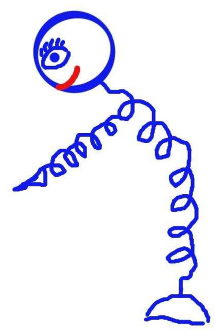
Computers Do Not Run on Logic

Mladen Pavičić

Physics Chair, Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia.

pavicic@grad.hr http://m3k.grad.hr/pavicic

The majority of books on classical computer logic and computation stop short of saying much about logic itself and dwell on the Boolean algebra—the main lattice model of classical logic—instead. Also when books and papers on quantum computers elaborate on quantum logic they actually speak of a Hilbert space algebra of quantum gates and circuits. It nevertheless often appears as if it were tacitly assumed that classical and quantum logics underlie classical and quantum computers, respectively. Can we really say that computers "run on logic?"



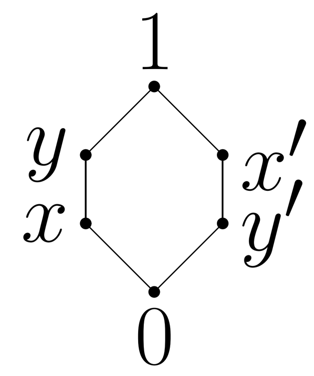
No, we cannot! Computers definitely "run on algebras" and not on logic. Until recently, however, the difference was subtle. Only one class of ortholattice models was known for either classical or quantum logic. Thus, for example, the numerical values of the Boolean algebra—a Boolean algebra is a distributive ortholattice—have been considered tantamount to truth values of logical propositions. Then in 1999 [1] we discovered that there is another ortholattice model for both classical and quantum logic. The new model for classical logic, e.g., is not numerical and logical propositions that correspond to its elements are non-numerical—they can be considered neither true nor false. Such an ortholattice obviously does not underlie today's computers and therefore logic in general also cannot be considered to underlie computers.

An *ortholattice* (OL), is defined by means of the following conditions: (1)$a\cup b = b\cup$; (2)$(a\cup b)\cup c = a\cup(b\cup c)$; (3)$a''=a$; (4)$a\cup(b\cup b')=b\cup b'$; (5)$a\cup(a\cap b)=a$; (6)$a\cap b=(a'\cup b')'$. We define the *greatest* (1) and *least* (0) *element* of the lattice: $1\overset{\rm def}{=}a\cup a'$, $0\overset{\rm def}{=}a\cap a'$, *ordering*: $a\le b \overset{\rm def}{\Longleftrightarrow} a\cap b=a$, *classical implication*: $a\to_0 b = a'\cup b$, *quantum implication*: $a\to_1 b=a'\cup(a\cap b)$, *equivalences*: $a\equiv_0 b=(a\to_0 b)\cap(b\to_0 a)$, $a\equiv b=(a\cap b)\cup(a'\cap b')$. Now, an OL is an *orthomodular lattice* (OML) if: $a\equiv_0 b \ \Leftrightarrow\ a=b$; a *Boolean algebra* (BA) if: $a\equiv b\ \Leftrightarrow\ a=b$, a non-orthomodular *weakly*-OML (WOML) *lattice* if $a\to_1 b=1\ \Rightarrow\ b'\to_1 a'=1$, and a non-distributive *weakly*-DL (WDL) *lattice* if $a\equiv_0 b=1\ \Rightarrow\ (a\cup c)\equiv_0(b\cup c)=1$. We discovered WOML in 1998 [1] and WODL in 1999 [2].

Quantum logic ($\mathcal{QL}$) contains the connectives $\to$, $\leftrightarrow$, $\equiv$, $\vee$, $\wedge$, $\neg$ which we represent with their lattice counterparts: $\to$, $\leftrightarrow$, $\equiv$, $\cup$, $\cap$, $'$. Its **axioms** are: (1) $\vdash A\vee B\equiv B\vee A$; (2) $\vdash A\vee(B\vee C)\equiv(A\vee B)\vee C$; (3) $\vdash A\equiv\neg\neg A$; (4) $\vdash\neg A\vee A\equiv(\neg A\vee A)\vee B$; (5) $\vdash A\vee(A\wedge B)\equiv A$; (6) $\vdash(A\wedge B)\equiv\neg(\neg A\vee\neg B)$; and the **rules of inference** (R1) $\vdash A\equiv B\Rightarrow\vdash A\vee C\equiv B\vee C$; (R2) $\vdash A\equiv B\ \&\vdash B\equiv C\Rightarrow\vdash A\equiv C$; (R3) $\vdash A\equiv B\Leftrightarrow\vdash\neg A\equiv\neg B$; (R4) $\vdash A\equiv B\Rightarrow\vdash B\equiv A$; (R5) $\vdash\neg A\vee A\equiv B\Leftrightarrow\vdash B$, where $\vdash A$ means "A is provable," i.e., "A is a theorem." *Classical logic* ($\mathcal{CL}$) has the **axioms**: (1) $\vdash A\vee A\to_0 A$; (2) $\vdash A\to_0 A\vee B$; (3) $\vdash A\vee B\to_0 B\vee A$; (4) $\vdash(A\to_0 B)\to_0(C\vee A\to_0 C\vee B)$; and the **rule of inference** (*Modus Ponens*) (R1) $\vdash A\ \&\ A\to_0 B\ \Rightarrow\vdash B$.

In 1999 we proved the soundness and completeness of $\mathcal{QL}$ for both of their ortholattice models: orthomodular OML and non-orthomodular WOML and of $\mathcal{CL}$ for both of their ortholattice models: distributive BA and non-distributive WDOL [2,3,4].

To prove the soundness of $\mathcal{QL}$ and $\mathcal{CL}$ for both WOML and OML and for both WDOL and BA, respectively, means to show that all axioms as well as the rules of inference (and therefore all theorems) from $\mathcal{QL}$ and $\mathcal{CL}$ hold in both WOML and OML and in both WDOL and BA, respectively. We carried out the proof in detail in 1999 in [2] and reviewed it in [3,4]. The task of proving the completeness of $\mathcal{QL}$ and $\mathcal{CL}$ is the opposite one: we have to impose the structure of both WOML and OML and of both WDOL and BA on the set of formulae of $\mathcal{QL}$ and $\mathcal{CL}$, respectively. We also carried out the proof in detail in 1999 in [2] and reviewed it in [3,4] but will still discuss some points here. It is well-known that an OL is an OML if and only if it fails in O6 lattice (*benzene ring*, *hexagon*) shown in the figure on the right. BA is orthomodular and it also fails in O6. However, neither WOML nor WDOL fail in O6. Moreover, in 2005, following our elaboration carried out in [2], E. Schechter proved [5] that O6 alone can serve as a model for classical logic. We use O6 in the completeness proofs of $\mathcal{QL}$ and $\mathcal{CL}$ for WOML and WDOL, respectively—O6 will filter out all orthomodular lattices.



The reasoning behind our unexpected proofs of soundness and completeness of quantum and classical logics for non-orthomodular and non-distributive ortholattice models is the following one. When a theorem A holds in $\mathcal{QL}$ or $\mathcal{CL}$, i.e., when A is provable in $\mathcal{QL}$ or $\mathcal{CL}$, then it is valid in one of their models ($\mathcal{M}$), i.e., in our ortholattices, and that means that $a=h(A)=1$ for valuations on the model $\mathcal{M}$. We often write that as $\vDash_{\mathcal{M}} A$ or simply $\vDash A$. So to prove the soundness it would actually suffice to map all logical axioms and rules of inference to $a=1$ conditions. However, the algebra we would obtain would be very weak—it would not be even an ortholattice. Therefore we use ortholattice OL instead. Thus we first map $\mathcal{QL}$ axioms 1-6 to $a\equiv b=1$ to OL. Since $a=b\ \Rightarrow\ a\equiv b=1$ holds in any OL it follows from OL 1-6 conditions that the $\mathcal{QL}$ axiom mappings hold in OL. Then we map R1-R5 in the form $a=1$ and arrive at WOML [2]. The standard approach uses OML which is much stronger then required. We apply an analogous procedure to $\mathcal{CL}$ to arrive at WDOL. To prove the completeness of $\mathcal{QL}$ we have to impose the structure of WOML on the set $\mathcal{F}^\circ$ of formulae of $\mathcal{QL}$. For that we first define $\mathcal{O}6$ as the set of mappings that do not let the orthomodularity through. Then we define the *relation of equivalence* as $A\approx B\overset{\rm def}{=}\Gamma\vdash A\equiv B\ \ (\forall o\in\mathcal{O}6)[(\forall X\in\Gamma)(o(X)=1)\Rightarrow o(A)=o(B)]$ and prove that it is a *relation of congruence* in the algebra $\mathcal{F}$, where $\Gamma\subseteq\mathcal{F}^\circ$. We also define the *equivalence class* $|A|=\{B\in\mathcal{F}^\circ: A\approx B\}$. We denote $\mathcal{F}^\circ/\approx\,=\{|A|: A\in\mathcal{F}^\circ\}$. The equivalence classes define the natural morphism $f:\mathcal{F}^\circ\longrightarrow\mathcal{F}^\circ/\approx$, which gives $f(A)\overset{\rm def}{=}|A|$. We write $a=f(A)$, $b=f(B)$, etc. Now, The relation $a=b$ on $\mathcal{F}^\circ/\approx$ is given by: $|A|=|B|\ \Leftrightarrow\ A\approx B$, and the *Lindenbaum algebra* $\mathcal{A}=\langle\mathcal{F}^\circ/\approx,\neg/\approx,\vee/\approx\rangle$ is a WOML, i.e., the conditions (1)-(6) and WOM condition hold for $\neg/\approx$ and $\vee/\approx$ as $'$ and $\cup$. Next we prove that in the Lindenbaum algebra $\mathcal{A}$, if $f(X)=1$ for all X in Γ implies $f(A)=1$, then $\Gamma\vdash A$ and we obtain the completeness: $\Gamma\vDash A\ \Rightarrow\ \Gamma\vdash A$. Analogous completeness we obtain for $\mathcal{CL}$ for its WDOL model.

Hence, we can prove all theorems from $\mathcal{QL}$ and $\mathcal{CL}$ in OLs that are much weaker than OML and BA, respectively, i.e., from WOML and WDOL and we can recover all these theorems again from WOML and WDOL (OM does not hold in the Lindenbaum algebra $\mathcal{A}$). Are WOML and WDOL special? Are there other such models? No, they are not special. We have shown in [6] that there are other OLs between WOML and OML and between WDOL and BA for which soundness and completeness of $\mathcal{QL}$ and $\mathcal{CL}$ can be proved. Possibly infinitely many of them. Actually we can say that logics are *valuation-nonmonotonic* [6] in the sense that their possible models (corresponding to their possible hardware implementations) and the valuations for them drastically change when we add new conditions to their defining conditions. Hardware implementations? With, e.g., 6 possible values for propositions as in O6? Well, it would not be difficult to make such logic gates. It is only a question whether such computers could find an application.

[1] M. Pavičić and N. D. Megill, Binary Orthologic with Modus Ponens Is either Orthomodular or Distributive, *Helvetica Physica Acta*, **72**, 189–210 1999; [2] M. Pavičić and N. D. Megill, Non-Orthomodular Models for Both Standard Quantum Logic and Standard Classical Logic: Repercussions for Quantum Computers, *Helvetica Physica Acta* **72**, 189–210 1999; [3] M. Pavičić, *Quantum Computation and Quantum Communication: Theory and Experiments*, Springer, New York (2005); [4] M. Pavičić and N. D. Megill, *Is Quantum Logic a Logic* in *Handbook of Quantum Logic and Quantum Structures: Quantum Logic*, Eds. K. Engesser, D. Gabbay and D. Lehmann, Elsevier, Amsterdam (2008); [5] E. Schechter, *Classical and Nonclassical Logics: An Introduction to the Mathematics of Propositions*, Princeton University Press, Princetom (2005); [6] M. Pavičić and N. D. Megill, Standard Logics Are Valuation-Nonmonotonic, *Journal of Logic and Computation*, **exn018**, 1-24 (2008);