MMP Algorithms for Kochen-Specker Vectors

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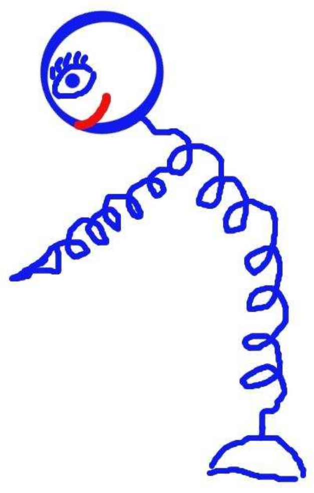
In 1967 S. Kochen and E. P. Specker formulated their famous theorem according to which there are quantum measurements that differ from any classically conceivable counterparts. Such setups might be important for the would-be quantum computation since they might filter out possible inapplicable classical solutions to particular quantum problems. However, until recently only a few dozens sets of Kochen-Specker vectors were known since their discovery depended on the ingenuity and intuition of the researchers who found them over the last 40 years. Can we find algorithms that could generate Kochen-Specker vectors exhaustively and at will?



Yes! We can. Of course a head-on approach (for instance by simply trying to solve zillions of equations describing mutual orthogonalities of Hilbert space vectors on all available classical supercomputers and clusters) cannot be applied here—at least not within a time comparable with the age of Universe. Our idea was to represent Hilbert space vectors and nonlinear equations by linear graphs and then generate only those graphs that would satisfy the appropriate orthogonality requirements, then filter out only those ones that satisfy the Kochen-Specker condition, turn them back into equations, and eventually apply linear interval analysis to obtain hundreds of Kochen-Specker vectors that of course include all known humanly discovered examples with the chosen parameters. [1-4]

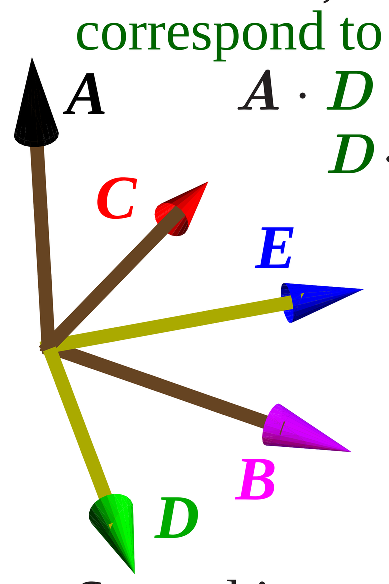
Kochen-Specker Vectors

In $\mathcal{H}^n$, $n \geq 3$, KS vectors are those vectors to which it is impossible to assign 1s and 0s in such a way that
(1) no two of mutually orthogonal vectors are both assigned 1;
(2) 0 is not assigned to all of them.

MMP Diagram Algorithm

1. Every vertex belongs to at least one edge;
2. Every edge contains at least 3 vertices;
3. Every edge, which intersects with another edge at most twice, contains at least 4 vertices.

Vectors below correspond to vertices shown in Fig. 1 and their orthogonalities: $\boldsymbol{A}\cdot\boldsymbol{B}=0$, $\boldsymbol{A}\cdot\boldsymbol{C}=0$, and $\boldsymbol{B}\cdot\boldsymbol{C}=0$ correspond to the edge $\boldsymbol{A}$—$\boldsymbol{B}$—$\boldsymbol{C}$ and $\boldsymbol{A}\cdot\boldsymbol{D}=0$, $\boldsymbol{A}\cdot\boldsymbol{E}=0$, and $\boldsymbol{D}\cdot\boldsymbol{E}=0$ to $\boldsymbol{A}$—$\boldsymbol{D}$—$\boldsymbol{E}$



Edges therefore also correspond to nonlinear equations because, e.g., $\boldsymbol{B}\cdot\boldsymbol{C} = B_xC_x + B_yC_y + B_zC_z = 0$, etc.

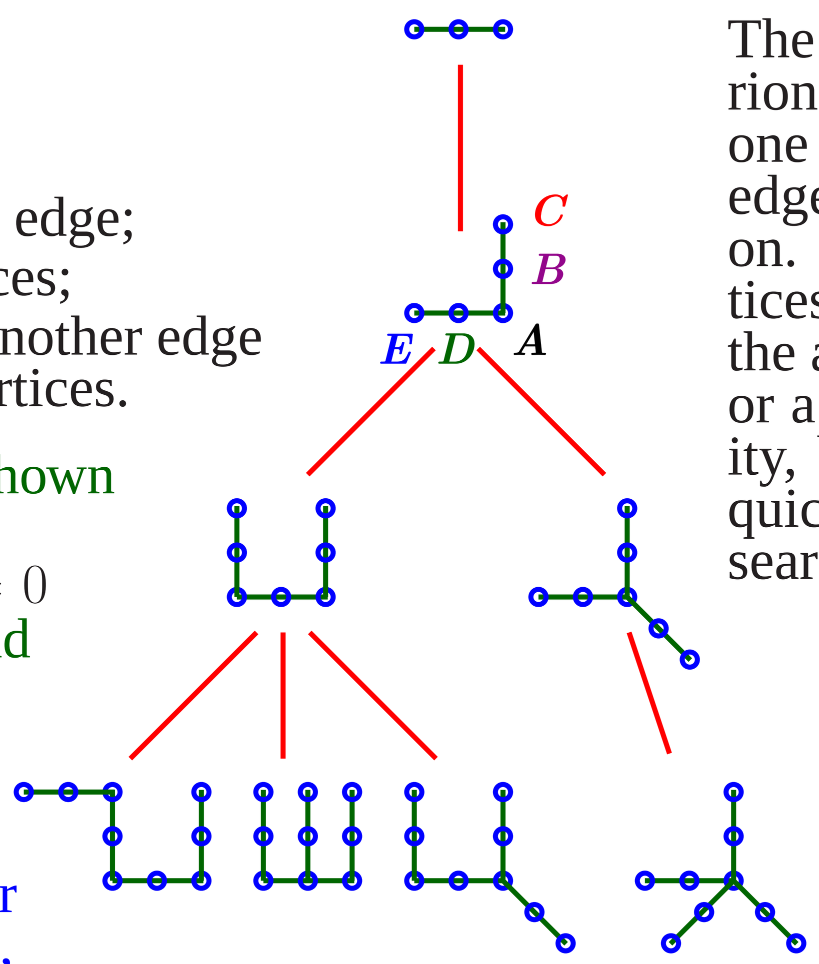


Fig. 1 Generation tree for non-isomorphic MMP diagrams

No 0-1 State Algorithm

The algorithm is exhaustive search of MMP diagrams with backtracking. The criterion for assigning non-dispersive (0-1) states is that each edge must contain exactly one vertex assigned to 1, with the others assigned to 0. As soon as a vertex on an edge is assigned a 1, all other vertices on that edge become constrained to 0, and so on. The algorithm scans the vertices in some order, trying 0 then 1, skipping vertices constrained by an earlier assignment. When no assignment becomes possible, the algorithm backtracks until all possible assignments are exhausted (no solution) or a valid assignment is found. In principle the search is of exponential complexity, but because the diagrams of interest are tightly coupled, constraints build up quickly. The algorithm uses this feature to avoid the exponential behaviour of the search.

The Main Non-Linear Orthogonality Algorithm

The number of vertices within edges corresponds to the dimension of $\mathbb{R}^n$ and that edges correspond to $n(n-1)/2$ equations resulting from inner products of vectors being equal to zero which mean their orthogonality. Each possible combination of edges for a chosen number of vectors (vertices) and edges makes a diagram which corresponds to a system of such nonlinear equations. However, before we come to this stage we first do not generate "impossible" systems at all and then we filter out those graphs that do not pass the above 0-1 state algorithm. So, only systems, i.e., set of equations that survive such massive filtering are then treated by the interval analysis and yet another filtering.

Interval Analysis Self-Teaching Algorithms for Solving Nonlinear Equations

Several interval analysis algorithms are used. For avoiding the exponential growth of the number of generated MMP diagrams KS-system are generated incrementally, i.e., sequentially, starting with a given m n-tuples before modifying the mth n-tuple. Thus as soon as the preliminary pass determines that an initial set of m n-tuples has no solution and that no further systems starting with this set will be generated. E.g., for 18 vectors and 12 quadruples without such a filter one should generate $> 2.9\cdot 10^{16}$ systems—what would require more than 30 million years on a 2 GHz CPU—while the filter reduces the generation to 100220 systems (obtainable within < 10 mins on a 2 GHz CPU). Thereafter no-0-1-state-algorithm gives us 22 systems without 0-1 states in < 5 secs. For the remaining systems two algorithms have been developed which gives 1 solution (see Fig. 2) within less then 1 sec.

Chosen results

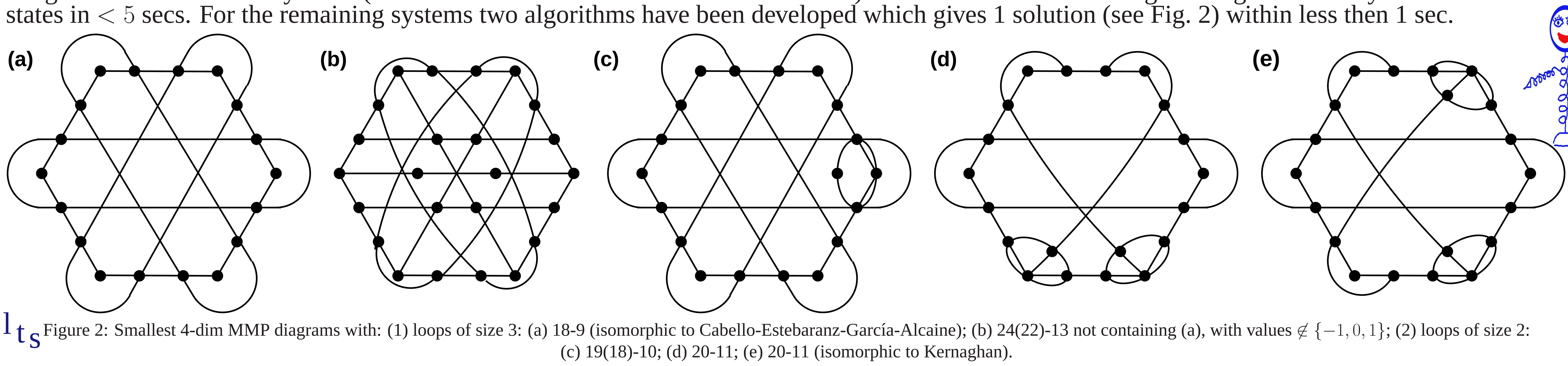


Figure 2: Smallest 4-dim MMP diagrams with: (1) loops of size 3: (a) 18-9 (isomorphic to Cabello-Estebaranz-García-Alcaine); (b) 24(22)-13 not containing (a), with values $\notin \{-1, 0, 1\}$; (2) loops of size 2: (c) 19(18)-10; (d) 20-11; (e) 20-11 (isomorphic to Kernaghan).

[1] Pavičić, M., Quantum Computers, Discrete Space, and Entanglement, *SCI 2002*, Orlando, **Vol. XVII**, pp. 65-70 (2002)
[2] Pavičić, M., J.-P. Merlet, B.D. McKay, and N.D. Megill, Kochen-Specker Vectors, *Journal of Physics* **A 38**, 1577-1592 (2005); **38**, 3709 (2005)
[3] Pavičić, M., J.-P. Merlet, and N.D. Megill, Exhaustive enumeration of Kochen-Specker vector systems, *INRIA* RR-5388 (2005)
[4] Pavičić, M., *Quantum Computation and Quantum Communication: Theory and Experiments*, Springer, New York (2005)