Quantum Algebras, Systems, and Computation

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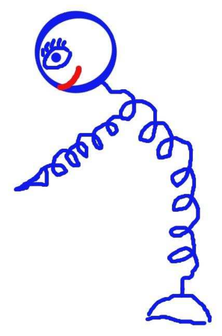
Classical computers are based on *bits* that correspond to two states of classical logic gates that are in effect just switches for the current passing through a computer: 1 (*open*) and 0 (*closed*). Quantum computers are based on infinitely many superpositions $\alpha|1\rangle+\beta|0\rangle$ of two basic states ($|1\rangle$ and $|0\rangle$) of quantum bits—*qubits*, i.e, quantum systems (photons, electrons, atoms,...). As opposed to digitalized classical computation and their exponentially growing 2^n gate passages for n bits, quantum computation can make use of that infinitely many possible states of a single qubit to speed up calculations exponentially. But so far that has been successfully devised only for specially designed algorithm for some problems. Can we find a general quantum algebra that would correspond to the Boolean algebra of classical computers?

A single axiom definition of *ortholattices* by means of the *Sheffer stroke* ($|$) reads [1]: $(((b|a)|(a|c))|d)|(a|((a|((b|b)|b))|c)) = a$.
Standard operations can be defined as follows: *disjunction*: $a\cup b = (a|a)|(b|b)$; *negation*: $a' = a|a$; *conjunction*: $a\cap b = (a'\cup b')'$; *classical implication*: $a\to_0 b = a'\cup b$; in any orthomodular lattice there are five (G. Kalmbach, 1974) *quantum implications*: $a\to_i b$, $i=1,\dots,5$, e.g., *Sasaki implication*: $a\to_1 b = a'\cup(a\cap b)$)—all five reduce to $a\to_0 b$ when we add the distributivity to an ortholattice, i.e., in a Boolean algebra. We also have $a\le b \Leftrightarrow^{\rm def} a\cup b = b$; $1 =^{\rm def} a\cup a'$; $0 =^{\rm def} 1'$.



Well, we do not know yet. We know that we can start with a very general algebra called *ortholattice* and then add axioms until we obtain an algebra called *Hilbert lattice* which is isomorphic to an *infinite dimensional Hilbert space* in which we can describe any quantum system. If we add more axioms we shall get *modular lattice* isomorphic to a *finite dimensional Hilbert space* of spin systems in which we can describe *qubits* as they pass *quantum gates* in any *quantum circuit*, i.e., *quantum computer*. Eventually, by adding the distributivity we get the *Boolean algebra*, i.e., *distributive lattice*. Now, every classical digitalized calculation can be implemented into a quantum computer because any Boolean algebra is modular. But with quantum calculation we might achieve much more, if we succeeded in approximating Hilbert lattices by modular lattices, i.e., substituting finite dimensional Hilbert space for infinite dimensional one. Hilbert space infinite dimensionality stems from space continuum, e.g., positions of electrons, protons, and neutrons in molecules. Hence, such an approximation would enable us to directly simulate molecules. In our approach to this program we find new lattice conditions and new infinite series of them that could eventually substitute for the standard definition of the Hilbert lattice. (Standard definition cannot be implemented into a quantum computer because it contains quantificators: *for all* and *there exists* which have no operational meaning.) If successful, we would consider cutting off series at particular points and carrying out the afore mentioned approximation.

Here are some "old" definitions. An ortholattice in which *distributivity* (D), *modularity* (M), *orthomodularity* (OM),
$\boxed{{\rm D}:\quad a\cap(b\cup c) = (a\cap b)\cup(a\cap c)}$, $\boxed{{\rm M}:\quad b\le a\ \Rightarrow\ a\cap(b\cup c) = (a\cap b)\cup(a\cap c)}$, $\boxed{{\rm OM}:\quad b\le a\ \&\ c\le a'\ \Rightarrow\ a\cap(b\cup c) = (a\cap b)\cup(a\cap c)}$,
hold is a *Boolean algebra* (BA), *modular lattice* (ML), *orthomodular lattice* (OML), respectively.

In 1987 [2] we discovered that an ortholattice in which

$\boxed{a\to_0 b\ \Leftrightarrow\ a\le b}$, $\boxed{a\to_i b\ \Leftrightarrow\ a\le b}$,

holds is a BA, OML, respectively, and vice versa.

In 1985 L. Beran proved that there are 96 one and two variable expressions in any orthomodular lattice.

In 1990 Malinowski proved that orthomodular lattices do not admit the deduction theorem. So, do we need operations of implication at all? Let us define *equivalences*:

$\boxed{a\equiv_0 b = (a\to_0 b)\cap(b\to_0 a)}$

$\boxed{a\equiv_i b = (a\cap b)\cup(a'\cap b'),\ i=1,..,5}$

Yes! In 1993 [3] and 1998 [4] we proved the conjecture: an ortholattice in which

$\boxed{a\equiv_0 b\ \Leftrightarrow\ a=b}$, $\boxed{a\equiv_i b\ \Leftrightarrow\ a=b}$, $i=1,..,5$

holds is a BA, OML, respectively, and vice versa. In any OML $a\equiv_i b = (a\to_i b)\cap(b\to_i a)$, $i=1,..,5$ hold. (This agrees with 1930 J. Herbrand's result that we can do without the deduction theorem in classical logic/lattices too.) In 1999 [5] we also proved

$\boxed{a\equiv_{ij} b\ \Leftrightarrow\ a=b}$, $i,j=1,..,5$

where $a\equiv_{ij} b = (a\to_i b)\cap(b\to_j a)$, $i,j=1,..,5$

In 2001 [6] we devised algorithms which enabled us to express two variable expression by means of all other five expressions of the same category in an identical way. We also constructed algebras which have the same axioms of an identical form for all five operations. For example $\boxed{a\cup b = ((((b\to_i a)\to_i (a\to_i b))\to_i b)\to_i a)\to_i a}$, $i=1,\dots,5$

In 2002 [7] we have discovered that all 80 one and two variable "quantum" expressions in any orthomodular lattice are fivefold defined. They all reduce to classical counterparts (16 altogether) in BA. So even constants (0,1) and variables (a,b) are fivefold defined. One of "quantum" 1's is, e.g., $((a\cap b)\cup(a\cap b'))\cup((a'\cap b)\cup(a'\cap b'))$ because this expression reduces to 1 in BA. $\boxed{((a\cup b)\cap(a\cup b'))\cap((a'\cup(a\cap b))\cup(a\cap b'))}$ is one of "quantum" variables because it reduces to a in BA. We also found algorithms which express both quantum and classical operations by means of any other in an identical way. For example $\boxed{a\cup_1 b = (a\cup_i(b\cap_i(a\cup_i(a\cap_i b))'))}$, $i=0,\dots,5$ Note that now i includes 0 as well, i.e., we can have identical expression of, e.g., $a\cup_1 b$ by means of $a\cup b$, $a\cup_1 b,\dots, a\cup_5 b$. We used this result to define algebras with such "merged" operations. We also obtained several other new algebras in 2003 [8,9].

From 2000 till 2008 [10,11,12,13] we worked on generation of infinite series of equations in Hilbert lattices and obtained several important new results for Godowski's and Mayet's equations and completely new infinite series of *generalised orthoarguesian equations* (nOA). We define nOA equations as follows:
$\boxed{(a_1\to a_3)\cap(a_1\overset{(n)}{\equiv}a_2)\le a_2\to a_3}$ where $a_1\overset{(3)}{\equiv}a_2 \overset{\rm def}{=} ((a_1\to a_3)\cap(a_2\to a_3))\cup((a_1'\to a_3)\cap(a_2'\to a_3))$ $a_1\overset{(n)}{\equiv}a_2 \overset{\rm def}{=} (a_1\overset{(n-1)}{\equiv}a_2)\cup((a_1\overset{(n-1)}{\equiv}a_n)\cap(a_2\overset{(n-1)}{\equiv}a_n))$, $n\ge 4$.
Previously known (Day's, Greechie's, Godowski's) orthoarguesian equations ("laws") are either our 3OA or 4OA. Using our algorithms and programs that run on our clusters we proved [13] that nOA are strictly stronger than $(n-1)$OA for $n\ge 7$. Some new recently obtained results will be published soon.

[1] W. McCune *et al.*, *Algebra universalis* **52**, 541-549 (2004); [2] M. Pavičić, *International Journal of Theoretical Physics (IJTP)*, **26**, 845–852 (1987); [3] M. Pavičić, *IJTP*, **32**, 1481–1505 (1993); [4] M. Pavičić, *IJTP*, **37**, 2099–2103 (1998); [5] M. Pavičić and N. D. Megill, *Helvetica Physica Acta* **72**, 189–210 1999; [6] N. D. Megill and M. Pavičić, *IJTP*, **40**, 1387–1410 (2001); [7] N. D. Megill and M. Pavičić, *Foundations of Physics*, **32**, 357–378 (2002); [8] N. D. Megill and M. Pavičić, *IJTP*, **42**, 2797–2805 (2003). [9] N. D. Megill and M. Pavičić, *IJTP*, **42**, 2825–2840 (2003). [10] N. D. Megill and M. Pavičić, *IJTP*, **39**, 2337–2379 (2000). [11] B. D. McKay and N. D. Megill and M. Pavičić, *IJTP*, **39**, 2381–2406 (2000). [12] M. Pavičić, *Quantum Computation and Quantum Communication: Theory and Experiments*, Springer, New York (2005); [13] M. Pavičić and N. D. Megill, *Quantum Logic and Quantum Computation* in *Handbook of Quantum Logic and Quantum Structures*, Eds. K. Engesser, D. Gabbay and D. Lehmann, pp. 751-787, Elsevier, Amsterdam (2007);