Bi-actegories

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early version!

Abstract

We define the $\triangleleft$ generalization of the Eilenberg-Moore category over a monad, where the monad is replaced by a coherent action of a monoidal category. Then we study the distributive laws from another monad to such a monoidal action, which are in a sense dual to the distributive laws from a monoidal action to a monad, studied in our earlier paper. The duality appears as a special case of study of distributive laws for two different actions of monoidal categories on the same category. When the actions are strong (coherences are invertible) and one action is left, then such distributive laws make a categorical analogue of a bimodule. For such bimodules, called bi-actegories, one can define a tensor product, using lax coequalizers. In the sequel we will show that the bi-actegories make a tricategory. Some special cases are of importance, for example for induction for module categories and associated bundles to categorified principal bundles.

Key words: monoidal category, action, distributive law, strict lifting, bi-actegories

1 Introduction. Modules in actegory.

1.1. We use the notation from part I. In particular, a monoidal category will be described as a 6-tuple $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, a, r, l)$ where $\mathbf{1}$ is the unit object, a is the associativity coherence, and r, l are right and left unit coherences. A left action $(\Diamond, \Psi, u)$ of monoidal category $\mathcal{C}$ on a category $\mathcal{M}$ is decribed by a bifunctor $\Diamond : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, $(C, M) \mapsto C\Diamond M$ with the action associativity coherence Ψ and the unit coherence u satisfying the usual pentagon and triangle axioms. The 4-tuple $(\mathcal{M}, \Diamond, \Psi, u)$ is also referred to as a $\mathcal{C}$-**actegory**.

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In part I [9] we studied the distributive laws from a $\mathcal{C}$-actegory $(\mathcal{M}, \Diamond, \Psi, u)$ to a monad $\mathbf{T} = (T, \mu, \eta)$ in $\mathcal{M}$ and proved that they are in bijective correspondence with the strict lifts of the action to the Eilenberg-Moore category $\mathcal{M}^{\mathbf{T}}$ of $\mathbf{T}$-modules. Now we present the preliminaries to establish a dual version.

1.2. (Modules in actegories.) Given a left $\mathcal{C}$-actegory $(\mathcal{M}, \Diamond, \Psi, u)$ a $\mathcal{C}$-**module in** $\mathcal{M}$ is a pair (M, ν) where M is an object in $\mathcal{M}$ and $\nu : \mathrm{Id}_{\mathcal{M}} \Diamond M \Rightarrow M$ is a natural transformation of functors $\mathcal{C} \to \mathcal{M}$ satisfying the following action axiom for components: for any two objects C, Q in $\mathcal{C}$, the diagram

$$\begin{tikzcd} Q\Diamond(C\Diamond M) \arrow[r, "Q\Diamond\nu_C"] \arrow[d, "(\Psi_M^{Q,C})^{-1}"'] & Q\Diamond M \arrow[d, "\nu_Q"] \\ (Q\otimes C)\Diamond M \arrow[r, "\nu_{Q\otimes C}"] & M \end{tikzcd}$$

commutes and $\nu_{\mathbf{1}} = u_M$. The naturality of ν amounts to requiring that $\nu_{C'} \circ (f \Diamond M) = \nu_C$ for any morphism $f : C \to C'$ in the monoidal category $\mathcal{C}$.

A **morphism** $g : (M, \nu) \to (N, \rho)$ **of** $\mathcal{C}$**-modules in** $\mathcal{M}$ is a morphism $g : M \to N$ in $\mathcal{M}$ such that, for all objects Q in $\mathcal{C}$, the following square

$$\begin{tikzcd} Q\Diamond M \arrow[r, "\nu_Q"] \arrow[d, "Q\Diamond g"'] & M \arrow[d, "g"] \\ Q\Diamond N \arrow[r, "\rho_Q"] & N \end{tikzcd} \tag{1}$$

commutes. Property $\rho_{\mathbf{1}} \circ (1 \Diamond g) = g \circ \nu_{\mathbf{1}}$ is automatic, namely it follows by naturality of $u : \mathbf{1} \Diamond \mathrm{Id} \Rightarrow \mathrm{Id}$ since $u_M = \nu_{\mathbf{1}}$ and $u_N = \rho_{\mathbf{1}}$.

One can require seemingly more general requirement that for every $q : Q \to Q'$ the diagram

$$\begin{tikzcd} Q\Diamond M \arrow[r, "\nu_Q"] \arrow[d, "q\Diamond g"'] & M \arrow[d, "g"] \\ Q'\Diamond N \arrow[r, "\rho_{Q'}"] & N \end{tikzcd}$$

commutes. But this follows from (1) using $q \Diamond g = (q \Diamond N) \circ (Q \Diamond g)$ and the naturality of ρ.

$\mathcal{C}$-modules in $\mathcal{M}$ and their morphisms make a category $\mathcal{M}^{(\mathcal{C}, \Diamond, \Psi, u)}$, often denoted simply by $\mathcal{M}^{\mathcal{C}}$ and called the **Eilenberg-Moore category of the** $\mathcal{C}$**-actegory** $\mathcal{M}$. It is equipped with a forgetful functor $U : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$, $(M, \nu) \to M$, which is in general, unlike in the case of usual Eilenberg-Moore categories not surjective on objects. In particular, there is no "free functor" $\mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ in general.

1.3. In the following $\mathcal{C}$ considered as a left $\mathcal{C}$-actegory via $\otimes$ and $\mathcal{M}$ is another left $\mathcal{C}$-actegory.

Proposition. *Suppose (Q,μ) is an object of $\mathcal{C}^{\mathcal{C}}$. Then, for every object M, the compositions*

$$\nu_C^{\mu,M} : C\Diamond(Q\Diamond M) \stackrel{(\Psi^{-1})^{C,Q}_M}{\longrightarrow} (C\otimes Q)\Diamond M \stackrel{\mu_C\Diamond M}{\longrightarrow} Q\Diamond M$$

are components of a natural transformation $\nu^{\mu,M}$ making $(Q\Diamond M,\nu^{\mu,M})$ an object in $\mathcal{M}^{\mathcal{C}}$. Moreover, the construction is functorial in M, yielding a functor $F^{(Q,\mu)} : \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ with the property $UF^{(Q,\mu)} = Q\Diamond\mathrm{Id}_{\mathcal{M}}$. For every morphism $g : (Q,\mu) \to (Q',\mu')$ in $\mathcal{C}^{\mathcal{C}}$, and morphism $n : N \to N'$ in $\mathcal{M}$, the morphisms $F(g,n) = F^g(n) := g\Diamond n : Q\Diamond M \to Q'\Diamond M'$ are in fact morphisms $(Q\Diamond M,\nu^{\mu,M}) \to (Q'\Diamond M,\nu^{\mu',M'})$ in $\mathcal{M}^{\mathcal{C}}$. The rule for morphisms $(g,n) \mapsto F(g,n)$ defines a bifunctor $F : \mathcal{C}^{\mathcal{C}} \times \mathcal{M} \to \mathcal{M}^{\mathcal{C}}$; in particular $F(\mathrm{id}_{(Q,\mu)},\mathrm{id}_M) = F^{(Q,\mu)}(M)$.

The proof is easy and we will only show the commutative diagram exhibiting the proof of the action axiom.

$$\begin{tikzcd}
(C_1\Diamond C_2)\Diamond(Q\Diamond M) \arrow[r, "C_1\Diamond(\Psi_M^{C_2,Q})^{-1}"] \arrow[dd, "(\Psi^{C_1,C_2}_{Q\Diamond M})^{-1}"] & C_1\Diamond((C_2\otimes Q)\Diamond M) \arrow[r, "C_1\Diamond(\mu_{C_2}\Diamond M)"] \arrow[d, "(\Psi_M^{C_1,C_2\otimes Q})^{-1}"] & C_1\Diamond(Q\Diamond M) \arrow[d, "(\Psi_M^{C_1,Q})^{-1}"] \\
 & (C_1\otimes(C_2\otimes Q))\Diamond M \arrow[r, "(C_1\otimes\mu_{C_2})\Diamond M"] \arrow[d, "a_{C_1,C_2,Q}\Diamond M"] & (C_1\otimes Q)\Diamond M \arrow[d, "\mu_{C_1}\Diamond M"] \\
(C_1\otimes C_2)\Diamond(Q\Diamond M) \arrow[r, "(\Psi_M^{C_1\otimes C_2,Q})^{-1}"] & ((C_1\otimes C_2)\otimes Q)\Diamond M \arrow[r, "\mu_{C_1\otimes C_2}\Diamond M"] & Q\Diamond M
\end{tikzcd}$$

1.4. Let us now clarify in which sense the above Eilenberg-Moore category is a generalization of the usual one (for monads). A monad $\mathbf{T}$ in $\mathcal{M}$ can be considered an actegory in two (or three if you like) ways. The first one is as an action of a trivial monoidal category $\mathbf{1}$ (with one object and one morphism) on $\mathcal{M}$; clearly Ψ^{-1} corresponds to the multiplication of the monad, and the unit of the monad can be extracted from the unit coherence. The second approach is to consider a strict action (Ψ and u trivial) of the monoidal subcategory $\mathbf{T}^*$ generated by $\mathbf{T} \subset \mathrm{End}(\mathcal{M})$, and morphisms μ, η. Or one can take an abstract version of this: a PRO (a category whose objects are natural numbers and tensor product of objects addition) for monoids with a strict representation in (=strict monoidal functor to) $\mathrm{End}(\mathcal{M})$ (a minor difference: some powers of T may coincide for the monad, so the representation may have a nonzero kernel, hence $\mathbf{T}^*$ is itself not necessarily a PRO).

Now it is an easy exercise that if a monad is understood as a monoidal category in any of the 3 ways above, the Eilenberg-Moore category will coincide: for the first recipe, the natural transformation ν boils down to a single action

$\nu : TM = 1\Diamond M \to M$ and in the other two we also have induced actions $\nu_n : T^n M \to M$ which are forced to be of the form $\nu_n = \nu_1 \circ T(\nu_1) \circ \ldots \circ T^{n-1}(\nu_1)$.

1.5. A **lax $\mathcal{C}$-equivariant functor** $(K,\gamma) : (\mathcal{M}, \Diamond^{\mathcal{M}}, \Psi^{\mathcal{M}}, u^{\mathcal{M}}) \to (\mathcal{N}, \Diamond^{\mathcal{N}}, \Psi^{\mathcal{N}}, u^{\mathcal{N}})$ **of $\mathcal{C}$-actegories** is an ordinary functor $K : \mathcal{M} \to \mathcal{N}$ together with a binatural transformation of bifunctors $\mathrm{Id}\Diamond^{\mathcal{N}} K \Rightarrow K \circ (\mathrm{Id}\Diamond^{\mathcal{M}}\mathrm{Id})$ satisfying the usual pentagon and unit coherences:

$$\begin{tikzcd}
(Q\otimes C)\Diamond^{\mathcal{N}} KM \arrow[rr, "\gamma_M^{Q\otimes C}"] \arrow[d, "(\Psi^{\mathcal{N}})^{Q,C}_{KM}"'] & & K((Q\otimes C)\Diamond^{\mathcal{M}} M) \arrow[d, "K((\Psi^{\mathcal{M}})^{Q,C}_M)"] \\
Q\Diamond^{\mathcal{N}}(C\Diamond^{\mathcal{N}} KM) \arrow[r, "Q\Diamond\gamma^C_M"'] & Q\Diamond^{\mathcal{N}} K(C\Diamond^{\mathcal{M}} M) \arrow[r, "\gamma^Q_{C\Diamond M}"'] & K(Q\Diamond^{\mathcal{M}}(C\Diamond^{\mathcal{M}} M))
\end{tikzcd} \tag{2}$$

A **natural transformation** $\alpha : (K,\zeta^K) \Rightarrow (L,\zeta^L)$ **of lax $\mathcal{C}$-equivariant functors** is a natural transformation $\alpha : K \Rightarrow K' : \mathcal{M} \to \mathcal{N}$ of underlying functors such that for all objects C in $\mathcal{C}$ and M in $\mathcal{M}$, the following square commutes:

$$\begin{tikzcd}
C\Diamond^{\mathcal{N}} KM \arrow[r, "\zeta^K_{C,M}"] \arrow[d, "C\Diamond\alpha_M"'] & K(C\Diamond^{\mathcal{M}} M) \arrow[d, "\alpha_{C\Diamond M}"] \\
C\Diamond^{\mathcal{N}} LM \arrow[r, "\zeta^L_{C,M}"] & L(C\Diamond^{\mathcal{M}} M)
\end{tikzcd}$$

1.6. Lax $\mathcal{C}$-equivariant functors of $\mathcal{C}$-actegories induce morphisms between their Eilenberg-Moore categories. Given (K,γ) as above, the pushforward $(K,\gamma)_* : \mathcal{M}^{\mathcal{C}} \to \mathcal{N}^{\mathcal{C}}$, or, abusing the notation, simply K_*, is defined by $(K,\gamma)_* : (M,\nu) \mapsto (KM, K\nu \circ \alpha)$. Here $K\nu \circ \alpha$ is the composition of natural transformation, written in components $(K\nu \circ \alpha)_C = K(\nu_C) \circ \alpha_C : C\Diamond^{\mathcal{N}} KM \to KM$. This natural transformation is indeed a $\mathcal{C}$-action in $(\mathcal{N}, \Diamond^{\mathcal{N}}, \Psi^{\mathcal{N}}, u^{\mathcal{N}})$, namely this follows by the commutativity of the diagram

$$\begin{tikzcd}
Q\Diamond^{\mathcal{N}}(C\Diamond^{\mathcal{N}} KM) \arrow[r, "Q\Diamond\gamma^C_M"'] \arrow[dd, "(\Psi^{Q,C}_{KM})^{-1}"'] & Q\Diamond^{\mathcal{N}} K(C\Diamond^{\mathcal{M}} M) \arrow[r, "Q\Diamond K(\nu_C)"'] \arrow[d, "\gamma^Q_{C\Diamond M}"] & Q\Diamond^{\mathcal{N}} KM \arrow[d, "\gamma^Q_M"] \\
 & K(Q\Diamond^{\mathcal{M}}(C\Diamond^{\mathcal{M}} M)) \arrow[r, "K(Q\Diamond\nu_C)"] \arrow[d, "K((\Psi^{\mathcal{M},Q,C}_M)^{-1})"] & K(Q\Diamond^{\mathcal{M}} M) \arrow[d, "K(\nu_Q)"] \\
(Q\otimes C)\Diamond^{\mathcal{N}} KM \arrow[r, "\gamma^Q_C"'] & K((Q\otimes C)\Diamond^{\mathcal{M}} M) \arrow[r, "K(\nu_{Q\otimes C})"'] & KM
\end{tikzcd}$$

for all objects Q, C in $\mathcal{C}$. The left pentagon is expressing the lax $\mathcal{C}$-equivariance of (K,γ), the right upper square is commutative by the naturality for γ, and the right lower square follows by the action axiom for ν and functoriality of K. Thus the external square is commutative what is the action axiom for $K\nu \circ \gamma$.

Clearly the square of functors

$$
\begin{tikzcd}
\mathcal{M}^{\mathcal{C}} \arrow[r, "(K,\gamma)_*"] \arrow[d, "U^{\mathcal{M}}"'] & \mathcal{N}^{\mathcal{C}} \arrow[d, "U^{\mathcal{N}}"] \\
\mathcal{M} \arrow[r, "K"] & \mathcal{N}
\end{tikzcd}
$$

commutes. (here add weak converse!)

1.7. A **distributive law from a monad T in $\mathcal{M}$ to the $\mathcal{C}$-actegory** $(\mathcal{M}, \Diamond, \Psi, u)$ is binatural transformation $d : \mathrm{Id}_{\mathcal{C}} \Diamond T \Rightarrow T \circ (\mathrm{Id}_{\mathcal{C}} \Diamond \mathrm{Id}_{\mathcal{M}})$ satisfying the pentagon and triangle identities (...)

1.8. Given a distributive law from $\mathbf{T}$ to $\mathcal{M}$ as above, define a lift $\mathbf{T}^{\mathcal{C}} = (T^{\mathcal{C}}, \mu^{\mathcal{C}}, \eta^{\mathcal{C}})$ by $T^{\mathcal{C}}(M, \nu) := (TM, T\nu \circ d_M)$. In components,

$$
\begin{tikzcd}
Q\Diamond TM \arrow[r, "d^Q_M"] \arrow[rr, bend right, "(T\nu\circ d_M)_Q"'] & T(Q\Diamond M) \arrow[r, "T(\nu_Q)"] & TM
\end{tikzcd}
$$

We have to show that $T\nu \circ d_M$ is indeed an action. The action axiom amounts to the commutativity of the external square in diagram

$$
\begin{tikzcd}
Q\Diamond(Q'\Diamond TM) \arrow[r, "Q\Diamond d^{Q'}_M"] \arrow[dd, "(\Psi^{Q,Q'}_{TM})^{-1}"'] & Q\Diamond T(Q'\Diamond M) \arrow[r, "Q\Diamond T(\nu_{Q'})"] \arrow[d, "d^Q_{Q'\Diamond M}"] & Q\Diamond TM \arrow[d, "d^Q_M"] \\
 & T(Q\Diamond(Q'\Diamond M)) \arrow[r, "T(Q\Diamond\nu_{Q'})"] \arrow[d, "T((\Psi^{Q,Q'}_M)^{-1})"] & T(Q\Diamond M) \arrow[d, "T(\nu_Q)"] \\
(Q\otimes Q')\Diamond M \arrow[r, "d^{Q\otimes Q'}_M"'] & T((Q\otimes Q')\Diamond M) \arrow[r, "T(\nu_{Q\otimes Q'})"'] & TM
\end{tikzcd}
$$

The left pentagon is one of the pentagons for the distributive law d, the upper right corner is commutative by the naturality of d, and the lower right corner follows by the action axiom for ν and the functoriality of T.

1.9. The general case are the distributive laws from an action of one monoidal category to an action of another monoidal category on the same category. If the distributive law is invertible and one action is left and another is right we have at hand an instance of the categorification of the notion of a bimodule: bi-actegories.

As, in the case of usual bimodules, there is a well-behaved notion of the tensor product of bi-actegories.

1.10. Given two monoidal categories $(\mathcal{A}, \otimes, \mathbf{1}, a^{\mathcal{A}}, l^{\mathcal{A}}, r^{\mathcal{A}})$ and $(\mathcal{B}, \otimes', \mathbf{1}', a^{\mathcal{B}}, l^{\mathcal{B}}, r^{\mathcal{B}})$, a category $\mathcal{M}$, a left coherent action $\triangleright : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ with action coherence

$\Psi_{\mathcal{A}}$ and unit coherence $u_{\mathcal{A}}$ and a right coherent action $\triangleleft : \mathcal{M} \times \mathcal{B} \to \mathcal{M}$ with coherences $\Psi_{\mathcal{B}}, u_{\mathcal{B}}$, we say that $\mathcal{M}$ is an $\mathcal{A}$-$\mathcal{B}$-bi-actegory if in addition it is equipped with a natural isomorphism of (tri)functors

$$l : \mathcal{A} \triangleright (\mathcal{M} \triangleleft \mathcal{B}) \Rightarrow (\mathcal{A} \triangleright \mathcal{M}) \triangleleft \mathcal{B}$$

satisfying some compatibility relations (making it into a distributive law) which will be written below.

1.10.1. A (co)lax $\mathcal{A}$-$\mathcal{B}$-biequivariant functor $(K, \gamma, \bar{\gamma})$ of two $\mathcal{A}$-$\mathcal{B}$-bi-actegories is a triple where $K : \mathcal{M} \to \mathcal{N}$ is an ordinary functor, the pair (K, γ) is a (co)lax $\mathcal{A}$-equivariant functor of left $\mathcal{A}$-actegories, $(K, \bar{\gamma})$ is a (co)lax $\mathcal{B}$-equivariant functor or right $\mathcal{B}$-actegories and an additional compatibility with the pair of distributive laws $l^{\mathcal{M}}, l^{\mathcal{N}}$ is required. More precisely, in the lax case, when the lax biequivariant functor has coherences $\gamma^A_M : A \triangleright KM \Rightarrow K(A \triangleright M)$ and $\bar{\gamma}^B_M : KM \triangleleft B \Rightarrow K(M \triangleleft B)$ in addition to the lax equivariance pentagons for γ and $\bar{\gamma}$, the hexagon

$$\begin{array}{ccccc} A \triangleright (KM \triangleleft B) & \xrightarrow{a \triangleright \bar{\gamma}^B_M} & A \triangleright K(M \triangleleft B) & \xrightarrow{\gamma^A_{M \triangleleft B}} & K(A \triangleright (M \triangleleft B)) \\ \downarrow {\scriptstyle l_{A,KM,B}} & & & & \downarrow {\scriptstyle K(l_{A,M,B})} \\ (A \triangleright KM) \triangleleft B & \xrightarrow{\gamma^A_M \triangleleft B} & K(A \triangleright M) \triangleleft B & \xrightarrow{\bar{\gamma}^B_{A \triangleright M}} & K((A \triangleright M) \triangleleft B) \end{array} \qquad (3)$$

is required to commute for all objects (A, M, B) in $\mathcal{A} \times \mathcal{M} \times \mathcal{B}$.

Remark: the columns are invertible, so even if we change the direction either of γ or $\bar{\gamma}$, but not both we have a good requirement (inverting l if needed). This points to the fact that mixed colax-lax and lax-colax biequivariant functors make sense.

1.10.2. A (natural) transformation of lax $\mathcal{A}$-$\mathcal{B}$-biequivariant functors $\alpha : (K, \gamma, \bar{\gamma}) \Rightarrow (L, \delta, \bar{\delta})$ from $\mathcal{M}$ to $\mathcal{N}$ such that $\alpha : (K, \gamma) \Rightarrow (L, \delta)$ is a transformation of lax $\mathcal{A}$-equivariant functors, and also $\alpha : (K, \bar{\gamma}) \Rightarrow (L, \bar{\delta})$ a transformation of $\mathcal{B}$-equivariant functors.

1.11. Given three $\mathcal{A}$-$\mathcal{B}$-bi-actegories $\mathcal{M}, \mathcal{N}, \mathcal{P}$ with compatibility transformations $l^{\mathcal{M}}, l^{\mathcal{N}}, l^{\mathcal{P}}$ respectively, and lax $\mathcal{A}$-$\mathcal{B}$-biequivariant functors $(K, \gamma, \bar{\gamma}) : \mathcal{M} \to \mathcal{N}$, and $(L, \delta, \bar{\delta}) : \mathcal{N} \to \mathcal{P}$, their composition is defined by the formula $(L \circ K, \delta \star \gamma, \bar{\gamma} \star \bar{\delta})$. Here of course $(\delta \star \gamma)^a_m = (L\gamma \circ \delta K)^a_m := L(\gamma^a_m) \circ \delta^a_{Km}$. It is a standard fact that $((L \circ K, \delta \star \gamma)$ and $L \circ K, \bar{\gamma} \star \bar{\delta})$ are colax $\mathcal{A}$- and $\mathcal{B}$-equivariant respectively. Thus we need to check that $(L \circ K, \delta \star \gamma, \bar{\gamma} \star \bar{\delta})$ is compatible with the pair of distributive laws $l^{\mathcal{M}}, l^{\mathcal{P}}$. This follows from the commutativity of the diagram

$$
\begin{tikzcd}
a \triangleright (LKm \triangleleft b) \arrow[r, "l^{\mathcal{P}}_{a,LKm,b}"] \arrow[d, "a\triangleright\bar{\delta}^b_{Km}"'] & (a \triangleright LKm) \triangleleft b \arrow[r, "\delta^A_{KM}\triangleleft b"'] \arrow[rr, bend left, "(\delta\star\gamma)^a_m\triangleleft b"] & L(a \triangleright Km) \triangleleft b \arrow[r, "L\gamma^a_m\triangleleft b"'] \arrow[d, "\bar{\delta}^b_{a\triangleright Km}"'] & LK(a \triangleright m) \triangleleft b \arrow[d, "\bar{\delta}^b_{K(a\triangleright m)}"] \\
a \triangleright L(Km \triangleleft b) \arrow[r, "\delta^a_{Km\triangleleft b}"] \arrow[d, "a\triangleright L\bar{\gamma}^b_m"'] & L(a \triangleright (Km \triangleleft b)) \arrow[r, "Ll^{\mathcal{N}}_{a,Km,b}"'] \arrow[d, "L(a\triangleright\bar{\gamma}^b_m)"] & L((a \triangleright Km) \triangleleft b) \arrow[r, "L(\gamma^a_m\triangleleft b)"'] & L(K(a \triangleright m) \triangleleft b) \arrow[d, "L\bar{\gamma}^b_{a\triangleright m}"] \\
a \triangleright LK(m \triangleleft b) \arrow[r, "\delta^a_{Km\triangleleft b}"] \arrow[rr, bend right, "(\delta\star\gamma)^a_{m\triangleleft b}"'] & L(a \triangleright K(m \triangleleft b)) \arrow[r, "L\gamma^a_{m\triangleleft b}"] & LK(a \triangleright (m \triangleleft b)) \arrow[r, "LKl^{\mathcal{M}}_{a,m,b}"'] & LK((a \triangleright m) \triangleleft b)
\end{tikzcd}
$$

for all objects a, m, b in $\mathcal{A}, \mathcal{M}, \mathcal{B}$ respectively. The vertical axes are $a \triangleright (\delta \star \gamma)^b_m$ and $(\bar{\delta} \star \bar{\gamma})^b_{a\triangleright m}$ respectively, so external sides form the compatibility hexagon.

1.12. (Here write about restriction of actions along monoidal functors (pulling back)...the inducton should be a pseudoadjoint to the restriction. But one needs the correct 2-category to work in.)

Let $(J, \zeta, \xi) : (\mathcal{B}, \otimes, \mathbf{1}, a, l, r) \to (\mathcal{G}, \otimes', \mathbf{1}', a', l', r')$ be a monoidal functor, where $J : \mathcal{B} \to \mathcal{G}$ is an ordinary functor, and $\zeta : J \otimes' J \Rightarrow J \circ (\mathrm{Id}_{\mathcal{B}} \otimes \mathrm{Id}_{\mathcal{B}})$, $\xi : J \otimes' \mathbf{1}' \Rightarrow J \circ (\mathrm{Id}_{\mathcal{B}} \otimes \mathbf{1})$ are the coherence natural isomorphisms. Let $\Diamond_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be the natural action of the monoidal category $\mathcal{G}$ on itself, then the action $\triangleright_{\mathcal{B}} = \Diamond \circ (J \Diamond \mathrm{Id}_{\mathcal{G}}) : \mathcal{B} \times \mathcal{G} \to \mathcal{G}$ makes $\mathcal{G}$ a left $\mathcal{B}$-actegory. The component $\Psi^{\mathcal{B}}_{b,b',g}$ of the coherence $\Psi^{\mathcal{B}}$ for this left action is the composition

$$b \triangleright_{\mathcal{B}} (b' \triangleright_{\mathcal{B}} g) = Jb \otimes' (Jb' \otimes' g) \stackrel{a^{\mathcal{G}}_{Jb,Jb',g}}{\longrightarrow} (Jb \otimes' Jb') \otimes' g \stackrel{\zeta_{b,b'} \otimes g}{\longrightarrow} J(b \otimes b') \otimes' g = (b \otimes b') \triangleright_{\mathcal{B}} g$$

When it is clear from context we will simply write $\triangleright$ for $\triangleright_{\mathcal{B}}$.

1.12.1. More generally, let $\mathcal{N}$ be a left $\mathcal{G}$-actegory. Then it becomes a left $\mathcal{B}$-actegory as follows. The action functor is of course $\mathcal{B} \triangleright_{\mathcal{B}} \mathcal{N} := J \triangleright_{\mathcal{G}} \mathcal{N} : \mathcal{B} \times \mathcal{N} \to \mathcal{N}$. The action coherence component $\Psi^{\mathcal{B}}_{b,b',n}$ is the composition

$$b \triangleright_{\mathcal{B}} (b' \triangleright_{\mathcal{B}} n) = Jb \triangleright_G (Jb' \triangleright_{\mathcal{G}} n) \stackrel{\Psi^{\mathcal{G}}_{Jb,Jb',n}}{\longrightarrow} (Jb \otimes' Jb') \triangleright_{\mathcal{G}} n \stackrel{\zeta_{b,b'} \otimes n}{\longrightarrow} J(b \otimes b') \triangleright_{\mathcal{G}} n = (b \otimes b') \triangleright_{\mathcal{B}} n$$

We say that $\mathcal{N}$ is equipped with a restricted action of $\mathcal{B}$ via (J, ζ, ξ) and denote it $(J, \zeta, \xi)_*(\mathcal{N})$ or simply $J_*(\mathcal{N})$ or even ${}_{\mathcal{B}}\mathcal{N}$. It is easy to check that any $\mathcal{G}$-equivariant functor $(K, \gamma) : \mathcal{N} \to \mathcal{P}$ of $\mathcal{G}$-actegories, restricts to the $\mathcal{B}$-equivariant functor $J_*(K, \gamma) := (K, \gamma^J) : {}_{\mathcal{B}}\mathcal{N} \to {}_{\mathcal{B}}\mathcal{P}$, where the restricted coherence $\gamma^J : \mathcal{B} \triangleright_{\mathcal{B}} K \to K \circ (\mathcal{B} \triangleright_{\mathcal{B}} \mathcal{N})$ has components $(\gamma^J)^b_n = \gamma^{Jb}_n : b \triangleright_{\mathcal{B}} Kn = Jb \triangleright_{\mathcal{G}} Kn \to K(Jb \triangleright_{\mathcal{G}} n) = K(b \triangleright_{\mathcal{B}} n)$.

1.13. (Definition of pseudocoequalizers) In an arbitrary 2-category $\mathfrak{A}$, given a parallel pair of 1-cells $F, G : \mathcal{A} \to \mathcal{B}$ their pseudocoequalizer is a triple $(\mathcal{C}, p, \sigma)$ where $p : \mathcal{B} \to \mathcal{C}$ is a functor, $\sigma : p \circ F \Rightarrow p \circ G$ is an invertible 2-cell such that

- (1-dimensional aspect) For any other 1-cell $r : \mathcal{B} \to \mathcal{D}$ and an invertible 2-cell $\tau : r \circ F \Rightarrow r \circ G$ there is a unique 1-cell $v : \mathcal{C} \to \mathcal{D}$ such that $r = v \circ p$ and $\tau = v\sigma$. We say that 1-cell v is induced by universality (of the pseudocoequilizer).
- (2-dimensional aspect) Suppose we are given 1-cells $r, r' : \mathcal{B} \to \mathcal{D}$, a 2-cell $\gamma : r \Rightarrow r'$ and an invertible 2-cells $\tau : rF \Rightarrow rG$, $\tau' : r'F \Rightarrow r'G$ such that the diagram
$$\begin{tikzcd} rF \arrow[r, "\gamma F"] \arrow[d, "\tau"'] & r'F \arrow[d, "\tau'"] \\ rG \arrow[r, "\gamma G"] & r'G \end{tikzcd} \tag{4}$$
commutes. Let $v, v' : \mathcal{C} \to \mathcal{D}$ be the 1-cells induced by universality of the pseudocoequalizer satisfying $vp = r$ and $v'p = r'$. Then there is a unique 2-cell $\delta : v \Rightarrow v'$ such that $\delta \circ p$.

1.13.1. (Alternative formulation of universality of pseudocoequalizers) Consider for every object $\mathcal{D}$ in $\mathfrak{A}$ the category $\mathfrak{Ps}_{\mathcal{D}}^{F,G}$ whose objects are triples $(\mathcal{D}, r, \tau)$ where $r : F \Rightarrow G$ and $\sigma : rF \Rightarrow rG$ is invertible and whose morphisms $\delta : (\mathcal{D}, r, \tau) \to (\mathcal{D}', r', \tau')$ are those 2-cells $\delta : r \Rightarrow r'$ in $\mathfrak{A}$ for which $\delta G \circ \tau = \tau' \circ \delta F$. There is a natural "precomposition" functor

$$\begin{aligned}
&_ \circ (\mathcal{D}, r, \sigma) = \mathfrak{A}(\mathcal{D}, \mathcal{D}') \circ (\mathcal{D}, r, r \overset{\sigma}{\Rightarrow} r') : \mathfrak{A}(\mathcal{D}, \mathcal{D}') \to \mathfrak{Ps}_{\mathcal{D}'}^{F,G}, \\
&(\mathcal{D} \xrightarrow{v} \mathcal{D}') \mapsto v \circ (\mathcal{D}, r, \sigma) := (\mathcal{D}', \mathcal{B} \xrightarrow{vr} \mathcal{D}', vr \overset{v\sigma}{\Longrightarrow} vr'), \\
&(v \overset{\gamma}{\Longrightarrow} v') \mapsto (v \overset{\gamma}{\Longrightarrow} v') \circ (\mathcal{D}, r, \sigma) := (vr \overset{\gamma r}{\Longrightarrow} v'r).
\end{aligned}$$

By naturality of γ, one has $\gamma rG \circ v\tau = v'\tau \circ \gamma rF$ hence $\delta := \gamma r$ is indeed a morphism in $\mathfrak{Ps}_{\mathcal{D}'}^{F,G}$. Now the pseudocoequilizer $(\mathcal{C}, p, \sigma)$ is characterized by the statement that $\_ \circ (\mathcal{C}, p, \sigma) : \mathfrak{A}(\mathcal{C}, \mathcal{D}') \to \mathfrak{Ps}_{\mathcal{D}'}^{F,G}$ is an isomorphism of categories for every object $\mathcal{D}'$ in $\mathfrak{A}$. This replaces both the 1-dimensional and the 2-dimensional aspect above.

1.14. (Pseudocoequalizers in $\mathfrak{Cat}$) When the 2-category is $\mathfrak{Cat}$, the pseudocoequalizers exist for all parallel pairs of functors. The following explicit construction does the job.

First form a graph $\mathcal{C}_0$ whose vertices (objects) are the objects of $\mathcal{B}$ and which has the set of arrows $\mathrm{Mor}\,\mathcal{B} \coprod S \coprod S^{-1}$ where

(i) morphisms from $\mathcal{B}$ form a copy of the set $\mathrm{Mor}\,\mathcal{B}$

(ii) formal arrows $F(a) \xrightarrow{s_a} G(a)$, for all $a \in \mathrm{Ob}\,\mathcal{A}$ form a set S

(iii) formal arrows $G(a) \xrightarrow{s_a^{-1}} F(a)$ for all $a \in \mathrm{Ob}\,\mathcal{A}$ form a set S^{-1}

Then form a free category $\mathcal{C}_f$ on this graph with composition $\circ_f$. The category $\mathcal{C}$ is the quotient of $\mathcal{C}_f$ by the smallest equivalence relation $\sim$ on the set of morphisms containing the relations from $\mathcal{B}$ (in other words $h \circ_f h' \sim h \circ_{\mathcal{B}} h'$ for all pairs h, h' of morphisms in $\mathcal{B}$), the relations $s_a \circ s_a^{-1} \sim \mathrm{id}_{G(a)}$, $s_a^{-1} \circ s_a \sim \mathrm{id}_{F(a)}$ for all $a \in \mathrm{Ob}\,\mathcal{A}$ and $G(f) \circ s_a \sim s_{a'} \circ F(f)$ for all morphism $f : a \to a'$ in $\mathcal{A}$.

Let $[f]$ be the class in $\mathcal{C}$ of the morphism $f \in \mathrm{Mor}\,\mathcal{B}$. The tautological map $p : f \mapsto [f]$ is in fact a functor (which is identity on objects). The maps s_a are in fact components of a natural isomorphism of functors $s : pF \Rightarrow pG$. The triple $(\mathcal{C}, p, s)$ is in fact a pseudocoequalizer of the parallel pair F, G.

Given $r : \mathcal{B} \to \mathcal{D}$ and invertible $\tau : rF \Rightarrow rG$ as above, one defines first the functor $v_0 : \mathcal{C}_0 \to \mathcal{D}$ by $v_0(b) = r(b)$ for all objects b in $\mathcal{B}$, $v_0(f) = r(f)$ for all $f \in \mathrm{Mor}\,\mathcal{B}$, $v_0(s_a) = \tau_a$ and $v_0(s_a^{-1}) = \tau_a^{-1}$ for all $a \in \mathrm{Ob}\,\mathcal{A}$. This functor trivially extends to a functor $v_f : \mathcal{C}_f \to \mathcal{D}$ on the free category $\mathcal{C}_f$. Finally one checks that the extension v_f factors down to a (unique) functor $v : \mathcal{C} \to \mathcal{D}$. Then $r = vp$ and $\tau = vs$.

For the 2-dimensional aspect in $\mathfrak{Cat}$ we proceed as follows. The components of δ are identical to the components of γ, i.e. for all $b \in \mathrm{Ob}\,\mathcal{B}$, one has $\delta_b = \gamma_b$. The fact that these components form a natural transformation $\delta : v \Rightarrow v'$, includes both the naturality of γ and the identity (4): for each object $a \in \mathcal{A}$, and for each morphism $g : b \to b'$ in $\mathcal{B}$ the following diagrams commute:

$$\begin{tikzcd} v(F(a)) \arrow[r, "\delta_{F(a)}"] \arrow[d, "v([s_a])"'] & v'(F(a)) \arrow[d, "v'([s_a])"] \\ v(G(a)) \arrow[r, "\delta_{G(a)}"] & v'(G(a)) \end{tikzcd} \qquad \begin{tikzcd} v(b_1) \arrow[r, "\delta_{b_1}"] \arrow[d, "v(g)"'] & v'(b_1) \arrow[d, "v'(g)"] \\ v(b_2) \arrow[r, "\delta_{b_2}"] & v'(b_2) \end{tikzcd}$$

1.15. Lemma. *In $\mathfrak{Cat}$ pseudocoequalizers commute with finite products: given pseudocoequalizers $(\mathcal{C}_i, p_i, s_i)$ of parallel pairs $F_i, G_i : \mathcal{A}_i \to \mathcal{B}_i$ for all $i = 1, \ldots, n$, the pseudocoequalizer of the pair $\prod_i F_i, \prod_i G_i : \prod_i \mathcal{A}_i \to \prod_i \mathcal{B}_i$ may be chosen in the form $(\prod_i \mathcal{C}_i, \prod_i p_i, \prod_i s_i)$.*

Proof. We may assume that all $(\mathcal{C}_i, p_i, s_i)$ are realized as in the explicit construction above and we also construct such an explicit pseudocoequalizer $(\mathcal{E}, r, t)$ of $\prod_i F_i, \prod_i G_i$. It is sufficient to exhibit an isomorphism $q : \mathcal{E} \to \prod_i \mathcal{C}_i$ of categories for which $qr = \prod_i p_i$ and $qt = \prod_i s_i$. By construction the classes of objects both for $\mathcal{E}$ and $\prod_i \mathcal{C}_i$ coincide with $\mathrm{Ob}(\prod_i \mathcal{B}_i) = \prod_i \mathrm{Ob}(\mathcal{B}_i)$ and we choose q to agree with the identity map on objects. To define q on morphisms we first define the appropriate functor $q_f : \mathcal{E}_f \to \prod_i \mathcal{C}_i$ on the free category $\mathcal{E}_f$ (from the construction of pseudocoequalizer above) and then we check relation by relation that it descends to the quotient $\mathcal{E}$. The generating morphisms of $\mathcal{E}_f$ are either morphisms $\prod_i b_i$ from $\prod_i \mathcal{B}_i$, for such the map $q_f(\prod_i b_i) = \prod_i [b_i]_i$ where $[b]_i = p_i(b_i)$ is the class in $\mathcal{C}_i$ of the morphism b_i in $(\mathcal{C}_i)_f$; or the morphisms $s_{\prod_i a_i}$ where $\prod_i a_i$ is an *object* in $\mathcal{A}_i$; in the latter case we define $q_f(\prod_i a_i) = \prod_i [(s_i)_{a_i}]_i = \prod_i p_i((s_i)_{a_i})$. We leave to the reader to check that q_f descends to the map $q : \mathcal{E} \to \prod_i \mathcal{C}_i$ and that it is an isomorphism of the categories, and to realize that $qr = r = \prod_i p_i$ and $qt = t = \prod_i s_i$.

1.16. In an arbitrary 2-category consider the following diagram

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, shift left, "F_1"] \arrow[r, shift right, "G_1"'] \arrow[d, "q_{\mathcal{A}}"] \arrow[dr, phantom, "\swarrow \alpha{,}\beta"] & \mathcal{B}_1 \arrow[r, "p_1"] \arrow[d, "q_{\mathcal{B}}"] & \mathcal{C}_1 \arrow[d, dotted, "v"] \\
\mathcal{A}_2 \arrow[r, shift left, "F_2"] \arrow[r, shift right, "G_2"'] & \mathcal{B}_2 \arrow[r, "p_2"] & \mathcal{C}_2
\end{tikzcd} \tag{5}$$

We say that such a diagram is **sequentially commutative** if $v \circ p_1 = p_2 \circ q_B$ and the two cells written as the two labels of the "double" 2-cell in the left-hand square have the appropriate sources and targets, i.e. $\alpha : q_{\mathcal{B}} \circ F_1 \Rightarrow F_2 \circ q_{\mathcal{A}}$ and $\beta : q_{\mathcal{B}} \circ G_1 \Rightarrow G_2 \circ q_{\mathcal{A}}$. We use the convention not to write the second label β for the 'double' two cell on the left if it is the identity 2-cell.

Suppose that the upper fork (that is the part of the diagram consisting of arrows (F_1, G_1, p_1) is the pseudocoequalizer sequence. In other words, this line is equipped with a given invertible 2-cell $\sigma_1 : p_1 F_2 \Rightarrow p_1 G_1$ such that $(\mathcal{C}_1, p_1, \sigma_1)$ is the pseudocoequalizer of the pair (F_1, G_1). Further suppose that the lower fork (F_2, G_2, p_2) is also equipped with some invertible 2-cell $\sigma_2 : p_2 F_2 \Rightarrow G_2$ which does not need however be universal, i.e. we do not require the pseudocoequalizer downstairs. However we assume now that α and β are invertible, and do not assume that v in the diagram exists, but we construct it by the universality of the pseudocoequalizer.

To this aim, we contruct an invertible 2-cell $\tau : p_2 q_{\mathcal{B}} F_1 \Rightarrow p_2 q_{\mathcal{B}} G_1$ as the composition $p_2(\beta^{-1} \circ (\sigma_2 q_{\mathcal{A}}) \circ \alpha)$. Then $v : \mathcal{C}_1 \to \mathcal{C}_2$ is uniquely characterized by $v \circ (\mathcal{C}_1, p_1, \sigma_1) = (\mathcal{C}_2, p_2 q_{\mathcal{B}}, p_2(\beta^{-1} \circ (\sigma_2 q_{\mathcal{A}}) \circ \alpha))$. In particular, for any object a in $\mathcal{A}_1$, the morphism $v((\sigma_1)_a)$ is the composition

$$q_{\mathcal{B}}(F_1(a)) \xrightarrow{\alpha_a} F_2(q_{\mathcal{A}}(a)) \xrightarrow{(\sigma_2)_{q_{\mathcal{A}}(a)}} G_2(q_{\mathcal{A}}(a)) \xrightarrow{\beta_a^{-1}} q_{\mathcal{B}}(G_1(a)).$$

1.17. We want to describe the compatibility of biactegory construction and induction for actions of monoidal categories. In a project with U. Schreiber, we are studying the categorified associated bundles which in the simplest case boil down to an induction of the following type.

In the setting of **1.12**, the pseudocoequalizer

$$\begin{tikzcd} \mathcal{M}\times\mathcal{B}\times\mathcal{G} \arrow[r, shift left, "\triangleleft\times\mathcal{G}"] \arrow[r, shift right, "\mathcal{M}\times\triangleright"'] & \mathcal{M}\times\mathcal{G} \arrow[r] & \mathcal{M}\otimes_{\mathcal{B}}\mathcal{G} \end{tikzcd}$$

is equipped with the natural right $\mathcal{G}$-action, namely the unique bifunctor $(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{G})\times\mathcal{G}\to\mathcal{M}\otimes_{\mathcal{B}}\mathcal{G}$ extending the action bifunctor $\mathcal{M}\times(\mathcal{G}\otimes'\mathcal{G}):\mathcal{M}\times\mathcal{G}\times\mathcal{G}\to\mathcal{M}\times\mathcal{G}\hookrightarrow\mathcal{M}\otimes_{\mathcal{B}}\mathcal{G}$ and such that for the additional s-type morphisms $([s_{m,b,g}],g')\mapsto[s_{m,b,g\otimes'g'}]$ and for any morphism $\gamma:g_1\to g_2$ in $\mathcal{G}$

$$([s_{m,b,g}],\gamma)\mapsto[s_{m,b,g\otimes'g_2}]\circ((m\triangleleft b)\times(g\otimes'\gamma))=(m\times(b\triangleright(g\circ\gamma)))\circ[s_{m,b,g\otimes'g_1}].$$

The components of the associativity coherence for this action are the maps

$$(m,g)\triangleleft(g_1\otimes'g_2)=(m,g\otimes'(g_1\otimes'g_2))\stackrel{m\times a^{-1}_{g,g_1,g_2}}{\longrightarrow}(m,(g\otimes'g_1)\otimes'g_2)=((m,g)\triangleleft g_1)\triangleleft g_2.$$

This way we obtained the **induced $\mathcal{G}$-actegory** from $\mathcal{B}$-actegory $(\mathcal{M},\triangleleft,\Psi,u)$ via the monoidal functor (J,ζ,ξ). Though the notation surpresses some data it is usually denoted by $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}\mathcal{M}=\mathcal{M}\otimes_{\mathcal{B}}\mathcal{G}$.

1.17.1. We now want to extend $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}$ to a 2-functor what is not completely trivial; moreover we can define $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}$ only on pseudo-$\mathcal{B}$-coequivariant morphisms and not for lax or colax ones. The reason is the universal property of pseudocoequalizers which allows us only this.

Let $(F,\gamma):\mathcal{M}\to\mathcal{N}$ be a pseudo-$\mathcal{B}$-equivariant functor. Thus for each object M in $\mathcal{M}$ and B in $\mathcal{B}$, the component $\gamma_{M,B}:FM\triangleleft B\to F(M\triangleleft B)$ is invertible. We want to define a functor $(F,\gamma)\otimes_{\mathcal{B}}\mathcal{G}=\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}(F,\gamma):\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}\mathcal{M}\to\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}\mathcal{N}$. It is defined by the 1-dimensional aspect of the universality of pseudocoequalizer via the following sequentially commutative diagram

$$\begin{tikzcd} \mathcal{M}\times\mathcal{B}\times\mathcal{G} \arrow[r, shift left, "\triangleleft\times\mathcal{G}"] \arrow[r, shift right, "\mathcal{M}\times\triangleright"'] \arrow[d, "F\times\mathcal{B}\times\mathcal{G}"'] & \mathcal{M}\times\mathcal{G} \arrow[r, "p_1"] \arrow[d, "F\times\mathcal{G}"] \arrow[dl, phantom, "\gamma\times\mathcal{G}"] & \mathcal{M}\otimes_{\mathcal{B}}\mathcal{G} \arrow[d, dotted, "(F,\gamma)\otimes_{\mathcal{B}}\mathcal{G}"] \\ \mathcal{N}\times\mathcal{B}\times\mathcal{G} \arrow[r, shift left, "\triangleleft\times\mathcal{G}"] \arrow[r, shift right, "\mathcal{N}\times\triangleright"'] & \mathcal{N}\times\mathcal{G} \arrow[r, "p_2"] & \mathcal{N}\otimes_{\mathcal{B}}\mathcal{G} \end{tikzcd}$$

It is clear that $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}(F,\gamma)$ is a *strict* $\mathcal{G}$-equivariant functor. In our standard representation of pseudocoequalizers in $\mathfrak{Cat}$, it agrees with $F\times G$ on the morphisms in the image of p_1 and it sends $[s^{\mathcal{M}}_{m,b,g}]\mapsto[s^{\mathcal{N}}_{F(m),b,g}]\circ(\gamma^{-1}_{m,b}\times\mathrm{id}_g)$.

Given a natural transformation of pseudo-$\mathcal{B}$-equivariant functors $\alpha:(F,\gamma)\Rightarrow(F',\gamma')$ and using the 2-dimensional aspect of the universal property of pseudocoequalizers we clearly have a 2-cell $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}\alpha=\alpha\otimes_{\mathcal{B}}\mathcal{G}:\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}(F,\gamma)\to\mathrm{Ind}^{\mathcal{G}}_{\mathcal{B}}(F',\gamma')$.

1.18. If $\mathcal{M}$ is in addition equipped with a $\mathcal{A}$-$\mathcal{B}$-biactegory structure then $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$ inherits a left $\mathcal{A}$-action. The left $\mathcal{A}$-action on $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$ is the vertical dotted arrow $v_{\mathcal{A}}$ in the sequentially commutative diagram

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{M} \times \mathcal{B} \times \mathcal{G} \arrow[r, shift left, "\mathcal{A}\times\triangleleft\times\mathcal{G}"] \arrow[r, shift right, "\mathcal{A}\times\mathcal{M}\times\triangleright"'] \arrow[d, "\triangleright\times\mathcal{B}\times\mathcal{G}"] & \mathcal{A} \times \mathcal{M} \times \mathcal{G} \arrow[r, "p_1"] \arrow[d, "\triangleright\times\mathcal{G}"] \arrow[dl, phantom, "\Swarrow l\times\mathcal{G},\mathrm{id}"] & \mathcal{A} \times \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G} \arrow[d, dotted, "v_{\mathcal{A}}"] \\
\mathcal{M} \times \mathcal{B} \times \mathcal{G} \arrow[r, shift left, "\triangleleft\times\mathcal{G}"] \arrow[r, shift right, "\mathcal{M}\times\triangleright"'] & \mathcal{M} \times \mathcal{G} \arrow[r, "p_2"] & \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}
\end{tikzcd}$$

where both the lower and the upper forks are the pseudocoequalizer forks. We have

$$l \times \mathcal{G} : (\triangleright \times \mathcal{G}) \circ (\mathcal{A} \times \triangleleft \times \mathcal{G}) \Rightarrow (\triangleleft \times \mathcal{G}) \circ (\triangleleft \times \mathcal{B} \times \mathcal{G}),$$

$$(\triangleright \times \mathcal{G}) \circ (\mathcal{A} \times \mathcal{M} \times \triangleright) = (\mathcal{M} \times \triangleright) \circ (\triangleleft \times \mathcal{B} \times \mathcal{G}),$$

$$v_{\mathcal{A}} p_1 = p_2(\triangleright \times \mathcal{G}).$$

Notice that I do not distinguish between $\mathcal{A} \times (\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G})$ and $(\mathcal{A} \times \mathcal{M}) \otimes_{\mathcal{B}} \mathcal{G}$ and hence I drop the bracket. Here $\mathcal{A} \times \mathcal{M}$ is acted upon by the right $\mathcal{B}$-action $\mathcal{A} \times \triangleleft_{\mathcal{M}}$ and the reader may check that $\mathcal{A} \times (\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G})$ and $(\mathcal{A} \times \mathcal{M}) \otimes_{\mathcal{B}} \mathcal{G}$ are canonically isomorphic, e.g. using lemma 1.15. Moreover, this canonical isomorphism is compatible with the further actions and other maps in the calculations below.

1.19. In more complicated diagrams we will omit the cartesian product sign. In order to find the coherences for our candidate for the monoidal action $v_{\mathcal{A}} : \mathcal{A} \times \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G} \to \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$. we first consider the following pasting diagram:

$$\begin{tikzcd}
\mathcal{AAMB} \arrow[rrr, "\mathcal{AA}\triangleleft"] \arrow[dr, "\mathcal{A}\triangleright\mathcal{B}"] \arrow[ddd, "\otimes\mathcal{MB}"'] & & & \mathcal{AAM} \arrow[dl, "\mathcal{A}\triangleright"'] \arrow[ddd, "\otimes\mathcal{M}"] \\
& \mathcal{AMB} \arrow[r, "\mathcal{A}\triangleleft"] \arrow[d, "\triangleright\mathcal{B}"'] \arrow[ur, phantom, "\Swarrow_{\mathcal{A}l}"] \arrow[dl, phantom, "\mathcal{A}\Psi\Nearrow"] & \mathcal{AM} \arrow[d, "\triangleright"] \arrow[dr, phantom, "\Psi\Nwarrow"] & \\
& \mathcal{MB} \arrow[r, "\triangleleft"] \arrow[ur, phantom, "\Swarrow_{l}"] & \mathcal{M} & \\
\mathcal{AMB} \arrow[rrr, "\mathcal{A}\triangleleft"'] \arrow[ur, "\triangleright\mathcal{B}"'] \arrow[urr, phantom, "l\Nearrow"] & & & \mathcal{AM} \arrow[ul, "\triangleright"]
\end{tikzcd} \tag{6}$$

whose external square commutes, i.e. it is filled with an identity 2-cell. This amounts to the following equation among the natural transformations of functors $\triangleright \circ (\otimes\triangleleft) \Rightarrow \triangleleft(\triangleright\mathcal{B})(\mathcal{A} \triangleright \mathcal{B})$ holds:

$$(\triangleleft \circ \mathcal{A}\Psi)(l \circ \otimes\mathcal{MB}) = l(\triangleright \circ \mathcal{A}l)(\Psi \circ \mathcal{AA}\triangleleft).$$

Componentwise, this means that for any $a_1, a_2 \in \mathrm{Ob}\,\mathcal{A}$, $m \in \mathrm{Ob}\,\mathcal{M}$, $b \in \mathrm{Ob}\,\mathcal{B}$, the diagram

$$\begin{tikzcd}
(a_1 \otimes a_2) \triangleright (m \triangleleft b) \arrow[r, "\Psi_{a_1,a_2,m\triangleleft b}"] \arrow[d, "l^{a_1\otimes a_2}_{m,b}"'] & a_1 \triangleright (a_2 \triangleright (m \triangleleft b)) \arrow[r, "a_1\triangleright l^{a_2}_{m,b}"] & a_1 \triangleright ((a_2 \triangleright m) \triangleleft b) \arrow[d, "l^{a_1}_{a_2\triangleright m,b}"] \\
((a_1 \otimes a_2) \triangleright m) \triangleleft b \arrow[rr, "(\Psi_{a_1,a_2,m})\triangleleft b"] & & (a_1 \triangleright (a_2 \triangleright m)) \triangleleft b
\end{tikzcd}$$

commutes. This is one of the two symmetric pentagons in the definition of the distributive law between two monoidal actions.

1.20. Consider the following diagram whose rows are pseudocoequalizer sequences

$$\begin{tikzcd}
\mathcal{AAMBG} \arrow[rr, shift left, "\mathcal{AA}\triangleleft\mathcal{G}"] \arrow[rr, shift right, "\mathcal{AAM}\triangleright"'] \arrow[d, "\otimes\mathcal{MB}"] \arrow[ddr, "\mathcal{A}\triangleright\mathcal{BG}"] & & \mathcal{AAMG} \arrow[r, "p_3"] \arrow[d, "\otimes\mathcal{MG}"] \arrow[ddr, "\mathcal{A}\triangleright\mathcal{G}"] & \mathcal{AAM} \otimes_{\mathcal{B}} \mathcal{G} \arrow[d, dotted, "v_b"] \arrow[ddr, dotted, "v_u"] & \\
\mathcal{AMBG} \arrow[rr, shift left, "\mathcal{A}\triangleleft\mathcal{G}"] \arrow[rr, shift right, "\mathcal{AM}\triangleright"'] \arrow[ddr, "\triangleright\mathcal{BG}"] \arrow[dr, phantom, "\mathcal{A}\Psi\Nearrow"] & & \mathcal{AMG} \arrow[r, "p_4"] \arrow[ddr, "\triangleright\mathcal{G}"] \arrow[dl, phantom, "\swarrow\mathcal{A}l\mathcal{G}"] \arrow[d, phantom, "\Psi\Nearrow"] & \mathcal{AM} \otimes_{\mathcal{B}} \mathcal{G} \arrow[ddr, dotted, "v_d"] \arrow[dr, phantom, "\tilde{\Psi}\Nearrow"] & \\
& \mathcal{AMBG} \arrow[rr, shift left, "\mathcal{A}\triangleleft\mathcal{G}"] \arrow[rr, shift right, "\mathcal{AM}\triangleright"'] \arrow[d, "\triangleright\mathcal{BG}"] \arrow[r, phantom, "\swarrow l\mathcal{G}"] & & \mathcal{AMG} \arrow[r, "p_1"] \arrow[d, "\triangleright\mathcal{G}"] & \mathcal{AM} \otimes_{\mathcal{B}} \mathcal{G} \arrow[d, dotted, "v_f"] \\
& \mathcal{MBG} \arrow[rr, shift left, "\triangleleft\mathcal{G}"] \arrow[rr, shift right, "\mathcal{M}\triangleright"'] \arrow[r, phantom, "\swarrow l\mathcal{G}"] & & \mathcal{MG} \arrow[r, "p_2"] & \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}
\end{tikzcd} \tag{7}$$

with pseudocoequalizer 2-cells $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. In particular $\sigma_2 : p_2 \circ (\triangleleft \times \mathcal{G}) \Rightarrow p_2 \circ (\mathcal{M} \times \triangleright) : \mathcal{M} \times \mathcal{B} \times \mathcal{G} \to \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$. We want to show the existence of an invertible 2-cell $\tilde{\Psi} : v_d v_b \Rightarrow v_f v_u$ filling the most right/dotted square using the 2-dimensional aspect of pseudocoequalizer $\mathcal{AAM} \otimes_{\mathcal{B}} \mathcal{G}$. For this we need to show that that the diagram (4) commutes where [1]

$$F := \mathcal{A} \times \mathcal{A} \times \triangleleft \times \mathcal{G} : \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{B} \times \mathcal{G} \to \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{G}$$

$$G := \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \triangleright : \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{B} \times \mathcal{G} \to \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{G}$$

$$r := p_2(\triangleright \times \mathcal{G})(\otimes \times \mathcal{M} \times \mathcal{G}) : \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{G} \to \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$$

$$r' := p_2(\triangleright \times \mathcal{G})(\otimes \times \mathcal{M} \times \mathcal{G}) : \mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{G} \to \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$$

$$\tau := (\sigma_2(\triangleright \times \mathcal{B} \times \mathcal{G})) \circ l \times \mathcal{G})(\otimes \times \mathcal{M} \times \mathcal{B})$$

$$\tau' := (\sigma_2(\triangleright \times \mathcal{B} \times \mathcal{G} \circ \mathcal{A} \times \triangleright \times \mathcal{B} \times \mathcal{G}))(p_2(l \times \mathcal{G})(\mathcal{A} \times \triangleleft \times \mathcal{B} \times \mathcal{G}))(p_2(\triangleright \times \mathcal{G})(\mathcal{A} \times l \times \mathcal{G}))$$

[1] Here we used our usual convention that when we concatenate a sequence of functors and one natural transformation we do the whiskering, and if we concatenate two natural transformations we do the vertical composition. When we write $\times$ (do not concatenate) then the concatenation of functors is the composition. Finally, $\times$ is stronger binding than the compositions.

Decriptively, τ is obtained by pasting $l \times \mathcal{G}, \sigma_2$ and the identity 2-cell $(\otimes \times \mathcal{M} \times \mathcal{G}) \circ (\mathcal{A} \times \mathcal{A} \times \triangleleft \times \mathcal{G}) \Rightarrow (\mathcal{A} \times \triangleleft \times \mathcal{G}) \circ (\otimes \times \mathcal{M} \times \mathcal{B})$. Similarly, τ' is obtained by pasting $\mathcal{A} \times l \times \mathcal{G}$, $l \times \mathcal{G}$ and σ_2. Thus the diagram (4) becomes

$$\begin{tikzcd}
r \circ (\mathcal{A}\mathcal{A} \triangleleft \mathcal{G}) \arrow[r, "p_2(\Psi\mathcal{G})(\mathcal{A}\mathcal{A}\triangleleft\mathcal{G})"] \arrow[d, "\tau"'] & r' \circ (\mathcal{A}\mathcal{A} \triangleleft \mathcal{G}) \arrow[d, "\tau'"] \\
r \circ (\mathcal{A}\mathcal{A}\mathcal{M}\triangleright) \arrow[r, "p_2(\Psi\mathcal{G})(\mathcal{A}\mathcal{A}\mathcal{M}\triangleright)"] & r' \circ (\mathcal{A}\mathcal{A}\mathcal{M}\triangleright)
\end{tikzcd} \tag{8}$$

and the role of $\gamma : r \Rightarrow r'$ from (4) is played by $p_2(\Psi \times \mathcal{G}) : r \Rightarrow r'$. From diagram (7) we infer that the lower horizontal arrow in (8), equals $p_2(\mathcal{M}\triangleright)(\mathcal{A}\Psi\mathcal{G})(\otimes\mathcal{M}\mathcal{B})$. Thus the left and bottom arrow compose to the pasting of $\mathcal{A}\Psi\mathcal{G}, l\mathcal{G}$ and σ_2, while the top and right arrows compose to the pasting of $\mathcal{A}l\mathcal{G}, l\mathcal{G}, \Psi\mathcal{G}$ and σ_2. Therefore the square (8) commutes, because all except σ_2 exactly correpond to the pasting diagram (6) times $\mathcal{G}$, and pasting with σ_2 is added to both sides. By universality, r induces exactly the composition $v_d v_b$, r' induces $v_f v_u$ and finally $\tilde{\Psi}$ is induced by $p_2(\Psi \times \mathcal{G})$. In the standard representation of the pseudocoequalizers in $\mathfrak{Cat}$, the component $\tilde{\Psi}_{(a,a',m,g)}$ agrees with $\Psi_{a,a',m} \times \mathrm{id}_g$ for any object (a, a', m, g) in $\mathcal{A} \times \mathcal{A} \times \mathcal{M} \times \mathcal{G}$. Taking into account the natural isomorphisms $\mathcal{A}\mathcal{A}(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}) \cong \mathcal{A}\mathcal{A}\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$ and alike it is clear that that $\tilde{\Psi}$ serves as the coherence for the left action of $\mathcal{A}$ on $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$. The coherence pentagon may be checked componentwise, and indeed it boils down to the coherence pentagon for Ψ, in fact the whole pentagon times $\mathcal{G}$.

Finally, the fact that $\mathcal{M} \times \mathcal{G}$ has commuting left $\mathcal{A}$-action and right $\mathcal{G}$-action, means that this is true for $\mathcal{A} \times \mathcal{M} \otimes_{\mathcal{B}} \mathcal{G}$ (assuming the standard representation of the pseudocoequalizer), hence the distributive law is the identity.

1.21. More generally, this analysis may be done with almost no changes, for a general $\mathcal{B}$-$\mathcal{G}$-biactegory $\mathcal{N}$ instead of $\mathcal{G}$, and where $\mathcal{B}$ and $\mathcal{G}$ are not related, instead of the case $\mathcal{N} = \mathcal{G}$ as a $\mathcal{B}$-$\mathcal{G}$-biactegory.

Theorem. *Given three monoidal categories, $\mathcal{A}, \mathcal{B}$ and $\mathcal{G}$, the above procedure defines a bi-2-functor*

$$\otimes_{\mathcal{B}} : (\mathcal{A} - \mathfrak{act}^p - \mathcal{B}) \times (\mathcal{B} - \mathfrak{act}^p - \mathcal{C}) \to (\mathcal{A} - \mathfrak{act}^p - \mathcal{C})$$

1.22. Given any object $\mathcal{M}$ in $\mathcal{G} - \mathfrak{act} - \mathcal{B}$ there is a biequivariant equivalence $\mathcal{G} \otimes_{\mathcal{B}} \mathcal{M} \cong \mathcal{M}$ induced by the action $\triangleleft : \mathcal{M} \times \mathcal{B} \to \mathcal{B}$. Notice that $\mathcal{G} \otimes_{\mathcal{B}} \mathcal{M}$ in our standard representation has the identity as the (structural) distributive law. As a corollary, *every biactegory is equivalent to a biactegory with commuting actions.*

The equivalence is in fact the map v induced by the universality in

$$\begin{tikzcd} \mathcal{G}\times\mathcal{G}\times\mathcal{N} \arrow[r, shift left, "\otimes\times\mathcal{N}"] \arrow[r, shift right, "\mathcal{G}\times\triangleright"'] & \mathcal{G}\times\mathcal{N} \arrow[r, "p"] \arrow[dr, "\triangleright"'] & \mathcal{G}\otimes_{\mathcal{G}}\mathcal{N} \arrow[d, dotted, "v"] \\ & & \mathcal{N} \end{tikzcd}$$

with the 2-cell $\Psi : \triangleright(\otimes\times\mathcal{N}) \to \triangleright(\mathcal{G}\times\triangleright)$. The quasiinverse of v is $p\circ(\mathbf{1}_{\mathcal{G}}\times\mathcal{N})$. Indeed, $v\circ p\circ(\mathbf{1}_{\mathcal{G}}\times\mathcal{N}) = 1_{\mathcal{G}}\triangleright\mathcal{N} \overset{u^{-1}}{\cong} \mathcal{N}$. We need to show also $p\circ(\mathbf{1}_{\mathcal{G}}\times\mathcal{N})\circ v \cong \mathcal{G}\otimes_{\mathcal{G}}\mathcal{N}$ for what we may use the universality of pseudocoequalizer provided that $p\circ(\mathbf{1}_{\mathcal{G}}\times\mathcal{N})\circ v\circ p \cong p$. Indeed, we do have $p\circ(\mathbf{1}_{\mathcal{G}}\times\mathcal{N})\circ v\circ p = p\circ(\mathbf{1}_{\mathcal{G}}\times\triangleright)$ and we may observe the isomorphisms

$$p(g,n) \overset{p(u_g\times\mathrm{id}_n)}{\longrightarrow} p((\mathbf{1}_{\mathcal{G}}\otimes g), n) \overset{\sigma_{\mathbf{1}_{\mathcal{G}},g,n}}{\longrightarrow} p(\mathbf{1}_{\mathcal{G}}, (g\triangleright n)).$$

natural in objects g in $\mathcal{G}$ and n in $\mathcal{N}$. Here σ is the two cell of the pseudocoequalizer row. Notice that $\mathcal{N}$ is just equivalent and not isomorphic to $\mathcal{G}\otimes_{\mathcal{G}}\mathcal{N}$, and in particular $(\mathcal{N}, p, \Psi)$ is not itself a pseudocoequalizer of

$$\begin{tikzcd} \mathcal{G}\times\mathcal{G}\times\mathcal{N} \arrow[r, shift left, "\otimes\times\mathcal{N}"] \arrow[r, shift right, "\mathcal{G}\times\triangleright"'] & \mathcal{G}\times\mathcal{N}, \end{tikzcd}$$

but rather a representative of its bicoequalizer.

1.23. Theorem. *Given any monoidal functor $(J,\zeta,\xi) : \mathcal{B}\to\mathcal{G}$, the restriction 2-functor $(J,\zeta,\xi)_* : \mathcal{B} - \mathfrak{act}^p \to \mathcal{G} - \mathfrak{act}^p$ is a right pseudoadjoint to the induction 2-functor $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}} = {}_{-}\otimes_{\mathcal{B}}\mathcal{G} : \mathcal{G} - \mathfrak{act}^p \to \mathcal{B} - \mathfrak{act}^p$.*

The proof is rather involved, and adjunction is really in pseudo sense. We just sketch the definition of the pseudoadjunctions and leave the coherences for the 2-cells involved as an exercise to the reader. In this proof we work with the left actegories.

The components of the counit of the pseudoadjunction are the 2-functors $\epsilon_{\mathcal{N}} : \mathcal{G}\otimes_{\mathcal{B}}(J,\zeta,\xi)_*\mathcal{N} \to \mathcal{N}$ for all left $\mathcal{G}$-actegories $\mathcal{N}$ are defined as follows (in terms of the standard representation of pseudocoequalizers). On the image of the natural projection $p : \mathcal{G}\times\mathcal{N}\to\mathcal{G}\otimes_{\mathcal{B}}\mathcal{N}$, the functor $\epsilon_{\mathcal{N}}$ agrees with the application of the left $\mathcal{G}$ action on $\mathcal{N}$, i.e. $\epsilon_{\mathcal{N}}(\_,\_) = \_\triangleright\_$. Then set $[s_{g,b,m}] \mapsto \Psi^{\mathcal{G}}_{g,J(b),m}$. This amounts to extend $\epsilon_{\mathcal{N}}$ to the whole $\mathcal{G}\otimes_{\mathcal{B}}\mathcal{N}$ by considering what happens to $\sigma_{(g,b,m)}$ by the universality in the diagram

$$\begin{tikzcd} \mathcal{G}\times\mathcal{B}\times\mathcal{N} \arrow[r, shift left, "\triangleleft\times\mathcal{G}"] \arrow[r, shift right, "\mathcal{G}\times J\triangleright"'] \arrow[d, "\mathcal{G}\times J\times\mathcal{N}"'] & \mathcal{G}\times\mathcal{N} \arrow[r, "p_1"] \arrow[d, "\mathrm{Id}"] & \mathcal{G}\otimes_{\mathcal{B}}(J,\zeta,\xi)_*\mathcal{N} \arrow[d, dotted, "\epsilon_{\mathcal{N}}"] \\ \mathcal{G}\times\mathcal{G}\times\mathcal{N} \arrow[r, shift left, "\otimes\times\mathcal{N}"] \arrow[r, shift right, "\mathcal{G}\times\triangleright"'] & \mathcal{G}\times\mathcal{N} \arrow[r, "\triangleright"] & \mathcal{N} \end{tikzcd}$$

whose row forks are equipped with invertible 2-cells σ and $\Psi^{\mathcal{G}}$ respectively.

This counit is just a pseudonatural transformation. Namely, given a (pseudo) $\mathcal{G}$-equivariant transformation $j : \mathcal{N} \to \mathcal{P}$ with coherence υ, the square

$$\begin{tikzcd}
\mathcal{G} \otimes_{\mathcal{B}} (J, \zeta, \xi)_* \mathcal{N} \arrow[rr, "\epsilon_{\mathcal{N}}"] \arrow[d, "\mathcal{G} \otimes_{\mathcal{B}} j"'] \arrow[drr, phantom, "\overset{\upsilon}{\nearrow}"] & & \mathcal{N} \arrow[d, "j"] \\
\mathcal{G} \otimes_{\mathcal{B}} (J, \zeta, \xi)_* \mathcal{P} \arrow[rr, "\epsilon_{\mathcal{P}}"] & & \mathcal{P}
\end{tikzcd}$$

commutes up to an invertible two cell $\upsilon : \_ \triangleright j(\_) \to j(\_ \triangleright \_)$. To see this notice that at the level of objects, $\_ \triangleright j = \epsilon_{\mathcal{P}} \circ (\_ \otimes_{\mathcal{B}} j)$ and that υ is still natural with respect to $[s]$-morphisms.

1.24. (Induction in stages) Given monoidal functors $(J, \zeta, \xi) : \mathcal{B} \to \mathcal{C}$ and $(J', \zeta', \xi') : \mathcal{C} \to \mathcal{G}$ two-stage induction is equivalent to the induction for the composed monoidal functor $(J' \circ J, \zeta' \star \zeta, \xi' \star \xi)$. In other words, there is a natural equivalence of 2-functors

$$\mathrm{Ind}_{\mathcal{C}}^{\mathcal{G}} \circ \mathrm{Ind}_{\mathcal{B}}^{\mathcal{C}} \cong \mathrm{Ind}_{\mathcal{B}}^{\mathcal{G}}$$

where the monoidal functors are surpressed from the notation, as usual.

1.25. $\otimes_{\mathcal{B}}$ as above gives a monoidal product helping to define a tricategory of monoidal categories, biactegories between them, (pseudo-)biequivariant functors of biactegories, and natural transformations of biequivariant functors.

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