

# A HOMOTOPY THEORY FOR STACKS

BY

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## ABSTRACT

We give a homotopy theoretic characterization of stacks on a site  $\mathcal{C}$  as the **homotopy sheaves** of groupoids on  $\mathcal{C}$ . We use this characterization to construct a model category in which stacks are the fibrant objects. We compare different definitions of stacks and show that they lead to Quillen equivalent model categories. In addition, we show that these model structures are Quillen equivalent to the  $S^2$ -nullification of Jardine's model structure on sheaves of simplicial sets on  $\mathcal{C}$ .

## 1. Introduction

The main purpose of this paper is to show that the classical definition of stacks [DM, Definition 4.1], [Gi, II.2.1], can be stated in terms of homotopy theory. From this point of view the definition appears natural, and places stacks into a larger homotopy theoretic context. Constructions that are commonly performed on stacks such as 2-category pullbacks, stackification, sheaves over a stack and others, have easy homotopy-theoretic interpretations.

The basic idea is this: a stack  $\mathcal{M}$  is an assignment to each scheme  $X$  of a groupoid  $\mathcal{M}(X)$ , which is required to satisfy 'descent conditions' [Gi, II.1.1].

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The descent conditions describe the circumstances under which we require that local data glue together to yield global data.

Naively, one might propose a “local-to-global” requirement that the assignment “isomorphism classes of -” satisfy the sheaf condition. However, for very fundamental reasons, this almost never happens in examples. Taking isomorphism classes is a localization process, and such processes rarely preserve limits such as those which arise in the statement of the sheaf condition.

Instead, one can ask that the assignment of groupoids satisfy a sheaf condition with respect to the **best functorial approximation to the limit** which respects isomorphism classes. This is called the homotopy limit, denoted  $\text{holim}$ , and is the total right derived functor of inverse limit, (see [DS, Sections 9–10]).

We prove that this **homotopy sheaf condition** is exactly the content of the descent conditions and so, in this sense, stacks are the **homotopy sheaves**.

**THEOREM 1.1:** *Let  $\mathcal{C}$  be a Grothendieck topology. Let  $\text{Grpd}/\mathcal{C}$  denote the category of categories fibered in groupoids over  $\mathcal{C}$  (see Definition 3.1). For each  $\mathcal{E} \in \text{Grpd}/\mathcal{C}$ , and  $X \in \mathcal{C}$ , denote by  $\mathcal{E}(X)$  the groupoid of maps  $\text{Hom}(\mathcal{C}/X, \mathcal{E})$ . Then  $\mathcal{E}$  is a stack [DM, Definition 4.1] if and only if for every cover  $\{U_i \rightarrow X\}$  in  $\mathcal{C}$  we have an equivalence of categories*

$$\mathcal{E}(X) \xrightarrow{\sim} \text{holim} \left( \prod \mathcal{E}(U_i) \rightrightarrows \prod \mathcal{E}(U_{ij}) \rightrightarrows \prod \mathcal{E}(U_{ijk}) \dots \right).$$

Here  $U_{i_0 \dots i_n}$  denotes the iterated fiber product  $U_{i_0} \times_X \dots \times_X U_{i_n}$ , and we take the homotopy limit of this cosimplicial diagram in the category of groupoids (see Section 2).

Classically stacks were defined either as categories fibered in groupoids or lax presheaves [Brn, Section 1] satisfying descent conditions. It is not hard to check that categories fibered in groupoids and lax presheaves are equivalent categories and that these two definitions agree [Holl, Appendix B]. The following results make precise the sense in which it suffices to work with actual presheaves of groupoids instead of lax presheaves or categories fibered in groupoids. Let  $P(\mathcal{C}, \text{Grpd})$  denote the categories of presheaves of groupoids on  $\mathcal{C}$ .

**THEOREM 1.2:** *There exists a model category structure on  $\text{Grpd}/\mathcal{C}$  in which weak equivalences are the fiberwise equivalences of groupoids. Similarly, there exists model category structure on  $P(\mathcal{C}, \text{Grpd})$  in which weak equivalences are the objectwise equivalences of groupoids. For these two model structures the*

adjoint pair (see Section 3.3)

$$\mathit{Grpd}/\mathcal{C} \begin{array}{c} \xrightarrow{\Gamma} \\ \xleftarrow{p} \end{array} P(\mathcal{C}, \mathit{Grpd})$$

is a Quillen equivalence.

This model structure on  $\mathit{Grpd}/\mathcal{C}$  encodes many of the classical constructions for categories fibered in groupoids and for stacks. The equivalences of stacks are just the weak equivalences. The 2-category pullback is the homotopy pullback. Writing these constructions in terms of standard homotopy theory makes clear their functoriality properties and relations with one another.

This adjunction sends the full subcategory of stacks in  $\mathit{Grpd}/\mathcal{C}$  to the full subcategory of  $P(\mathcal{C}, \mathit{Grpd})$  of those presheaves which satisfy descent, which we call stacks.

*Definition 1.3:* Let  $\mathcal{C}$  be a Grothendieck topology. A presheaf of groupoids,  $F$  on  $\mathcal{C}$  is a **stack** if for every cover  $\{U_i \rightarrow X\}$  in  $\mathcal{C}$ , we have an equivalence of categories

$$F(X) \xrightarrow{\sim} \operatorname{holim} \left( \prod F(U_i) \rightrightarrows \prod F(U_{ij}) \rightrightarrows \prod F(U_{ijk}) \dots \right).$$

The fact that  $(\Gamma, p)$  is a Quillen equivalence tells us that it is equivalent to make any homotopy theoretic construction in one of these categories or the other, (i.e. we can use  $\Gamma$  to translate our problem in  $\mathit{Grpd}/\mathcal{C}$  into one about presheaves, solve it and then apply  $p$  to the result).

Finally, we show that the model category structures of Theorem 1.2 can be localized and the result is a model structure even better suited to the study of stacks. One can also consider the category of sheaves of groupoids on  $\mathcal{C}$ , denoted  $Sh(\mathcal{C}, \mathit{Grpd})$ , instead of presheaves.

**THEOREM 1.4:** *There are simplicial model category structures  $\mathit{Grpd}/\mathcal{C}_L$ ,  $P(\mathcal{C}, \mathit{Grpd})_L$  and  $Sh(\mathcal{C}, \mathit{Grpd})_L$ , in which the stacks are the fibrant objects.*

The adjoint pairs

$$\mathit{Grpd}/\mathcal{C}_L \begin{array}{c} \xrightarrow{\Gamma} \\ \xleftarrow{p} \end{array} P(\mathcal{C}, \mathit{Grpd})_L \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{sh} \end{array} Sh(\mathcal{C}, \mathit{Grpd})_L,$$

are Quillen equivalences (where the right adjoints point to presheaves). All of these functors take stacks as defined in the domain category to stacks as defined in the range category and thus restrict to give adjoint pairs between the stacks in each of these categories.

**COROLLARY 1.5:** *In each of the above model categories the fibrant replacement functor gives a stackification functor.*

Of the different categories mentioned above, the simplest to analyze is presheaves of groupoids which is closely related to presheaves of simplicial sets. Here a form of Dugger’s local lifting conditions [DHI, Section 3], modified for groupoids, (see Definition 5.6) allows us to characterize weak equivalences in a simple way (Theorem 5.7). The comparison with the homotopy theory of simplicial sets is encapsulated in the following result.

**PROPOSITION 1.6:** *The above model structure on  $P(\mathcal{C}, Grpd)$  is Quillen equivalent to Joyal’s model category structure on  $P(\mathcal{C}, sSet)$  localized with respect to the maps  $\partial\Delta^n \otimes X \rightarrow \Delta^n \otimes X$ , for each  $X \in \mathcal{C}$  and  $n > 2$ .*

This theorem tells us that the homotopy theory of stacks is recovered from Joyal’s model category by eliminating all higher homotopies. A direct corollary of this Quillen equivalence is the following result which nicely generalizes the usual criterion for a map to be an isomorphism between two sheaves of sets.

**COROLLARY 1.7:** *If the topology on  $\mathcal{C}$  has enough points [MM, p. 521], the weak equivalences in  $P(\mathcal{C}, Grpd)$  are the stalkwise equivalences of groupoids.*

The characterization of stacks as the homotopy sheaves of groupoids is the source and inspiration for all of the above results. Furthermore this characterization generalizes naturally to a definition of  $n$ -stack as follows

**Definition 1.8:** A presheaf of simplicial sets  $F$  on  $\mathcal{C}$  is an  $n$ -**stack** if for every  $X$  in  $\mathcal{C}$ ,  $F(X)$  is a Kan complex, and for every hypercover  $U_\bullet \rightarrow X$  in  $\mathcal{C}$  [DHI, Definition 4.2], we have an equivalence of categories

$$F(X) \xrightarrow{\sim} \text{holim } F(U_\bullet)$$

where the homotopy limit is taken in the category of  $(S^{n+1})^{-1}sSet$ , the  $S^{n+1}$  nullification of simplicial sets.

The reason for considering hypercovers as opposed to covers is discussed in the introduction of [DHI]. There a model category is presented for presheaves of simplicial sets on  $\mathcal{C}$  in which these  $n$ -stacks are the fibrant objects, and the analogues of 1.6 and 1.7 are proven.

For  $n = 1$ , Definition 1.8 is equivalent to 1.3, by [DHI, Theorem 1.1], and Theorems 1.4 and 5.7. This is the foundation for the recent work of Toen and Vezzosi on “homotopical algebraic geometry” (see [TV, Section 3]), following Simpson who first showed that model categories are useful in understanding the theory of higher stacks, see [HS].

**1.1. RELATION TO OTHER WORK.** In [JT], Joyal and Tierney introduce a model structure on sheaves of groupoids on a site where the fibrant objects satisfy a strengthening of the stack condition, and are called strong stacks. It follows from Proposition 5.10 that our model structures for stacks are Quillen equivalent to Joyal and Tierney’s.

The main difference of our treatment is that we show that the **descent conditions**, and hence **the classical definition of stack** can be described in terms of a natural homotopy theoretic generalization of the sheaf condition. It is this characterization which leads to a model category structure where the fibrant objects are precisely the stacks. In addition, our construction of the model structure draws a precise connection between the classical theory of stacks and the homotopy theory of simplicial presheaves.

**1.2. CONTENTS.** In section 2 we define the model structure on groupoids, and prove that it is Quillen equivalent to a localization of simplicial sets with respect to the map  $S^2 \rightarrow *$ , called the  $S^2$  nullification of  $sSet$ . We then present formulas for homotopy limits and colimits in  $\mathcal{G}rpd$  and prove that the descent category is a model for the homotopy inverse limit of a cosimplicial diagram of groupoids.

In Section 3, we review the definition of categories fibered in groupoids over a fixed base category  $\mathcal{C}$ , denoted  $\mathcal{G}rpd/\mathcal{C}$ . We construct an adjoint pair of functors between  $\mathcal{G}rpd/\mathcal{C}$  and the category of presheaves of groupoids on  $\mathcal{C}$ . We show that these functors send the subcategory of stacks in  $\mathcal{G}rpd/\mathcal{C}$  to the subcategory of stacks in  $P(\mathcal{C}, \mathcal{G}rpd)$ . Using this adjoint pair we prove Theorem 1.1.

In Section 4, we put model structures on  $\mathcal{G}rpd/\mathcal{C}$  and  $P(\mathcal{C}, \mathcal{G}rpd)$ . In the basic model structure on each of these categories the weak equivalences are defined to be objectwise (or fiberwise). We note that the adjoint pair  $(p, \Gamma)$  between these categories is a Quillen equivalence. We also observe that these model structures

can be localized with respect to the local equivalences  $\{\text{holim } U_\bullet \rightarrow X\}$ . In these local model structures the fibrant objects are the stacks, and the adjoint pair  $(p, \Gamma)$  is still a Quillen equivalence. This proves most of Theorem 1.4.

In Section 5, we define the local lifting conditions and prove Proposition 1.6. Finally, we show that there is a local model structure on sheaves of groupoids on  $\mathcal{C}$  for which  $(sh, i)$  is a Quillen equivalence, which completes the proof of Theorem 1.4.

Appendix A contains proof that limits and colimits exist in the category  $\mathcal{G}rpd/\mathcal{C}$  of categories fibered in groupoids, which is needed to show that one can put a model structure on  $\mathcal{G}rpd/\mathcal{C}$ .

Appendix B contains a proposition about pushouts of categories needed for the proof of the left properness of the model structure on  $\mathcal{G}rpd$  and for that on  $\mathcal{G}rpd/\mathcal{C}$ .

1.3. NOTATION AND ASSUMPTIONS. For a general introduction to Grothendieck topologies, see [Ta], [MM]. So as not to run into set theoretic problems, we assume that the Grothendieck topology  $\mathcal{C}$  is a small category.

We write  $Pre(\mathcal{C})$  for the category of presheaves of sets on  $\mathcal{C}$ . For  $\{U_i \rightarrow X\}$  a cover in  $\mathcal{C}$ , and  $F$  a presheaf on  $\mathcal{C}$ , let

- the  $n + 1$ -fold product  $U \times_X U \times_X \cdots \times_X U$  denote the coproduct  $\coprod U_{i_0} \times_X \cdots \times_X U_{i_n}$ , of the representable functors in  $Pre(\mathcal{C})$ , where the coproduct is taken over all multi-indices  $(i_0, \dots, i_n)$ .
- $U_\bullet$  denote the simplicial diagram in  $Pre(\mathcal{C})$  with  $(U_\bullet)_n$  equal to the  $n + 1$ -fold product  $U \times_X \cdots \times_X U$ , with face and boundary maps defined by the various projection and diagonal maps. This is called the **nerve** of the cover  $\{U_i \rightarrow X\}$ .
- $F(U \times_X U \times_X \cdots \times_X U)$  denote the product  $\prod F(U_{i_0} \times_X \cdots \times_X U_{i_n})$ , and  $F(U_\bullet)$  the cosimplicial diagram  $\text{Hom}_{Pre(\mathcal{C})}(U_\bullet, F)$ .

For model categories and their localizations our references are [DS], [Ho], [Hi].

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**2. Homotopy limits and colimits of groupoids**

We discuss here a model structure on the category of groupoids, denoted  $\mathcal{G}rpd$  and prove some results about homotopy limits and colimits which will enable us in the next section to interpret the descent conditions in a homotopy theoretic manner and prove Theorem 1.1. The proofs of the results in this section are not hard. For more details the reader is referred to [Holl].

2.1. MODEL STRUCTURE ON  $\mathcal{G}rpd$ . The model structure we discuss on  $\mathcal{G}rpd$  is derived from that on  $s\mathcal{S}et$  via the adjoint pair  $(\pi_{oid}, N)$ , where  $\pi_{oid}$  denotes the fundamental groupoid and  $N$  the nerve construction. Recall,  $\pi_{oid}X$  is the groupoid with objects  $X_0$ , and morphisms freely generated by  $X_1$ , subject to the relations  $s_0a = id_a$  for each  $a \in X_0$ , and  $d_2x \circ d_0x = d_1x$  for each  $x \in X_2$ . It follows that all morphisms in  $\pi_{oid}X$  are isomorphisms.

We will sometimes abuse notation and denote the groupoid  $\pi_{oid}(\Delta^i)$  by  $\Delta^i$ . This is the groupoid with  $i + 1$  objects and unique isomorphisms between them. Similarly, we will sometimes denote  $\pi_{oid}(\partial\Delta^i)$  by  $\partial\Delta^i$ . Let  $BG$  denote the groupoid with one object whose automorphism group is the group  $G$ .

Note that the morphisms  $\partial\Delta^i \rightarrow \Delta^i, i = 0, 1, 2$ , are

$$\emptyset \rightarrow \star, \quad \{\star, \star\} \rightarrow \Delta^1, \quad \Delta^2 \times (B\mathbb{Z} \rightarrow \star).$$

**THEOREM 2.1:** *There is a left proper, simplicial, cofibrantly generated model category structure on  $\mathcal{G}rpd$  in which:*

- *weak equivalences are functors which induce an equivalence of categories,*
- *fibrations are the functors with the right lifting property with respect to the map  $\Delta^0 \rightarrow \Delta^1$ ,*
- *cofibrations are functors which are injections on objects.*

*The generating trivial cofibration is the morphism  $\Delta^0 \rightarrow \Delta^1$ , and the generating cofibrations are the morphisms  $\partial\Delta^i \rightarrow \Delta^i, i = 0, 1, 2$ .*

**Note 2.2:** In this model category structure all objects are both fibrant and cofibrant, so all weak equivalences are homotopy equivalences.

This model category structure appears in [An, Bo], a detailed description and proof can be found in [St], Section 6. The fact that it is simplicial and

cofibrantly generated is easy to check, and the left properness follows from the fact that all objects are cofibrant, see B.1.

COROLLARY 2.3: *With this model structure on  $\mathit{Grpd}$ , the adjoint pair*

$$\pi_{oid}: s\mathit{Set} \leftrightarrow \mathit{Grpd}: N$$

*is a Quillen pair.*

The following are important observations which we will use freely.

LEMMA 2.4: *Let  $G \xrightarrow{f} H$  be a map of groupoids. The following are equivalent:*

- *$f$  is a weak equivalence in  $\mathit{Grpd}$ ;*
- *$Nf$  is weak equivalence in  $s\mathit{Set}$ .*

*Similarly, the following are equivalent:*

- *$f$  is a (trivial) fibration in  $\mathit{Grpd}$ ;*
- *$Nf$  is a (trivial) fibration in  $s\mathit{Set}$ ;*
- *$f$  has the right lifting property with respect to  $\Delta^0 \rightarrow \Delta^1$  (with respect to  $\partial\Delta^n \rightarrow \Delta^n$  for  $n = 0, 1, 2$ ).*

Consider the model structure on  $s\mathit{Set}$  which is the localization [Hi, Definition 3.3.1.1] of the usual model structure with respect to the map  $\partial\Delta^3 \rightarrow \Delta^3$ . We will call this the  $S^2$ -**nullification** of  $s\mathit{Set}$  [DF], denoted  $(S^2)^{-1}s\mathit{Set}$ . Notice that the maps  $\partial\Delta^n \rightarrow \Delta^n, n > 2$ , are all weak equivalences in  $(S^2)^{-1}s\mathit{Set}$ , and so  $(S^2)^{-1}s\mathit{Set}$  is also the localization of  $s\mathit{Set}$  with respect to this set of maps.

LEMMA 2.5: *In  $(S^2)^{-1}s\mathit{Set}$ , weak equivalences are the maps which induce an isomorphism on  $\pi_0$  and  $\pi_1$  at all base points.*

THEOREM 2.6: *The adjoint pair  $\pi_{oid}: (S^2)^{-1}s\mathit{Set} \leftrightarrow \mathit{Grpd}: N$  is a Quillen equivalence.*

2.2. HOMOTOPY LIMITS AND COLIMITS. In [Hi, 18.1.2, 18.1.8, 18.5.3], explicit constructions are given of homotopy limit and colimit functors in arbitrary simplicial model categories. Explicit formulas for homotopy limits and colimits in simplicial sets go back to [BK, Section XI.4]. Here we will give simplified formulas for homotopy limits and colimits in case the simplicial structure on the category derives from an enrichment with tensor and cotensor (see [Db]) over  $\mathit{Grpd}$ .



*Definition 2.7:* Let  $M$  be a category enriched with tensor and cotensor over  $sSet$ . We say that the simplicial structure on  $M$  **derives from an enrichment over  $Grpd$** , if  $M$  is enriched with tensor and cotensor over  $Grpd$ , and if for all  $A, B \in M, K \in sSet$ , there are natural isomorphisms

$$sSet(A, B) \cong N(Grpd(A, B)), \quad A^K \cong A^{\pi_{oid}K}, \quad A \otimes K \cong A \otimes \pi_{oid}K.$$

compatible with the natural isomorphism for each pair of simplicial sets  $\pi_{oid}(K \times K') \cong \pi_{oid}K \times \pi_{oid}K'$ .

Our main concern will be the homotopy limit of a cosimplicial diagram, and dually the homotopy colimit of a simplicial diagram. Our simplified formula for the former will allow us in the next section to interpret the descent conditions for stacks in a homotopy-theoretic manner.

Let  $\mathcal{C}$  be a simplicial model category, and  $I$  a small category. The homotopy limit of an  $I$ -diagram  $X$  in  $\mathcal{C}$  with each  $X(i)$  fibrant is the equalizer of the two natural maps

$$\prod_i X(i)^{N(I/i)} \rightrightarrows \prod_{j \xrightarrow{\alpha} i} X(i)^{N(I/j)},$$

where  $I/i$  denotes the category of objects over  $i$ . Similarly, the homotopy colimit of an objectwise cofibrant  $I$ -diagram  $X$  is the coequalizer of the two maps

$$\prod_{i \xrightarrow{\alpha} j} X(i) \otimes N(j/I) \rightrightarrows \prod_i X(i) \otimes N(i/I),$$

where  $j/I$  denotes the category of objects under  $j$ .

For  $Y$  a fibrant object and  $\mathbf{X} \in \mathcal{C}^I$  objectwise cofibrant, these functors satisfy the equation

$$(2.8) \quad sSet(\text{hocolim } \mathbf{X}, Y) \cong \text{holim } sSet(\mathbf{X}, Y).$$

**THEOREM 2.9:** *Let  $\mathcal{C}$  be a simplicial model category whose simplicial structure derives from an enrichment over  $Grpd$ , and let  $X^\bullet$  be a cosimplicial object in  $\mathcal{C}$ , with each  $X^i$  fibrant. Then a model for the homotopy inverse limit of  $X^\bullet$  is given by the equalizer of the natural maps*

$$\prod_{i=0}^2 (X^i)^{\Delta^i} \rightrightarrows \prod_{\substack{i \leq 2, j \leq 1 \\ [j] \rightarrow [i]}} (X^i)^{\Delta^j}.$$

*Proof.* First, notice that writing  $\text{sk}_2 \Delta^\bullet$  for the 2-skeleton of  $\Delta^\bullet$ , the inclusion  $\pi_{oid} \text{sk}_2 \Delta^\bullet \rightarrow \pi_{oid} \Delta^\bullet$  is an isomorphism. It follows that  $\text{Tot } X^\bullet$ , the space

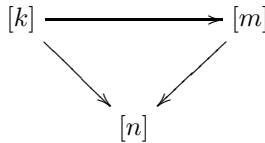
of maps from  $\Delta^\bullet$  to  $X^\bullet$ , is isomorphic to  $\text{Tot}_2 X^\bullet$ , the space of maps from the restriction  $\Delta^\bullet|_{\Delta[2]}$  to  $X^\bullet|_{\Delta[2]}$  where  $\Delta[2]$  denotes the full subcategory of  $\Delta$  with objects  $[0], [1]$  and  $[2]$ . Since the map  $\pi_{oid} \text{sk}_1 \Delta^2 \rightarrow \pi_{oid} \Delta^2$  is surjective,  $\text{Tot}_2 X^\bullet$  is given by the equalizer in the statement of the theorem.

It now suffices to show that the homotopy limit of  $X^\bullet$  is naturally homotopy equivalent to  $\text{Tot } X^\bullet$ . Using the definition of the homotopy limit in a simplicial model category given above, this is an easy consequence of the following Proposition. ■

PROPOSITION 2.10: *There is a homotopy equivalence of cosimplicial groupoids*

$$F: \pi_{oid}N(\Delta/[\bullet]) \leftrightarrow \pi_{oid}\Delta^\bullet: G.$$

*Proof.* Morphisms in  $\pi_{oid}N(\Delta/[n])$  are generated by the commutative triangles



and their formal inverses. Since the spaces  $N(\Delta/[n])$  and  $\Delta^n$  are contractible in their fundamental groupoids there is a unique isomorphism between any two objects. Therefore when defining functors between them it suffices to set the values on objects.

Let  $\pi_{oid}N(\Delta/[n]) \xrightarrow{F_n} \pi_{oid}\Delta^n$  be the functor which sends the object  $[m] \rightarrow [n]$  to the vertex  $[0] \xrightarrow{e_m} [m] \rightarrow [n]$ , where  $e_k: [0] \rightarrow [k]$  sends 0 to  $k$ . One can check easily that  $F$  is natural in  $n$ , and so defines a morphism

$$\pi_{oid}N(\Delta/[\bullet]) \xrightarrow{F} \pi_{oid}\Delta^\bullet \in \mathcal{C}\mathcal{G}rpd.$$

Let  $G_n$  be the functor which is defined on objects by including  $[0] \rightarrow [n]$  in  $\Delta/[n]$ . Again it is easy to check that  $G_n$  is natural in  $[n]$ , and so defines a morphism  $\pi_{oid}\Delta^\bullet \xrightarrow{G} \pi_{oid}N(\Delta/[\bullet])$  in  $\mathcal{C}\mathcal{G}rpd$ .

The composition  $F \circ G$  is the identity. There are unique natural transformations  $G_n \circ F_n \xrightarrow{\alpha_n} id$  which must commute with the simplicial operations. ■

The groupoid  $\text{Tot}_2(X^\bullet)$  will also be called the **descent category** of  $X^\bullet$ . From now on, when we refer to the homotopy limit of a cosimplicial groupoid  $X^\bullet$  we will mean the simpler model  $\text{Tot}_2(X^\bullet)$ . The following corollary gives an explicit description of this groupoid.

COROLLARY 2.11: *The homotopy inverse limit of a cosimplicial groupoid  $X^\bullet$  is the groupoid whose*

- *objects are pairs  $(a, d^1(a) \xrightarrow{\alpha} d^0(a))$ , with  $a \in obX^0, \alpha \in morX^1$ , such that  $s^0(\alpha) = id_a$ , and  $d^0(\alpha) \circ d^2(\alpha) = d^1(\alpha)$ ,*
- *morphisms  $(a, \alpha) \rightarrow (a', \alpha')$  are maps  $a \xrightarrow{\beta} a'$ , such that the following diagram commutes*

$$\begin{array}{ccc}
 d^1(a) & \xrightarrow{d^1(\beta)} & d^1(a') \\
 \downarrow \alpha & & \downarrow \alpha' \\
 d^0(a) & \xrightarrow{d^0(\beta)} & d^0(a')
 \end{array}$$

Dually we have the following theorem giving a formula for homotopy colimits of simplicial diagrams.

THEOREM 2.12: *Let  $\mathcal{C}$  be a simplicial model category whose simplicial structure derives from a groupoid action and let  $X_\bullet \in s\mathcal{C}$ , be such that each  $X_i$  is cofibrant. Then the homotopy colimit of  $X_\bullet$  is naturally homotopy equivalent to the coequalizer of the maps*

$$\coprod_{\substack{n \leq 1, m \leq 2 \\ [n] \rightarrow [m]}} X_m \otimes \Delta^n \rightrightarrows \coprod_{n=0}^2 X_n \otimes \Delta^n.$$

### 3. Stacks

There are different categories in which the descent condition can be formulated and so in which stacks can be defined. In this section we will discuss stacks in the context of categories fibered in groupoids over  $\mathcal{C}$ , denoted  $Grpd/\mathcal{C}$  [DM, Gi].

After discussing some important properties of  $Grpd/\mathcal{C}$ , we will construct an adjoint pair  $p: P(\mathcal{C}, Grpd) \leftrightarrow Grpd/\mathcal{C}: \Gamma$ , satisfying the following properties:

- For  $F$  in  $P(\mathcal{C}, Grpd)$ , the map  $F(X) \rightarrow \Gamma pF(X)$  is an equivalence of groupoids, for all  $X \in \mathcal{C}$ .
- For  $\mathcal{E} \in Grpd/\mathcal{C}$ , the map  $p\Gamma\mathcal{E} \rightarrow \mathcal{E}$  is an equivalence of categories over  $\mathcal{C}$ .

In the following subsection we will discuss the classical definition of stacks in  $Grpd/\mathcal{C}$  used in algebraic geometry [DM]. We show that it can be reformulated

in terms of homotopy limits of groupoids, expressing stacks as those objects satisfying the **homotopy sheaf condition**.

3.1. CATEGORIES FIBERED IN GROUPOIDS OVER  $\mathcal{C}$ .

*Definition 3.1* ([DM]): The category  $\mathcal{G}rpd/\mathcal{C}$  is the full subcategory of  $\mathcal{C}at/\mathcal{C}$  whose objects are functors  $\mathcal{E} \xrightarrow{F} \mathcal{C}$  satisfying the following properties:

- (1) Given  $Y \xrightarrow{f} X \in \mathcal{C}$ , and  $X' \in \mathcal{E}$  such that  $F(X') = X$ , there exists  $Y' \xrightarrow{f'} X' \in \mathcal{E}$  such that  $F(f') = f$ .
- (2) Given a diagram in  $\mathcal{E}$ , over the commutative diagram in  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & Y' & \\
 & \downarrow f' & \\
 Z' & \xrightarrow{g'} & X'
 \end{array} & \xrightarrow{F} & \begin{array}{ccc}
 & Y & \\
 & \downarrow f & \\
 Z & \xrightarrow{g} & X,
 \end{array}
 \end{array}$$

with  $F(f') = f, F(g') = g$ , there exists a unique  $h'$  such that  $g' \circ h' = f'$  and  $F(h') = h$ .

This definition may seem involved but it becomes very simple when we look at the functors  $F_{X'}$  induced by  $F$  on the over categories

$$\mathcal{E}/X' \xrightarrow{F_{X'}} \mathcal{C}/X,$$

where  $X' \in \mathcal{E}$ , and  $F(X') = X$ . The conditions for  $\mathcal{E} \xrightarrow{F} \mathcal{C}$  to be a category fibered in groupoids over  $\mathcal{C}$  are equivalent to the simple requirement that each functor  $F_{X'}$  induce a surjective equivalence of categories.

Let  $\mathcal{E}_X$  denote the fiber category over  $X$ . This has objects those of  $\mathcal{E}$  lying over  $X$  and morphisms those of  $\mathcal{E}$  lying over  $id_X$ . It is easy to see that if  $\mathcal{E} \rightarrow \mathcal{C} \in \mathcal{G}rpd/\mathcal{C}$ , the fiber categories  $\mathcal{E}_X$  are groupoids.

*Example 3.2:* The simplest examples of categories fibered in groupoids over  $\mathcal{C}$  are the projection functors  $\mathcal{C}/X \rightarrow \mathcal{C}$  for each  $X \in \mathcal{C}$ . If  $Y \xrightarrow{f} X$  is an object of  $\mathcal{C}/X$ , then  $(\mathcal{C}/X)/f \cong \mathcal{C}/Y$ , and so conditions 1. and 2. above are trivially satisfied. Notice that  $(\mathcal{C}/X)_Y$  is the discrete groupoid whose set of objects is  $\text{Hom}_{\mathcal{C}}(Y, X)$ .

Another class of simple examples is  $\mathcal{C} \times G \xrightarrow{pr} \mathcal{C}$ , for  $G \in \mathcal{G}rpd$ . Here the fibers over each  $X \in \mathcal{C}$  are canonically isomorphic to  $G$ .

The proofs of the following results are straightforward.

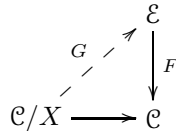
LEMMA 3.3:  $\mathit{Grpd}/\mathcal{C}$  is enriched with tensor and cotensor over  $\mathit{Grpd}$ . The objects of  $\mathit{Grpd}(\mathcal{E}, \mathcal{E}')$  are the functors  $\mathcal{E} \rightarrow \mathcal{E}'$  over  $\mathcal{C}$ , and the morphisms are the natural isomorphisms between such functors covering the identity natural automorphism of  $id_{\mathcal{C}}$ . Moreover, the tensor is given by the formula

$$\mathcal{E} \otimes G := \mathcal{E} \times_{\mathcal{C}} (\mathcal{C} \times G),$$

and the cotensor  $\mathcal{E}^G$  is the category of functors from  $(G \rightarrow *)$  to  $(\mathcal{E} \rightarrow \mathcal{C})$ .

PROPOSITION 3.4: Let  $\mathcal{E} \xrightarrow{F} \mathcal{C} \in \mathit{Grpd}/\mathcal{C}$ , and  $X \in \mathcal{C}$ , then

- (1) for any  $X' \in \mathcal{E}$  with  $F(X') = X$  there is a section



such that  $G(id_X) = X'$ .

- (2) If  $G, G': \mathcal{C}/X \rightarrow \mathcal{E}$  are two sections and  $G(id_X) \xrightarrow{f} G'(id_X)$  is a morphism  $\mathcal{E}_X$ , then there is a unique natural isomorphism  $G \xrightarrow{\phi} G'$  over  $id_{\mathcal{C}}$ , with  $\phi(id_X) = f$ .

It follows that, for each  $X \in \mathcal{C}$ , the natural map

$$\mathit{Grpd}(\mathcal{C}/X, \mathcal{E}) \rightarrow \mathcal{E}_X$$

given by evaluation at  $id_X$  is a surjective equivalence of groupoids. There is a left inverse which is unique up to unique natural isomorphism.

This says that given  $\mathcal{E} \rightarrow \mathcal{C}$  there is a functorial “rigidification” of the fibers. Later we will use this method of rigidification to construct a functor from  $\mathit{Grpd}/\mathcal{C}$  to  $P(\mathcal{C}, \mathit{Grpd})$ .

The following observation, which can be proven in a similar fashion, will be used in the next subsection.

PROPOSITION 3.5: Let  $\mathcal{E} \rightarrow \mathcal{C}$  be a category fibered in groupoids, and  $Y \xrightarrow{f} X$  morphism in  $\mathcal{C}$ . There are “pullback” functors  $\mathcal{E}_X \xrightarrow{f^*} \mathcal{E}_Y$  which are unique up to a unique natural isomorphism.

3.2. STACKS. Let  $\mathcal{E} \rightarrow \mathcal{C}$  be a category fibered in groupoids, and assume that for each  $X \xrightarrow{f} Y$  we have chosen pullback functors  $\mathcal{E}_Y \xrightarrow{f^*} \mathcal{E}_X$ .

Given a morphism  $U_i \rightarrow U \in \mathcal{C}$ , we will sometimes abuse notation and denote the pullback of an element  $a \in \mathcal{E}_U$  to  $\mathcal{E}_{U_i}$  by  $a|_{U_i}$ . In defining some of the maps below, we will also make implicit use of the natural isomorphisms  $(a|_{U_i})|_{U_{ij}} \cong a|_{U_{ij}}$ .

*Definition 3.6* ([Gi],[DM]): A **stack** in  $\mathit{Grpd}/\mathcal{C}$  is an object  $\mathcal{E} \rightarrow \mathcal{C}$  which satisfies the following properties for any cover  $\{U_i \rightarrow X\}$  :

- (1) given  $a, b \in \mathcal{E}_X$ , the following is an equalizer sequence

$$\mathrm{Hom}_{\mathcal{E}_X}(a, b) \rightarrow \prod \mathrm{Hom}_{\mathcal{E}_{U_i}}(a|_{U_i}, b|_{U_i}) \rightrightarrows \prod \mathrm{Hom}_{\mathcal{E}_{U_{ij}}}(a|_{U_{ij}}, b|_{U_{ij}}),$$

- (2) given  $a_i \in \mathcal{E}_{U_i}$  and isomorphisms

$$a_i|_{U_{ij}} \xrightarrow{\alpha_{ij}} a_j|_{U_{ij}},$$

satisfying the **cocycle condition**

$$\alpha_{jk}|_{U_{ijk}} \circ \alpha_{ij}|_{U_{ijk}} = \alpha_{ik}|_{U_{ijk}},$$

then there exist  $a \in \mathcal{E}_X$ , and isomorphisms  $a|_{U_i} \xrightarrow{\beta_i} a_i$ , such that the following square commutes

$$(3.7) \quad \begin{array}{ccc} a|_{U_{ij}} & \xrightarrow{\beta_i|_{U_{ij}}} & a_i|_{U_{ij}} \\ = \downarrow & & \downarrow \alpha_{ij} \\ a|_{U_{ij}} & \xrightarrow{\beta_j|_{U_{ij}}} & a_j|_{U_{ij}}. \end{array}$$

In this case, we say that  $\mathcal{E} \rightarrow \mathcal{C}$  **satisfies descent**.

*Note 3.8:* The cocycle condition applied to indices  $(i, i, j)$  implies that  $\alpha_{ii}|_{U_{ii}} = id$  which implies that  $\alpha_{ii}$  is itself the identity by (1) of 3.6.

This definition seems very complicated, but it can be considerably simplified if we recall the description of the homotopy inverse limit of a cosimplicial groupoid given in Corollary 2.11.

**THEOREM 3.9** (Theorem 1.1): *A category fibered in groupoids  $\mathcal{E} \rightarrow \mathcal{C}$  is a stack if and only if for all covers  $\{U_i \rightarrow X\}$*

$$(3.10) \quad \mathit{Grpd}(\mathcal{C}/X, \mathcal{E}) \rightarrow \mathrm{holim} \mathit{Grpd}(\mathcal{C}/U_\bullet, \mathcal{E})$$

*is an equivalence.*

*Proof.* We begin by showing that condition (1) in Definition 3.6 is equivalent to the requirement that for objects  $F_a, F_b \in \mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$ , the set of morphisms  $F_a \rightarrow F_b$  is in bijective correspondence with the set of morphisms between their images in  $\text{holim } \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$ .

Consider objects  $F_a, F_b \in \mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$ , and let  $a = F_a(id_X)$  and  $b = F_b(id_X)$  in  $\mathcal{E}_X$ . Evaluation at  $id_{(-)}$  induces bijections

$$\begin{array}{ccccc} \text{Hom}(F_a, F_b) & \longrightarrow & \prod \text{Hom}(F_a|_{U_i}, F_b|_{U_i}) & \Longrightarrow & \prod \text{Hom}(F_a|_{U_{ij}}, F_b|_{U_{ij}}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \text{Hom}_{\mathcal{E}_X}(a, b) & \longrightarrow & \prod \text{Hom}_{\mathcal{E}_{U_i}}(a|_{U_i}, b|_{U_i}) & \Longrightarrow & \prod \text{Hom}_{\mathcal{E}_{U_{ij}}}(a|_{U_{ij}}, b|_{U_{ij}}). \end{array}$$

It follows that the top line is an equalizer if and only if the bottom one is. By corollary 2.11, the top line is an equalizer if and only if  $\text{Hom}(F_a, F_b)$  is in bijective correspondence with the set of maps from the image of  $F_a$  to the image of  $F_b$  in  $\text{holim } \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$ . The requirement that the bottom line be an equalizer is condition (1) in Definition 3.6.

To finish the proof we have to show that condition (2) is equivalent to the requirement that every object in  $\text{holim } \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$  be isomorphic to one in the image of  $\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$ . This follows from the description of morphisms in Corollary 2.11 once we show that specifying an object in  $\text{holim } \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$  is equivalent to specifying descent datum as in condition (2) of Definition 3.6.

By corollary 2.11, an object of  $\text{holim } \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$ , consists of an object  $F_c \in \prod \mathcal{G}rpd(\mathcal{C}/U_i, \mathcal{E})$ , and an isomorphism  $d^1 F_c \xrightarrow{\alpha} d^0 F_c$ , satisfying

$$d^0(\alpha) \circ d^2(\alpha) = d^1(\alpha) \quad \text{and} \quad s^0(\alpha) = id_{F_c}.$$

For any  $U \xrightarrow{f} V$ , and  $F_a \in \mathcal{G}rpd(\mathcal{C}/V, \mathcal{E})$  with  $F_a(id_V) = a$ , the evaluation  $F_a|_U(id_U)$  is a choice of pullback of  $a$  along  $f$ , and so  $F_a|_U(id_U)$  is canonically isomorphic to the pullback  $f^*a$ , which we chose in advance. Evaluating at  $id_{U_i}$  determines  $c \in \prod \mathcal{E}_{U_i}$ , and isomorphisms  $\alpha_{ij} = \alpha(id_{U_{ij}})$  satisfying the cocycle condition. Composing with the canonical isomorphisms  $c|_{U_{ij}} \cong F_c|_{U_{ij}}(id_{U_{ij}})$ , we obtain isomorphisms  $c|_{U_i} \xrightarrow{\alpha_{ij}} c|_{U_j}$ , satisfying the cocycle condition.

Conversely, given  $c \in \prod \mathcal{E}_{U_i}$  and  $\alpha_{ij}$ , as in condition (2) satisfying  $\Delta^*(\alpha_{ii}) = id_{U_i}$  (see Note 3.8), we can lift them to an object  $F_c \in \prod \mathcal{G}rpd(\mathcal{C}/U_i, \mathcal{E})$ , and an isomorphism  $d^1 F_c \xrightarrow{\alpha} d^0 F_c$ . Since these lifts are essentially unique they must also satisfy the cocycle condition and  $s^0(\alpha) = id_{F_c}$  and hence determine an object of  $\text{holim } \mathcal{G}rpd(\mathcal{C}/U_\bullet, \mathcal{E})$ . ■

3.3. ADJOINT PAIR BETWEEN  $\mathcal{G}rpd/\mathcal{C}$  AND  $P(\mathcal{C}, \mathcal{G}rpd)$ . Let  $\mathcal{E} \rightarrow \mathcal{C}$  be a category fibered in groupoids. By Corollary 3.4, the assignment to each  $X \in \mathcal{C}$  of the sections  $\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E})$  is a functor such that  $\mathcal{G}rpd(\mathcal{C}/X, \mathcal{E}) \xrightarrow{\sim} \mathcal{E}_X$ .

*Definition 3.11:* Let  $\Gamma: \mathcal{G}rpd/\mathcal{C} \rightarrow P(\mathcal{C}, \mathcal{G}rpd)$  be the functor which sends  $\mathcal{E} \rightarrow \mathcal{C}$  to the presheaf  $\Gamma\mathcal{E}(X) := \mathcal{G}rpd_{\mathcal{G}rpd/\mathcal{C}}(\mathcal{C}/X, \mathcal{E})$ .

Let  $p: P(\mathcal{C}, \mathcal{G}rpd) \rightarrow \mathcal{G}rpd/\mathcal{C}$  be the functor defined by setting  $pF$  to be the category whose

- objects are pairs  $(X, a)$  with  $a \in F(X)$ ,
- morphisms  $(X, a) \rightarrow (Y, b)$  are pairs  $(f, \alpha)$  where  $X \xrightarrow{f} Y \in \mathcal{C}$  and  $a \xrightarrow{\alpha} F(f)(b)$  is an isomorphism in  $F(X)$ .

The composition of two morphisms  $(X, a) \xrightarrow{(f, \alpha)} (Y, b) \xrightarrow{(g, \beta)} (Z, c)$  is the pair  $(g \circ f, F(f)(\beta) \circ \alpha)$ .

It is easy to check that both  $p$  and  $\Gamma$  preserve the enrichment over  $\mathcal{G}rpd$  with tensor and cotensor.

**THEOREM 3.12:** *The functors  $p: P(\mathcal{C}, \mathcal{G}rpd) \leftrightarrow \mathcal{G}rpd/\mathcal{C} : \Gamma$  form an adjoint pair. The unit of the adjunction is an objectwise equivalence, and the counit is a fiberwise equivalence of groupoids.*

This adjoint pair restricts to an adjunction between the subcategory of stacks and  $\mathcal{G}rpd/\mathcal{C}$  and the subcategory of presheaves  $P \in P(\mathcal{C}, \mathcal{G}rpd)$  which satisfy the following condition:

- For any cover  $\{U_i \rightarrow X\}$  in  $\mathcal{C}$  the induced map  $P(X) \rightarrow \text{holim } P(U_\bullet)$  is an equivalence of groupoids.

With this motivation we will also call a presheaf of groupoids satisfying the above condition a stack, (Definition 1.3).

#### 4. Model structures

In this section we put model structures on  $P(\mathcal{C}, \mathcal{G}rpd)$  and  $\mathcal{G}rpd/\mathcal{C}$ . We first construct basic model structures, then we localize them so that the local weak equivalences detect the topology on  $\mathcal{C}$ . We then observe that in these **local** model structures, the fibrant objects are the stacks, and the weak equivalences are the maps which locally are an equivalence of groupoids. Isomorphisms between sheaves can be detected locally and this property characterizes the



subcategory of sheaves. It follows from our analysis that analogously global equivalences between stacks can be detected locally, and this property is a characterization of stacks.

We will use the notation of [Hi, Definition 7.1.3, 9.1.5] for the model category axioms.

4.1. THE BASIC MODEL CATEGORY STRUCTURES. Henceforth we will abuse notation and denote by  $X$  the representable functor  $\text{Hom}_{\mathcal{C}}(-, X)$  considered as a discrete groupoid.

PROPOSITION 4.1: *There is a left proper, cofibrantly generated, model category structures on  $P(\mathcal{C}, \text{Grpd})$ , where*

- $f$  is a weak equivalence or a fibration if  $\text{Grpd}(X, f)$  is one for all  $X \in \mathcal{C}$ ,
- cofibrations are the maps with the left lifting property with respect to trivial fibrations.

The maps of the form  $X \rightarrow X \otimes \Delta^1$ , for  $X \in \mathcal{C}$ , form a set of generating trivial cofibrations. The maps of the form  $X \otimes \partial\Delta^i \rightarrow X \otimes \Delta^i$  for  $X \in \mathcal{C}$  and  $i = 0, 1, 2$  form a set of generating cofibrations.

*Proof.* The proof is an easy exercise. ■

Now we construct a model category on  $\text{Grpd}/\mathcal{C}$  relative using the set of generators  $\mathcal{C}/X \rightarrow \mathcal{C}$ .

THEOREM 4.2: *There is a left proper, cofibrantly generated, simplicial model category structure on  $\text{Grpd}/\mathcal{C}$  in which*

- $f$  is a weak equivalence or a fibration if  $\text{Grpd}_{\text{Grpd}/\mathcal{C}}(\mathcal{C}/X, f)$  is one for all  $X \in \mathcal{C}$ ,
- cofibrations are the maps with the left lifting property with respect to trivial fibrations.

The maps of the form  $\mathcal{C}/X \rightarrow (\mathcal{C}/X \otimes \Delta^1)$ , for  $X \in \mathcal{C}$ , form a set of generating trivial cofibrations. The maps of the form  $(\mathcal{C}/X \otimes \partial\Delta^i) \rightarrow (\mathcal{C}/X \otimes \Delta^i)$ , for  $X \in \mathcal{C}$  and  $i = 0, 1, 2$  form a set of generating cofibrations.

*Proof.* For M1, see Appendix A. M2–M4(1) are obvious. In order to apply the small object argument to prove M5, we need to check that the objects  $\mathcal{C}/X \otimes G \rightarrow \mathcal{C}$  with  $G = (\partial)\Delta^i, i = 0, 1, 2$ , are small with respect to the colimits which appear in the small object argument. First notice that sequential colimits

in  $\mathit{Grpd}/\mathcal{C}$  agree with sequential colimits in  $\mathit{Cat}/\mathcal{C}$ . For convenience, in the construction of the factorization for M5(1) we will take pushouts along both the generating cofibrations and the generating trivial cofibrations.

Let  $\mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$  be constructed as usual, using the small object argument, and let consider a map  $F : \mathcal{C}/X \rightarrow \mathit{colim} \mathcal{E}_i$ .  $F(id_X)$  lifts to some element  $X'$  in some  $\mathcal{E}_i$ , and we can extend this to a map  $F'_i : \mathcal{C}/X \rightarrow \mathcal{E}_i$ . Let  $F'$  be the composition  $\mathcal{C}/X \rightarrow \mathcal{E}_i \rightarrow \mathit{colim} \mathcal{E}_i$ . Then  $F'(id_X) = F(id_X)$ , and so there is a unique natural isomorphism  $\phi : F \rightarrow F'$  making the following diagram commute

$$\begin{array}{ccc}
 \mathcal{C}/X & \xrightarrow{F'_i} & \mathcal{E}_i \\
 \downarrow & \searrow^{F'} & \downarrow \\
 \mathcal{C}/X & \xrightarrow{\phi} & \mathit{colim} \mathcal{E}_i \\
 \uparrow \scriptstyle F & \swarrow & \\
 \mathcal{C}/X & \xrightarrow{\quad} & \mathcal{C}/X \otimes \Delta^1
 \end{array}$$

The map  $\mathcal{C}/X \rightarrow \mathcal{C}/X \otimes \Delta^1$  is one of the generating trivial cofibrations, so by construction we obtain a lift

$$\begin{array}{ccc}
 \mathcal{C}/X & \xrightarrow{F'_i} & \mathcal{E}_i \\
 \downarrow & & \downarrow \\
 \mathcal{C}/X & \xrightarrow{\quad} & \mathcal{C}/X \otimes \Delta^1 \\
 \uparrow \scriptstyle F & \swarrow & \uparrow \scriptstyle \phi \\
 \mathcal{C}/X & \xrightarrow{\quad} & \mathit{colim} \mathcal{E}_i \\
 & & \downarrow \\
 & & \mathcal{E}_{i+1}
 \end{array}$$

Thus  $\mathcal{C}/X$  is small with respect  $\mathit{colim} \mathcal{E}_i$ . Since natural transformations between sections are determined uniquely by their evaluation on  $id_X$ , a similar argument shows that  $\mathcal{C}/X \otimes (\partial)\Delta^i$  is small with respect to  $\mathit{colim} \mathcal{E}_i$ . This completes the proof of M5(1).

For M5(2) use the small object argument for the generating trivial cofibrations. We need to show that the first map in the factorization is a weak equivalence. Note that if  $\mathcal{E} \rightarrow \mathcal{E}'$  has the left lifting property with respect to all fibrations, then in particular it has the left lifting property with respect to  $\mathcal{E} \rightarrow \mathcal{C}$  and  $(\mathcal{E}')^{\Delta^1} \rightarrow (\mathcal{E}')^{\partial\Delta^1}$ , and therefore it is an equivalence of categories over  $\mathcal{C}$ . An equivalence of categories over  $\mathcal{C}$  is clearly a weak equivalence. It

follows that the cofibration constructed using the small object argument for M5(2) is also a weak equivalence.

M4(2) now follows by the same argument given in the proof of Theorem 4.1. M7 follows immediately from the definition of (trivial) fibration in  $\mathit{Grpd}/\mathcal{C}$  and the adjunction formulas given by the simplicial structure.

To show left properness, it suffices to show that the pushout of a trivial fibration along a cofibration is a weak equivalence. We begin by noting that trivial fibrations are surjective equivalences of categories. Let  $F: \mathcal{E}' \rightarrow \mathcal{E}''$  be a trivial fibration and let  $X', Y' \in \mathcal{E}', X'' = F(X'), Y'' = F(Y')$ . Clearly  $F$  is surjective on objects and morphisms. We will show that the map

$$\mathrm{Hom}_{\mathcal{E}'}(X', Y') \rightarrow \mathrm{Hom}_{\mathcal{E}''}(X'', Y'')$$

is a bijection. If  $F(f') = F(g')$  then  $f'$  and  $g'$  have the same image in  $\mathcal{C}$  and so there is a unique isomorphism  $h'$  filling in the following triangle in  $\mathcal{E}'$ :

$$\begin{array}{ccc} X' & & \\ \downarrow & \searrow f' & \\ \downarrow h' & & \\ X' & \xrightarrow{g'} & Y'. \end{array}$$

By the uniqueness of the lifting  $h'$ ,  $F(h') = \mathrm{id}_{X''} \in \mathcal{E}''$ . Since  $F$  is a trivial fibration it follows that  $h' = \mathrm{id}_{X'}$ .

Now note that cofibrations in  $\mathit{Grpd}/\mathcal{C}$  are inclusions on objects as this is the case for the generating cofibrations. Proposition B.1 implies that the pushout in  $\mathrm{Cat}/\mathcal{C}$  of a surjective equivalence of categories along an inclusion on objects is still an equivalence of categories over  $\mathcal{C}$ . This simultaneously implies that the pushout in  $\mathrm{Cat}/\mathcal{C}$  coincides in this case with the pushout in  $\mathit{Grpd}/\mathcal{C}$  (see the proof of Theorem A.1) and completes the proof. ■

**COROLLARY 4.3:** *The adjoint pair  $p: P(\mathcal{C}, \mathit{Grpd}) \leftrightarrow \mathit{Grpd}/\mathcal{C}: \Gamma$  is a Quillen equivalence.*

**4.2. LOCAL MODEL CATEGORY STRUCTURES.** For convenience, we will now also denote by  $X$  the category fibered in groupoids  $\mathcal{C}/X \rightarrow \mathcal{C}$ . In the  $P(\mathcal{C}, \mathit{Grpd})$  or  $\mathit{Grpd}/\mathcal{C}$ , let  $S$  denote the set of maps

$$S = \{ \mathrm{hocolim} U_{\bullet} \rightarrow X : \{U_i \rightarrow X\} \text{ is a cover in } \mathcal{C} \}$$

where  $U_{\bullet}$  denotes the nerve of the covering  $\{U_i \rightarrow X\}$ .

PROPOSITION 4.4: *Let  $\mathcal{M}$  be  $P(\mathcal{C}, \mathit{Grpd})$  or  $\mathit{Grpd}/\mathcal{C}$ . There is a model category structure on  $\mathcal{M}$  which is the localization of the model structure of Theorems 4.1 or 4.2 with respect to the set of maps  $S$ .*

We call these weak equivalences **local weak equivalences**.

*Proof.* Since homotopy colimits of cofibrant objects are cofibrant, the domains and ranges of the morphisms in the localizing set are cofibrant. By Theorems 4.1 and 4.2, the model category structures on  $P(\mathcal{C}, \mathit{Grpd})$  and  $\mathit{Grpd}/\mathcal{C}$  satisfy the hypothesis of [Hi, Theorem 4.1.1], so the proposition follows. ■

Let  $\mathcal{M}$  be  $P(\mathcal{C}, \mathit{Grpd})$  or  $\mathit{Grpd}/\mathcal{C}$ . We will write  $\mathcal{M}_L$  for the category  $\mathcal{M}$  with the model structure given by the previous proposition.

COROLLARY 4.5: *The adjoint pair  $p: P(\mathcal{C}, \mathit{Grpd})_L \leftrightarrow (\mathit{Grpd}/\mathcal{C})_L: \Gamma$  is a Quillen equivalence.*

Since in the old model structure on  $\mathcal{M}$  every object is fibrant, and  $X \in \mathcal{C}$  is cofibrant, an object  $F \in \mathcal{M}_L$  is fibrant if and only if

$$\mathit{Grpd}(X, F) \rightarrow \mathit{Grpd}(\mathit{hocolim} U_\bullet, F) = \mathit{holim} \mathit{Grpd}(U_\bullet, F)$$

is a weak equivalence for all covers. This happens if and only if  $F$  is a stack. It follows that **a fibrant replacement functor for  $\mathcal{M}_L$  is a stackification functor**. One of the properties of localizations of model categories is that local equivalences between fibrant objects are just the old equivalences. It follows that **a local equivalence between stacks is just an objectwise weak equivalence**.

*Remark 4.6:* Since stacks are the fibrant objects, and representables are cofibrant, it follows that when  $\mathcal{M}$  is a stack,  $h\mathit{Hom}(X, \mathcal{M})$  is equivalent to the groupoid  $\mathcal{M}(X)$ . In particular,  $[X, \mathcal{M}]$  is the set of isomorphism classes of  $\mathcal{M}(X)$ .

*Remark 4.7:* It is not hard to check that a small presentation (in the sense of [Dg, Definition 6.1]) of  $P(\mathcal{C}, \mathit{Grpd})_L$  is given by the Yoneda embedding of  $\mathcal{C}$  in  $P(\mathcal{C}, \mathit{Grpd})$  and the set of maps

- $X \otimes \partial\Delta^n \rightarrow X \otimes \Delta^n$ , for all  $X \in \mathcal{C}, n > 2$ ,
- $\mathit{hocolim} U_\bullet \rightarrow X$  for all covers  $\{U_i \rightarrow X\}$  in  $\mathcal{C}$ .

This means that the local model category structure is the “quotient” of the universal model category generated by  $\mathcal{C}$  by the relations given by the maps above.

### 5. Characterization of local equivalences

In this section we prove that a morphism  $f$  is a local weak equivalence in the model structure of proposition 4.4 if and only if it satisfies one of the following equivalent properties:

- $f$  is an isomorphism on sheaves of homotopy groups;
- $f$  satisfies the local lifting conditions, (Definition 5.6);
- $f$  is a stalkwise weak equivalence (when  $\mathcal{C}$  has enough points).

It follows that a map between stacks satisfying one of the above properties is actually an objectwise equivalence.

We use this to prove that our local model structure  $P(\mathcal{C}, \mathcal{G}rpd)_L$  is Quillen equivalent to the  $S^2$ -nullification of Joyal’s model structure on presheaves of simplicial sets [Ja].

Using the characterization of local weak equivalences we also prove that there is a local model category structure on  $Sh(\mathcal{C}, \mathcal{G}rpd)$  such that the adjoint pair  $sh: P(\mathcal{C}, \mathcal{G}rpd)_L \leftrightarrow Sh(\mathcal{C}, \mathcal{G}rpd): i$  is a Quillen equivalence.

5.1. JOYAL’S MODEL STRUCTURE. For a simplicial set  $X$ , and basepoint  $a \in X_0$ ,  $\pi_n(X, a)$  denotes the  $n$ -th homotopy group of the fibrant replacement of  $X$  with basepoint the image of  $a$ .

*Definition 5.1* ([Ja]): Let  $F$  be a presheaf of simplicial sets or groupoids. Then

- $\pi_0 F$  is the presheaf of sets defined by  $(\pi_0 F)(X) := \pi_0(F(X))$ .
- For  $F \in P(\mathcal{C}, sSet)$  and  $a \in F(X)_0$ ,  $\pi_n(F, a)$  is the presheaf of groups on  $\mathcal{C}/X$  defined by

$$\pi_n(F, a)(Y \xrightarrow{f} X) = \pi_n(F(Y), f^*a).$$

For  $F \in P(\mathcal{C}, \mathcal{G}rpd)$  and  $a \in \text{ob } F(X)$ ,  $\pi_n(F, a) := \pi_n(NF, a)$ .

We say that a map  $F \xrightarrow{\phi} G$  of presheaves of simplicial sets or groupoids is an **isomorphism on sheaves of homotopy groups** if the induced maps  $sh\pi_0(\phi)$  and  $sh\pi_n(\phi, a)$  are isomorphisms for all  $a \in F(X)$ , and all  $X \in \mathcal{C}$ .

Note that if  $F$  is a presheaf of groupoids then  $\pi_i(F, a) = 0$  for  $i > 1$ , and  $\pi_1(F, a)$  is the presheaf of groups  $\text{Aut}_F(a)$  on  $\mathcal{C}/X$ , where

$$\text{Aut}_F(a)(Y \xrightarrow{f} X) := \text{Aut}_{F(Y)}(f^*a).$$

Note also that if  $F \rightarrow G$  is an objectwise weak equivalence, then the induced map of presheaves of homotopy groups is an isomorphism.

*Note 5.2:* If  $\mathcal{C}$  has enough points then a map induces an isomorphism on sheaves of homotopy groups if and only if it induces an isomorphism on the stalks of the sheaves of homotopy groups, which is equivalent to inducing an weak equivalence on the stalks.

*Reference 5.3* (Joyal’s Model Structure [Ja]): There is a left proper, cofibrantly generated, simplicial model structure on  $P(\mathcal{C}, sSet)$  where

- cofibrations are the maps which are objectwise cofibrations,
- weak equivalences are the maps which are isomorphisms on sheaves of homotopy groups,
- fibrations are the maps with the right lifting property with respect to the trivial cofibrations.

The Joyal model category will be denoted by  $P(\mathcal{C}, sSet)_J$ . Note that in the  $S^2$ -nullification of  $P(\mathcal{C}, sSet)_J$  the weak equivalences are the maps which induce isomorphisms on the  $sh\pi_0$  and  $sh\pi_1$ .

**THEOREM 5.4:** *There is a Quillen equivalence between the model categories  $P(\mathcal{C}, Grpd)_L$  and the  $S^2$ -nullification of  $P(\mathcal{C}, sSet)_J$  given by the adjoint pair  $(\pi_{oid}, N)$  and the identity adjunction of  $P(\mathcal{C}, sSet)$ .*

*Proof.* The result follows from localizing the Quillen equivalence in Theorem 1.2 of [DHI] combined with the application of Corollary A.9 of [DHI]. ■

**COROLLARY 5.5:** *The weak equivalences in  $P(\mathcal{C}, Grpd)_L$  are the image under  $\pi_{oid}$  of those in  $(S^2)^{-1}P(\mathcal{C}, sSet)_J$ . In particular, a morphism  $f \in P(\mathcal{C}, Grpd)$  is a local weak equivalence if and only if it induces an isomorphism on sheaves of homotopy groups.*

**5.2. CHARACTERIZATION OF LOCAL WEAK EQUIVALENCES.** The following definition is the restriction to groupoids of the local lifting conditions of Section 3 of [DHI] for simplicial sets.

**Definition 5.6:** A map  $F \xrightarrow{\phi} G \in P(\mathcal{C}, \mathcal{G}rpd)$  is said to satisfy the **local lifting conditions** if:

- (1) Given a commutative square

$$\begin{array}{ccc} \emptyset & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ \star & \longrightarrow & G(X) \end{array} \Rightarrow \exists \text{ cover } U \rightarrow X, \begin{array}{ccccc} \star & \xleftarrow{\quad} & \emptyset & \longrightarrow & F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta^1 & \xleftarrow{\quad} & \star & \longrightarrow & G(X) & \longrightarrow & G(U). \end{array}$$

- (2) For  $A \rightarrow B$ , one of the generating cofibrations  $\partial\Delta^1 \rightarrow \Delta^1, B\mathbb{Z} \rightarrow \star$ , given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ B & \longrightarrow & G(X) \end{array} \Rightarrow \exists \text{ cover } U \rightarrow X, \begin{array}{ccccc} A & \longrightarrow & F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & G(X) & \longrightarrow & G(U). \end{array}$$

**THEOREM 5.7:** A map  $F \xrightarrow{\phi} G \in P(\mathcal{C}, \mathcal{G}rpd)$  is an equivalence on sheaves of homotopy groups if and only if it satisfies the local lifting conditions.

*Proof.* Recall that for  $F$  a presheaf, its sheafification  $shF$ , can be constructed by setting

$$shF(X) = \text{colim}(\text{eq } F(U) \rightrightarrows F(V))$$

where the colimit is taken over all covers  $U \rightarrow X$  and  $V \rightarrow U \times_X U$ . It follows that if  $a \in shF(X)$  then there exists a cover  $U \rightarrow X$  such that  $a$  lifts to an element of  $F(U)$ . Similarly if  $a, b \in F(X)$  have the same images in  $shF(X)$  there exists a cover  $U \rightarrow X$  so that they have the same image in  $F(U)$ . Conversely these two properties are enough to characterize the sheafification. It follows that the lifting conditions for  $\emptyset \rightarrow \star$  and  $\partial\Delta^1 \rightarrow \Delta^1$  are equivalent to  $sh\pi_0\phi$  being an isomorphism, and the lifting conditions for  $B\mathbb{Z} \rightarrow \star$  and  $\star \rightarrow B\mathbb{Z}$  (which is implied by that for  $\partial\Delta^1 \rightarrow \Delta^1$ ) are equivalent to  $sh \text{Aut}_\phi(a)$  being an isomorphism for all  $a \in F(X), X \in \mathcal{C}$ . ■

The following two Corollaries are straightforward exercises using the local lifting conditions.

**COROLLARY 5.8:** In  $P(\mathcal{C}, \mathcal{G}rpd)_L$ :

- The pullback of a weak equivalence by a levelwise fibration is again a weak equivalence.

- The pullback of a weak equivalence which is a levelwise fibration is weak equivalence which is a levelwise fibration.

In particular  $P(\mathcal{C}, \mathcal{G}rpd)_L$  is right proper.

COROLLARY 5.9: If  $F \xrightarrow{\phi} G \in P(\mathcal{C}, \mathcal{G}rpd)$  is a local weak equivalence and  $F$  is a stack then  $\phi$  is a levelwise weak equivalence.

5.3. LOCAL MODEL CATEGORY STRUCTURE ON SHEAVES OF GROUPOIDS.

PROPOSITION 5.10: There exists a local model category structure on  $Sh(\mathcal{C}, \mathcal{G}rpd)$ , denoted  $Sh(\mathcal{C}, \mathcal{G}rpd)_L$ , in which a morphism  $f$  is a weak equivalence (resp., fibration) if and only if  $i(f)$  is a weak equivalence (resp., fibration) in  $P(\mathcal{C}, \mathcal{G}rpd)_L$ . Furthurmore, the adjoint pair

$$P(\mathcal{C}, \mathcal{G}rpd)_L \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{sh} \end{array} Sh(\mathcal{C}, \mathcal{G}rpd)_L$$

induce Quillen equivalences between the local model structures.

*Proof.* To see that the model structure  $Sh(\mathcal{C}, \mathcal{G}rpd)_L$  is well defined it suffices to show that given a generating trivial cofibration  $f$  in  $P(\mathcal{C}, \mathcal{G}rpd)_L$ , that  $sh(f)$  is a weak equivalence, and that the pushout of  $sh(f)$  along any morphism in  $Sh(\mathcal{C}, \mathcal{G}rpd)$  is still a weak equivalence. Both of these statments follow since the natural transformation  $F \rightarrow i(shF)$  satisfies the local lifting conditions, and so is a weak equivalence in  $P(\mathcal{C}, \mathcal{G}rpd)_L$ . This also implies that  $(sh, i)$  is a Quillen equivalence. ■

COROLLARY 5.11: A morphism  $X \xrightarrow{f} Y \in Sh(\mathcal{C}, \mathcal{G}rpd)_L$  is a weak equivalence if and only if it is objectwise full and faithful, and satisfies 5.6(1).

The following is a consequence of Corollary 5.9:

COROLLARY 5.12: Let  $F \in P(\mathcal{C}, \mathcal{G}rpd)$  be a stack, then  $sh(F)$  is also a stack.

**Appendix A. Limits and colimits in  $\mathcal{G}rpd/\mathcal{C}$**

The goal of this section is to prove the following

THEOREM A.1: Categories fibered in groupoids over  $\mathcal{C}$  are closed under small limits and colimits.



In order to prove this, we will need a few preliminaries.

*Definition A.2:*  $F: \mathcal{E} \rightarrow \mathcal{C} \in \text{Cat}/\mathcal{C}$  is pre-fibered in groupoids if

- (1) Given  $f: Y \rightarrow X \in \mathcal{C}$  and  $X' \in \mathcal{E}$  such that  $F(X') = X$ , there exists  $f' \in \mathcal{E}$ , with target  $X'$ , such that  $F(f') = f$ .
- (2) Given a diagram in  $\mathcal{E}$ , over the commutative diagram in  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 & Y' & \xrightarrow{F} & Y \\
 & \downarrow f' & & \downarrow f \\
 Z' \xrightarrow{g'} & X' & \xrightarrow{F} & Z \xrightarrow{g} X, \\
 & & & \nearrow h
 \end{array}$$

with  $F(f') = f, F(g') = g$ , there exists  $h'$  such that  $g' \circ h' = f'$  and  $F(h') = h$ . Moreover, given two such maps  $h'_1, h'_2$ , there exists an automorphism  $\phi \in \text{Aut}_{\mathcal{E}}(Y')$  such that  $F(\phi) = id_Y$  and  $h'_1 \circ \phi = h'_2$ .

Thus, the difference between fibered and pre-fibered is that categories which are pre-fibered in groupoids satisfy a weaker condition than the uniqueness in Condition (2) of Definition 3.1.

**PROPOSITION A.3:** *Let  $I$  be a small category, and  $F: I \rightarrow \text{Grpd}/\mathcal{C}$ , a diagram. Then the colimit of the diagram  $F$  in  $\text{Cat}/\mathcal{C}$  is pre-fibered in groupoids.*

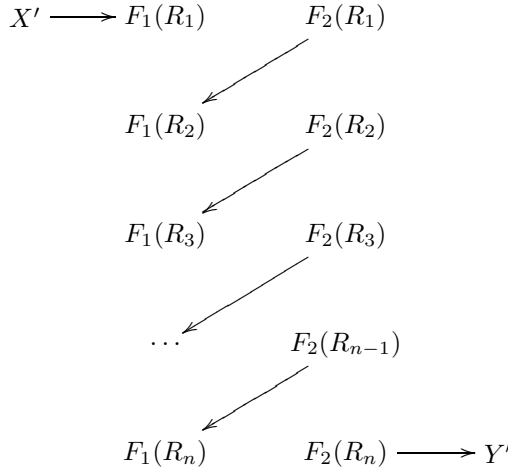
*Proof.* The coproduct in  $\text{Cat}/\mathcal{C}$  of a set of objects in  $\text{Grpd}/\mathcal{C}$  is again in  $\text{Grpd}/\mathcal{C}$  so it suffices to consider the case of a coequalizer diagram. Consider the diagram

$$\begin{array}{ccccc}
 \mathcal{R} & \xrightarrow{F_1} & \mathcal{E} & \longrightarrow & \bar{\mathcal{E}} \\
 & \searrow F_2 & \downarrow & \swarrow & \\
 & & \mathcal{C} & & 
 \end{array}$$

where  $F_1, F_2$  are maps in  $\text{Grpd}/\mathcal{C}$  and  $\bar{\mathcal{E}}$  is the coequalizer of the two arrows in  $\text{Cat}$ . Recall that the coequalizer in  $\text{Cat}$  has objects the coequalizer of the sets of objects, and morphisms the formal compositions of the coequalizer of the morphisms, modulo the relations given by composition in  $\mathcal{E}$ . Thus the map  $\bar{\mathcal{E}} \rightarrow \mathcal{C}$  clearly satisfies Condition (1) of Definition A.2. The proof that Condition (2) holds follows below:

**LEMMA A.4:** *Let  $X'$  and  $Y'$  be in  $\mathcal{E}$ , and suppose  $\bar{f}: [X'] \rightarrow [Y']$  is a map between their images in  $\bar{\mathcal{E}}$ . Write  $X = F(X')$  and  $Y = F(Y')$  in  $\mathcal{C}$ . Then there*

is a sequence of objects  $R_1, R_2, \dots, R_n \in \mathcal{R}$  with  $F(R_i) = X$  and maps in  $\mathcal{E}$ :



with all but the last covering the identity map of  $X$  in  $\mathcal{C}$ , such that  $\bar{f}$  is the composite of images

$$\begin{aligned}
 [X'] \rightarrow [F_1(R_1)] &= [F_2(R_1)] \rightarrow [F_1(R_2)] \\
 &= [F_2(R_2)] \rightarrow [F_1(R_3)] \rightarrow \dots [F_2(R_{n-1})] \rightarrow [F_1(R_n)] \\
 &= [F_2(R_n)] \rightarrow [Y'] \text{ in } \bar{\mathcal{E}}.
 \end{aligned}$$

*Proof.* Use properties of groupoids over  $\mathcal{C}$  to simplify the expression of a map in  $\bar{\mathcal{E}}$  as formal composition of maps in  $\mathcal{E}$ , modulo the relations given by  $\mathcal{R}$ . ■

Note that the maps  $X' \rightarrow F_1(R_1), F_2(R_1) \rightarrow F_1(R_2), F_2(R_2) \rightarrow F_1(R_3), \dots, F_2(R_{n-1}) \rightarrow F_1(R_n)$ , are isomorphisms in  $\mathcal{E}$  because they cover the identity of  $X$  in  $\mathcal{C}$ . Also note that  $[F_2(R_n)] \rightarrow [Y]$  covers the same maps as  $\bar{f}$ .

**COROLLARY A.5:** *Suppose  $\bar{f}$  and  $\bar{g}$  are maps  $[X] \rightarrow [Y]$  in  $\bar{\mathcal{E}}$  covering the same map  $f$  in  $\mathcal{C}$ . Then there is an automorphism  $\phi$  of  $[X']$  such that  $\bar{f} \circ \phi = \bar{g}$ .*

*Proof.* Factor  $\bar{f}$  and  $\bar{g}$  as in Lemma A.4:

$$\begin{aligned}
 [X'] \rightarrow [F_1(R_1)] &= [F_2(R_1)] \rightarrow [F_1(R_2)] = [F_2(R_2)] \rightarrow [F_1(R_3)] \rightarrow \dots \\
 &\quad \dots [F_2(R_{n-1})] \rightarrow [F_1(R_n)] = [F_2(R_n)] \rightarrow [Y'], \\
 [X'] \rightarrow [F_1(S_1)] &= [F_2(S_1)] \rightarrow [F_1(S_2)] = [F_2(S_2)] \rightarrow [F_1(S_3)] \rightarrow \\
 &\quad \dots [F_2(S_{k-1})] \rightarrow [F_1(S_k)] = [F_2(S_k)] \rightarrow [Y'].
 \end{aligned}$$

Consider the diagram in  $\mathcal{E}$ , over the diagram in  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 F_2(S_k) & \xrightarrow{F} & X \\
 \downarrow & & \downarrow f \\
 F_2(R_n) \longrightarrow Y' & \xrightarrow{F} & X \xrightarrow{f} Y.
 \end{array}$$

As  $\mathcal{E}$  is fibered in groupoids over  $\mathcal{C}$  there is a unique isomorphism

$$F_2(S_k) \xrightarrow{h} F_2(R_n)$$

covering the identity of  $X$  so that

$$\begin{array}{ccc}
 & F_2(S_k) & \\
 & \downarrow & \\
 h \swarrow & & \downarrow \\
 F_2(R_n) & \longrightarrow & Y'
 \end{array}$$

commutes in  $\mathcal{E}$ . Let  $\phi$  be the composite

$$\begin{aligned}
 [X'] \rightarrow [F_1(S_1)] &= [F_2(S_1)] \rightarrow \cdots [F_1(S_k)] = [F_2(S_k)] \xrightarrow{\bar{h}} [F_2(R_n)] \\
 &= [F_1(R_n)] \rightarrow [F_2(R_{n-1})] \rightarrow \cdots [F_1(R_2)] \rightarrow [F_2(R_1)] \\
 &= [F_1(R_1)] \rightarrow [X']
 \end{aligned}$$

where the second set of maps are the inverses of the isomorphisms in the factorization of  $\bar{f}$ . ■

PROPOSITION A.6: *The coequalizer  $\bar{\mathcal{E}}$  satisfies Condition (2) of Definition A.2.*

*Proof.* Given a diagram in  $\bar{\mathcal{E}}$ , over the commutative diagram in  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 \bar{Y}' & \xrightarrow{F} & Y \\
 \downarrow \bar{f}' & & \downarrow f \\
 \bar{Z}' \xrightarrow{\bar{g}'} \bar{X}' & \xrightarrow{F} & Z \xrightarrow{g} X,
 \end{array}$$

with  $F(\bar{f}') = f, F(\bar{g}') = g$ , factor  $\bar{f}'$  and  $\bar{g}'$  as in Lemma A.4

$$\begin{aligned}
 [Y'] \rightarrow [F_1(R_1)] &= [F_2(R_1)] \rightarrow [F_1(R_2)] = [F_2(R_2)] \rightarrow [F_1(R_3)] \rightarrow \cdots \\
 & \quad [F_2(R_{n-1})] \rightarrow [F_1(R_n)] = [F_2(R_n)] \rightarrow [X'], \\
 [Z'] \rightarrow [F_1(S_1)] &= [F_2(S_1)] \rightarrow [F_1(S_2)] = [F_2(S_2)] \rightarrow [F_1(S_3)] \\
 & \quad \cdots [F_2(S_{k-1})] \rightarrow [F_1(S_k)] = [F_2(S_k)] \rightarrow [X'].
 \end{aligned}$$

Then there is a unique lift of  $X \xrightarrow{f} Z$  in the diagram in  $\mathcal{E}$ , over the commutative diagram in  $\mathcal{C}$ ,

$$\begin{array}{ccc}
 & F_2(R_n) & \\
 & \swarrow u & \downarrow \\
 F_2(S_k) & \longrightarrow & X' \\
 & & \\
 & \xrightarrow{F} & \\
 & & \\
 & & \begin{array}{ccc} & Y & \\ & \swarrow h & \downarrow f \\ Z & \xrightarrow{f} & X. \end{array}
 \end{array}$$

The map

$$\begin{aligned}
 [Y'] \rightarrow [F_1(R_1)] &= [F_2(R_1)] \rightarrow \cdots [F_1(R_n)] = [F_2(R_n)] \xrightarrow{\bar{u}} [F_2(S_k)] \\
 &= [F_1(S_k)] \rightarrow [F_2(S_{k-1})] \rightarrow \cdots [F_2(S_1)] = [F_1(S_1)] \rightarrow [Z']
 \end{aligned}$$

provides the desired lift of  $h$  in  $\bar{\mathcal{E}}$ . ■

This completes the proof of Proposition A.3 ■

PROPOSITION A.7: *Let  $\mathcal{E} \rightarrow \mathcal{C}$  be pre-fibered in groupoids. Let  $\sim$  be the equivalence relation on  $\mathcal{E}$  generated by setting  $\alpha \sim id$  for the automorphisms  $\alpha \in \mathcal{E}$  which satisfy:*

- (1)  $\alpha$  maps to an identity morphism in  $\mathcal{C}$ ,
- (2) there exists  $f \in \mathcal{E}$  such that  $f \circ \alpha = f$ .

Then  $(\mathcal{E}/\sim) \rightarrow \mathcal{C}$  is also pre-fibered in groupoids.

*Proof.* The proof follows from the fact that the map  $\mathcal{E} \rightarrow (\mathcal{E}/\sim)$  is surjective on morphisms and bijective on objects. ■

*Proof of Theorem A.1.* Let  $I$  be a small category and  $F: I \rightarrow \mathcal{Grpd}/\mathcal{C}$  be a diagram. We denote by  $F'$  the composite  $I \xrightarrow{F} \mathcal{Grpd}/\mathcal{C} \rightarrow \mathcal{Cat}/\mathcal{C}$ .

COLIMITS: Let  $\mathcal{E}_0$  denote the colimit of  $F'$  in  $\mathcal{Cat}$ . We will show that the desired colimit of  $F$  in  $\mathcal{Grpd}/\mathcal{C}$  is the directed colimit  $\mathcal{E}$  in  $\mathcal{Cat}/\mathcal{C}$  of the categories  $\mathcal{E}_i$ , where  $\mathcal{E}_i = \mathcal{E}_{i-1}/\sim$ .

$$(A.8) \quad \mathcal{E}_0 \rightarrow (\mathcal{E}_0/\sim) = \mathcal{E}_1 \rightarrow (\mathcal{E}_1/\sim) = \mathcal{E}_2 \rightarrow \cdots$$

Propositions A.3 and A.7 imply that Condition (1) and the existence part in Condition (2) of Definition 3.1 are still satisfied by  $\mathcal{E}$ .

To show the uniqueness part in Condition (2), suppose given a commutative diagram in  $\mathcal{E}$ :

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & X \\
 \Downarrow h_2 & \nearrow g & \\
 Z & & 
 \end{array}$$

such that  $h_1$  and  $h_2$  project to the same map in  $\mathcal{C}$ . We can pick lifts  $h'_1$  and  $h'_2$  of  $h_1$  and  $h_2$  in some  $\mathcal{E}_i$ . They also project to the same map in  $\mathcal{C}$  and as  $\mathcal{E}_i$  is pre-fibered in groupoids by Proposition A.3, there is an automorphism  $\alpha$  of  $Y$  in  $\mathcal{E}_i$  mapping to an identity in  $\mathcal{C}$  such that  $h'_2 \circ \alpha = h'_1$ . It follows that  $h'_1 = h'_2 \in \mathcal{E}_{i+1}$  and so  $h_1$  and  $h_2$  agree in  $\mathcal{E}$ .

To show that  $\mathcal{E}$  is the colimit in  $\mathit{Grpd}/\mathcal{C}$ , observe that if  $\mathcal{F}$  is pre-fibered in groupoids, and  $\mathcal{E}'$  is fibered in groupoids, then any map  $\mathcal{F} \rightarrow \mathcal{E}' \in \mathit{Cat}/\mathcal{C}$ , factors uniquely through  $\mathcal{F}/\sim$ .

LIMITS: Consider the inverse limit of our diagram in  $\mathit{Cat}/\mathcal{C}$ . The objects and morphisms of  $\lim F'$  are the inverse limits of the sets of objects and morphisms, so for each object  $X' \in \lim F'$ , the category  $(\lim F')/X'$ , is the inverse limit of categories  $F(i)/X'_i, i \in I$ . It is easy to see that the map  $(\lim F')/X' \rightarrow \mathcal{C}/X$

- is a bijection on Hom-sets, since this is the case for each of the functors  $F(i)/X'_i \rightarrow \mathcal{C}/X$ ,
- is not necessarily a surjection on objects, even though each of the functors  $F(i)/X'_i \rightarrow \mathcal{C}/X$  is.

Consider the full subcategory of  $\lim F'$  with objects all those  $X'$  such that  $(\lim F')/X' \rightarrow \mathcal{C}/X$  is surjective on objects. This subcategory is clearly fibered in groupoids and satisfies the universal property of the limit. ■

### Appendix B. Pushouts in $\mathit{Cat}$

The goal of this section is to prove

PROPOSITION B.1: *Let  $A, B, C$  be small categories, and  $A \xrightarrow{i} B$  be a functor which is a monomorphism on objects, and  $A \xrightarrow{j} C$  a surjective equivalence of categories. Then the induced functor to the pushout in  $\mathit{Cat}$ ,  $B \rightarrow P := C \coprod_A B$  is also a surjective equivalence of categories.*

*Proof.* First note that the universal map  $B \xrightarrow{p} P$  is surjective on objects. If  $b, b' \in \text{ob } B$ , then  $p(b) = p(b')$  if and only there exist  $a, a' \in \text{ob } A$  with  $i(a) = b, i(a') = b'$  and  $j(a) = j(a')$ . So there is a unique map  $a \rightarrow a' \in A$  which maps to the identity of  $j(a)$  and we will call the image of this map in  $B$  the canonical map  $b \rightarrow b'$ . For  $b$  not in the image of  $A$  the canonical map  $b \rightarrow b$  is defined to be the identity.

It is clear that  $p$  induces an isomorphism on components so it remains to show that  $p$  induces an isomorphism

$$\text{Hom}_B(b, b') \longrightarrow \text{Hom}_P(p(b), p(b')).$$

For  $\beta, \beta'$  objects of  $P$ , let  $W(\beta, \beta')$  denote the set of words formed by formal compositions of morphisms in  $B$  and  $C$  such that the first map in the word has domain representing  $\beta$ , the last map has range representing  $\beta'$  and consecutive maps have domains and ranges whose images in  $P$  agree. Recall that  $\text{Hom}_P(\beta, \beta')$  is the quotient of  $W(\beta, \beta')$  by the equivalence relation generated by the composition in  $B$ , composition in  $C$  and  $i(f) \sim j(f)$  for  $f$  a morphism in  $A$ .

Let  $b, b'$  be objects of  $B$  and write  $\beta = p(b), \beta' = p(b')$ . We will define functions  $\phi_{b,b'}: W(\beta, \beta') \rightarrow \text{Hom}_B(b, b')$  which are constant on the equivalence classes of  $W(\beta, \beta')$  and so determine functions  $\text{Hom}_P(\beta, \beta') \rightarrow \text{Hom}_B(b, b')$ . It will be immediate from the construction that these are inverse to  $p$  and this will complete the proof.

The functions  $\phi_{b,b'}$  are defined by induction on the length of words as follows. Let  $w$  be a word of length 1. If  $w$  is a morphism  $c \xrightarrow{f} c' \in C$  then let  $a, a'$  be the unique objects in  $A$  such that  $i(a) = b, i(a') = b', j(a) = c, j(a') = c'$  and let  $a \xrightarrow{g} a'$  denote the unique morphism in  $A$  such that  $j(g) = f$ . Define  $\phi_{b,b'}(w) = i(g)$ . If  $w$  is a morphism  $b_1 \xrightarrow{f} b_2 \in B$  define  $\phi_{b,b'}(w)$  to be the composite  $b \rightarrow b_1 \xrightarrow{f} b_2 \rightarrow b'$  where the unlabeled arrows are canonical morphisms.

Now suppose  $\phi_{b,b'}$  has been defined on words of length  $\leq n$  and let  $w = w'f$  where  $w'$  is a word of length  $n$  and  $f$  is a morphism in  $B$  or in  $C$ . Let  $b''$  be an arbitrary object of  $B$  mapping to the range of  $w'$  and define  $\phi_{b,b'}(w)$  as the composite  $b \xrightarrow{\phi_{b,b''}(w')} b'' \xrightarrow{\phi_{b'',b'}(f)} b'$ . It follows from the construction that the value of  $\phi_{b,b'}$  is independent of the choice of  $b''$  and that  $\phi_{b,b'}$  is constant on the equivalence classes of  $W(\beta, \beta')$ . ■

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