. In a work with my student M. Bašić, we had some progress in a program of finding a correct category of 'groups' for integration of Leibniz algebras. In this section, $\mathbf{k}$ is a field of characteristic zero, but many intermediate statements allow for a commutative unital ring. A right (left) Leibniz $\mathbf{k}$-algebra is an $\mathbf{k}$-module L with a $\mathbf{k}$-bilinear product $[,] : L \otimes L \to L$ such that the operators $\mathrm{ad}_r l = [\_, l]$ (resp. $\mathrm{ad}_l l = [l, \_]$) are derivations for all $l \in L$.

Here is a short outline of our attempt to integration of Leibniz $\mathbf{k}$-algebras.

0.2. (Background on LP-categories) ([3, 4, 2, 1]) Given a category $\mathcal{V}$, the category of arrows $\mathrm{Arr}\mathcal{V}$ is defined as follows: the objects of $\mathrm{Arr}\mathcal{V}$ are morphisms of $\mathcal{V}$ and the morphisms from $\underline{V} = (V_1 \xrightarrow{f_V} V_0)$ to $\underline{W} = (W_1 \xrightarrow{f_W} W_0)$ are pairs $\alpha : V_1 \to W_1$, $\beta : V_0 \to W_0$ making the obvious square commutative, $f_W \circ \alpha = \beta \circ f_V$. If $\mathcal{V}$ is a $\mathbf{k}$-linear abelian symmetric monoidal category, we can view objects of $\mathrm{Arr}\mathcal{V}$ as two term (= length 1) chain complexes in $\mathrm{Arr}\mathcal{V}$, concentrated always in fixed degrees, say 0 and 1. The tensor product of such complexes given by $(V \otimes W)_n = \oplus_i V_i \otimes W_{n-i}$ (with obvious differential) is a 3-term chain complex concentrated in degrees $0, 1, 2$. Loday and Pirashvili observed that if one truncates this tensor product leaving out degree 2 term, one still has nice tensor product for good $\mathcal{V}$; $\mathrm{Arr}\mathcal{V}$ with this "infinitesimal" tensor product is an abelian symmetric monoidal category, which we call **LP-(tensor) category**, $\mathcal{V}_{\mathrm{L}P}$ (and when $\mathcal{V}$ is understood, we will just refer to this category as LP). If $\mathcal{V} = \mathbf{k}$-Mod (and for some other closed $\mathcal{V}$) this category is closed: inner homs $\mathbf{Hom}(\underline{V}, \underline{W}) = (\mathrm{hom}_1(\underline{V}, \underline{W}) \xrightarrow{\mathrm{p}} \mathrm{hom}(\underline{V}, \underline{W}))$ are constructed as follows: $\mathrm{hom}_1(\underline{V}, \underline{W})$ is the subobject of $\mathbf{Hom}_{\mathcal{V}}(V_1, W_1) \oplus \mathbf{Hom}_{\mathcal{V}}(V_0, W_0) \oplus \mathbf{Hom}_{\mathcal{V}}(V_0, W_1)$ whose underlying set is formed by formal triples $\alpha + \beta + \tilde{\beta}$ where $f_W \circ \alpha = \beta \circ f_V$ and $f_W \circ \tilde{\beta} = \beta$; the natural projection p is forgetting the lift $\tilde{\beta}$ (p is neither epi nor mono in general). Similarly, $(n+1)$-term chain complexes with a truncated tensor product form a symmetric monoidal weak n-category which we call generalized LP-category $\mathcal{V}_{\mathrm{L}Pn}$.

Lie algebras in the LP-category $(\mathbf{k}-\mathrm{Mod})_{\mathrm{LP}}$ are simply objects $(M \xrightarrow{f} \mathfrak{g})$ in $(\mathbf{k} - \mathrm{Mod})_{\mathrm{LP}}$ where $\mathfrak{g}$ is a $\mathbf{k}$-Lie algebra, with a $\mathfrak{g}$-action on M which is making $(M \xrightarrow{f} \mathfrak{g})$ $\mathfrak{g}$-equivariant. The category of (say, right) Leibniz $\mathbf{k}$-algebras canonically embeds into the category of Lie algebras in $(\mathbf{k} - \mathrm{Mod})_{\mathrm{LP}}$ by

$$(L, [,]) \mapsto (L \xrightarrow{\mathrm{pr}} L_{\mathrm{Lie}}), \quad L_{\mathrm{Lie}} := L/\langle [l, l], l \in L \rangle,$$

where pr is the natural projection of L to the quotient Lie $\mathbf{k}$-algebra $L' = L/\langle l^2, l \in L \rangle$; moreover $(L \xrightarrow{\mathrm{pr}} L_{\mathrm{Lie}})$ has a canonical structure of a Lie algebra in $(\mathbf{k} - \mathrm{Mod})_{\mathrm{LP}}$. On the other hand, if $M \to \mathfrak{g}$ is any Lie algebra in $(\mathbf{k} - \mathrm{Mod})_{\mathrm{LP}}$ then M has a right Leibniz bracket given by the right action of $\mathfrak{g}$ on M, namely $[m, m']_M := m f(m')$.

Similarly, an object $f : M \to A$ has a structure of **associative algebra in LP** if A is an associative $\mathbf{k}$-algebra, M is an A-A-bimodule, and if f is a map of bimodules. The category of associative dialgebras is canonically embedded as a full subcategory of the category of inner associative algebras in LP. In every symmetric **k**-linear monoidal category, there is a functor from associative to Lie algebras (bracket is given by $[,] = \mu \circ (\mathrm{id} \circ \mathrm{id} - \tau)$, where τ is the symmetry and μ the multiplication). In **k**-Mod, this functor has a left adjoint: the enveloping associative algebra of a Lie algebra in LP. Loday introduces the notion of **associative k-dialgebra** as a **k**-module with two associative bilinear products, left $\dashv$, and right $\vdash$, satisfying 3 additional identities involving both products. The underlying **k**-module of an associative dialgebra D is a Leibniz algebra with bracket

$$[d, d'] = d \dashv d' - d' \vdash d.$$

This 'underlying Leibniz algebra' functor has a left adjoint, the universal enveloping dialgebra of the Leibniz algebra. Finally, if we start with a Leibniz algebra then we define its "envelope in LP" by first taking a universal enveloping dialgebra and then consider this dialgebra as an associative algebras in LP; or one can first embed the Leibniz algebras as Lie algebras in LP and then take the enveloping algebra in LP; with isomorphic result. Other important constructions are given: the (internal) symmetric algebra $S(\underline{V})$ and the (internal) tensor algebra $T(\underline{V})$ generated by any object $\underline{V} = (V_1 \to V_0) \in \mathcal{V}_{\mathrm{L}P}$. If **k** is a field of characteristics zero then one has a PBW-theorem for Lie algebras in LP and a version of Ado's theorem. Namely, the inner end $\mathbf{End}(\underline{V}) = \mathbf{Hom}(\underline{V}, \underline{V})$ is an associative algebra in LP (by general nonsense on inner homs); one can therefore form a Lie algebra in LP out of it. This internal Lie algebra is denoted $\mathbf{gl}(\underline{V})$. Ado's theorem in LP ([2]) says that, if $\mathrm{char}\, \mathbf{k} = 0$, every Lie algebra $M \to \mathfrak{g}$ in LP with finite dimensional M and $\mathfrak{g}$ is an internal Lie subalgebra of $\mathbf{gl}(\underline{V})$ for $\underline{V} = (V_1 \to V_0)$ with finite dimensional V_0, V_1; the proof uses the usual Ado's theorem (crucial in classical approaches to the proof of Lie-Cartan's theorem on the existence of a Lie group whose tangent Lie algebra is given).

0.3. We would like to construct a nice category generalizing Lie groups (or algebraic groups) to integrate finite dimensional Lie algebras in LP-category over $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$. As a Lie algebra in LP $M \to \mathfrak{g}$ is a usual Lie **k**-algebra $\mathfrak{g}$ with a $\mathfrak{g}$-equivariant map from a $\mathfrak{g}$-module M and a usual Lie algebra integrates to a Lie group, one may target some category of Lie groups with an additional structure, say involving a G-equivariant sheaf of $\mathcal{O}$-modules $\mathcal{M}$ and a map of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{O}$. But to get some feeling of what kind of sheaves and integration should be involved we try first developing an algebraic group version.

0.3.1. First we consider the category $\mathrm{Aff}_{\mathrm{L}P,\mathbf{k}}$ of "affine schemes in LP", which is simply the opposite to the category of commutative associative algebras in LP. To have change of base, one needs to consider LP-categories over different bases **k**; in particular over integers where one talks about commutative associative LP-rings. The very definition of $\mathrm{Aff}_{\mathrm{L}P,\mathbf{k}}$ has built in the fact that the tensor product in LP was used in defining the internal associative algebra. However, the categorical coproduct of internal commutative algebras $V \to A$, $W \to B$, is still formed using the usual $V \otimes W \to A \otimes B$, rather than infinitesimal tensor products of underlying objects; this implies a recipe for the fibered products in the category of affine LP schemes. However, we are interested in noncategorical fiber product which corresponds to the opposite of internal tensor product of algebras in LP (hence infinitesimal tensor product of underlying objects).

0.3.2. Next one considers some notion of Zariski topology to glue affine schemes in LP. The most sensible approach would be to take the Rosenberg's spectrum of abelian category of modules in LP over a commutative algebra in LP; for Rosenberg spectrum there is a standard choice of Zariski topology. We however so far considered only a weaker, naive topology: localization with respect to the classical Zariski topology of the usual bottom algebra (if $M \to A$ is a commutative algebra in LP, we consider the localizations of A, induced localization of $M \to A$, and its effect on the category of internal modules over $M \to A$) . Then we glue the schemes from affine schemes in this topology.

0.3.3. In the category of LP-schemes which we obtain, we can extend the fiber product from $\mathrm{Aff}_{LP,\mathbf{k}}$ as in the classical case: gluing the fiber products of affine covers. This may be done for categorical fiber products as well as for the LP-fiber products described above. **LP-group schemes** are the group objects in the category of LP-schemes (in classical Zariski topology of bottom algebras) with respect to the LP-product.

0.4. Affine examples of LP-group scheme may be constructed from any Hopf algebra in LP (notice that the dualization changes the directions of arrows; bad dualizations and nonprojective objects are often a trouble in this context). Regarding that we know a Lie algebra $\mathbf{gl}(\underline{V})$ in LP, we would like to have also an LP-group scheme $GL(\underline{V})$ whenever $\underline{V}$ in LP. I have constructed a good candidate for $GL(\underline{V})$ at IHÉS in 2007, and M.B. is studying this example now. The construction is very simple and follows Manin's approach to linear quantum groups. One first constructs the inner end of the symmetric algebra $S(\underline{V})$ in LP of $\underline{V}$; this inner end is a bialgebra in LP, which we denote $\mathcal{O}(M(\underline{V}))$; in suitable basis this algebra is rather easy to construct very explicitly. Then one looks for a Hopf envelope in LP; to this aim one introduces additional generators forcing existence of the antipode (in LP sense), similarly to the construction of general linear groups in various familiar categories. The new relations look cubic, but case by case examination in a base shows that they actually cut out an ideal in a linear way. The quotient is denoted $\mathcal{O}(GL(\underline{V}))$. However, the structure of $\mathcal{O}(GL(\underline{V}))$ is relatively complicated and we still do not see exactly the correspondence with $\mathbf{gl}(\underline{V})$ at the tangent level.

0.5. Of course, to get to the tangent level one needs the theory of invariant differential operators. For a general LP-scheme the study of regular differential operators is in progress. We tried to understand the analogue of the Weyl algebra, as presumably the local case first. It is not enough to consider the $\mathbf{k}$-derivations of an algebra $\mathbf{A}$ in LP: derivations form just a usual Lie $\mathbf{k}$-algebra. Instead, we consider the module of derivations of $\mathbf{A}$ as a $\mathbf{k}$-submodule of the inner end $\mathbf{End}(\mathbf{A})$, where we cut out the inner *derivative submodule* in LP in a categorical way. In "components", this is equivalent to the following definition.

Definition. *Let* $\mathbf{A} = (M \xrightarrow{f} A)$ *be an algebra in LP. A* **lower derivation** *of* $\mathbf{A}$ *is a pair* $d = (d_M, d_A)$ *of a* $\mathbf{k}$*-linear maps* $d_M : M \to M$, $d_A : A \to A$, *with* $f \circ d_M = d_A \circ f$, *such that* d_A *is a derivation in the usual sense,* $d_M(am) = d_A(a)m + ad_M(m)$, $d_M(ma) = md_A(a) + d_M(m)a$ *where* $(a \in A, m \in M)$, *and* $f \circ d_M = d_A \circ f$. *Lower derivations form a* $\mathbf{k}$*-submodule* $\mathrm{Der}_0\mathbf{A}$ *of* $\mathrm{end}(\mathbf{A})$ *An element* $b = (b_M, b_A, b_\phi)$ *in the upper part* $\mathrm{end}_1(\mathbf{A})$ *of the inner end* $\mathbf{End}(\mathbf{A})$ (cf. **0.2**: this means that $b_M : M \to M$, $b_A : A \to A$, $b_\phi : A \to M$, $f \circ b_M = b_A \circ f$ and $f \circ b_\phi = b_A$) *is the* **upper derivation** *if* (b_M, b_A) *is a lower derivation and* $b_\phi \in \mathrm{Der}_{\mathbf{k}}(A, M)$ (where M is understood as an A-bimodule). *The upper derivations make a* $\mathbf{k}$*-submodule* $\mathrm{Der}_1\mathbf{A}$ *of* $\mathrm{hom}_1(\mathbf{A}, \mathbf{A})$. *The inner object of derivations is* $\mathbf{DerA} = (\mathrm{Der}_1\mathbf{A} \xrightarrow{p} \mathrm{Der}_0\mathbf{A})$ *where the natural projection is a restriction of the natural projection in* $\mathbf{End}(\mathbf{A})$ (hence it is neither injective nor surjective in general).

DerA is naturally a Lie algebra object in LP. For $\mathbf{A} = S(\underline{V})$ the projection p is not injective (for general $\underline{V}$), and M. Bašić has determined ([1]) a basis of $\mathrm{Der}_i S(\underline{V})$ for $i = 0, 1$; there are 6 combinatorially distinct types of upper derivations and 4 distinct types of lower derivations. The **inner Weyl algebra in LP** is the smallest inner subalgebra in $\mathbf{End}\, S(\underline{V})$ which contains $S(\underline{V})$ and $\mathbf{Der}S(\underline{V})$; conjecturally all relations are of the commutation type (hence quadratic; all quadratic relations have been computed in [1]); however the proof that there are no high order additional relations relies on a small combinatorial conjecture which we did not resolve so far. It is not known if the Grothendieck style definition of differential operators in this context would give the lower part of the inner Weyl algebra as we defined it.

0.6. For a (commutative but not cocommutative) Hopf algebra $(\mathbf{H} : H_1 \to H_0, \Delta)$ in $(\mathrm{Vec}_{\mathbf{k}})_{LP}$ we (tentatively) define an **invariant part** of the internal endomorphism algebra $\mathbf{End}(\mathbf{H})$ of $\mathbf{H}$ in LP. Denote the components of the coproduct by $\Delta_0 : H_0 \to H_0 \otimes H_0$ and $\Delta_1 = \Delta_{1l} + \Delta_{1r}$, where $\Delta_{1l} : H_1 \to H_0 \otimes H_1$ and $\Delta_{1r} : H_1 \to H_1 \otimes H_0$; denote $\mathrm{id}_i = \mathrm{id}_{T_i}$, $i = 1, 2$. Let $T_i : H_i \to H_i$, $i = 1, 2$ be components of an endomorphism T of $\mathbf{H}$. Then T is invariant if $\Delta_1 \circ T_1 = (\mathrm{id}_0 \otimes T_1 \oplus \mathrm{id}_1 \otimes T_0) \circ \Delta_1$ and $\Delta_0 \circ T_0 = (\mathrm{id}_0 \otimes T_0) \circ \Delta_0$. Let $T_\phi : H_0 \to H_1$ be a $\mathbf{k}$-linear map such that $T' = (T_1, T_0, T_\phi)$ is an element of $\mathrm{end}_1(\mathbf{H})$. Then we proclaim T' invariant if $T = (T_1, T_0)$ is invariant. One checks that the invariant part of $\mathbf{End}(\mathbf{H})$ is a Lie algebra in LP.

Thus one could in principle check if the inner Lie algebra of derivations in LP of our inner Hopf algebra $\mathcal{O}(GL(\underline{V}))$ has the invariant part isomorphic (as a Lie algebra in LP) to the Lie algebra $\mathbf{gl}(\underline{V})$ defined by Kurdiani in [2]. If this is so, the Ado's theorem in LP ([2]) could help us reduce integration of other finite-dimensional Lie algebras in LP to the special example (via subalgebra/subgroup correspondence which needs separate treatment in LP), and the construction of a spectrum of $\mathcal{O}(GL(\underline{V}))$ in the category of LP-schemes could give us geometrical integration.

References

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