Massive Generation of Contextual Quantum Sets

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EMN Meeting on QCQI, Aug 23-26, 2016, Berlin, Germany.

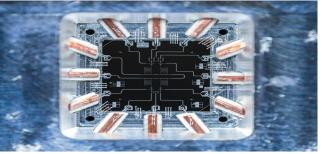
EMN QCQI-2016



Quantum Computation Perspectives

NEWS IN FOCUS

426 | NATURE | VOL 532 | 28 APRIL 2016



A €1-billion (US\$1.1-billion) European flagship project could advance the state of quantum computing

_ F U ND ING

Billion-euro boost for quantum tech

Third European Union flagship project will be similar in size and ambition to graphene and human-brain initiatives.

Quantum Computation Importance

QUANTUM COMPUTING

Powered by magic

What gives quantum computers that extra oomph over their classical digital counterparts? An intrinsic, measurable aspect of quantum mechanics called contextuality, it now emerges. SEE ARTICLE P.351

STEPHEN D. BARTLETT

or decades, researchers have struggled with the question of what makes quantum computers so powerful, and the answer has been as elusive as an understanding of quantum physics itself. Is there some unique feature of quantum physics that is responsible for enabling quantum computers to perform certain computations faster than their conventional digital counterparts? Many of the more exotic properties of quantum mechanics have

been put forward as possible candidates, but so far none has held up to scrutiny. On page 351 of this issue, Howard *et al.* 1 uncover a remarkable connection between the power of quantum computers and one of the stranger properties of quantum theory known as contextuality.

Designs for quantum computers often mirror those of conventional computers, in that they are built out of basic components such as logic gates that perform elementary operations on quantum bits of information. A commonly used set of operations for a quantum processor

19 JUNE 2014 | VOL 510 | NATURE | 345

Quantum Computation Magic

ARTICLE

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doi:10.1038/nature13460

Contextuality supplies the 'magic' for quantum computation

 $Mark\ Howard^{1,2},\ Joel\ Wallman^2,\ Victor\ Veitch^{2,3}\ \&\ Joseph\ Emerson^2$

Quantum computers promise dramatic advantages over their classical counterparts, but the source of the power in quantum computing has remained elusive. Here we prove a remarkable equivalence between the onset of contextuality and the possibility of universal quantum computation via 'magic state' distillation, which is the leading model for experimentally realizing a fault-tolerant quantum computer. This is a conceptually satisfying link, because contextuality, which precludes a simple 'hidden variable' model of quantum mechanics, provides one of the fundamental characterizations of uniquely quantum phenomena. Furthermore, this connection suggests a unifying paradigm for the resources of quantum information: the non-locality of quantum theory is a particular kind of contextuality, and non-locality is already known to be a critical resource for achieving advantages with quantum communication. In addition to clarifying these fundamental issues, this work advances the resource framework for quantum computation, which has a number of practical applications, such as characterizing the efficiency and trade-offs between distinct theoretical and experimental schemes for achieving robust quantum computation, and putting bounds on the overhead cost for the classical simulation of quantum alzorithms.

Quantum Computation Magic Hypergraph

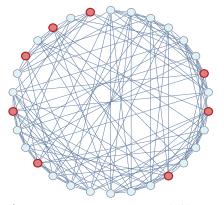
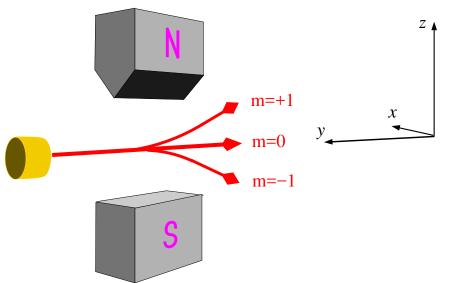


Figure 2 | Our construction applied to two qubits. Each of the 30 vertices in this graph Γ corresponds to a two-qubit stabilizer state; connected vertices correspond to orthogonal states. A maximum independent set (representing mutually non-orthogonal states) of size $\alpha(\varGamma)=8$ is highlighted in red. As described in Theorem 1 (main text), this value of α identifies all states $\rho \notin \mathcal{P}_{SIM}$ as exhibiting contextuality with respect to the stabilizer measurements in our construction.

Stern-Gerlach (SG) Experiment



Noncontextuality vs. Contextuality

Classical theories do not depend on arrangements in which measurements are carried out, i.e., on their "context," and we say that classical theories are *non-contextual* and that all their observables can be ascribed predetermined values.

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Quantum theories do depend on arrangements in which measurements are carried out and we say that quantum theories are *contextual* and that their observables cannot be ascribed predetermined values.

Contextuality can be used to reveal quantum nonlocality



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Contextuality can improve security of quantum communication

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Its implementation develops quantum information techniques of handling, manipulating, and measuring of qubits by means of quantum gates and circuits

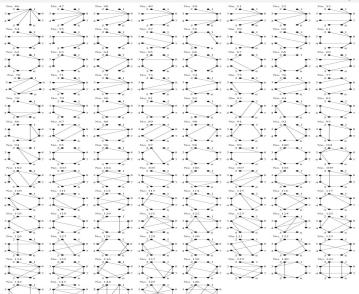
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Contextual sets are likely to find applications in the field of quantum information, similar to ones recently found for the graph states implementing entanglements for one-way quantum computing and quantum error correction.

Graph States—Cabello et. al., Phys. Rev. A (2009)



Algorithmic resources for the one-way quantum computation.

KS Theorem Cited in Recent Papers on Contextuality

Kochen-Specker sets are the most important examples of contextual sets. Each one of them proves the Kochen-Specker theorem.

A QUANTUM REVIVAL

Citations of the 1967 Kochen–Specker theorem have soared since physicists have been able to test it with specially prepared atoms and photons.



KS Contextual Application



NATURE | NEWS

Photons test quantum paradox

Contextuality theorem could improve secure communication.

Eugenie Samuel Reich

15 April 2013

Photon Experiment

PHYSICAL REVIEW X 3, 011012 (2013)

Experimental Implementation of a Kochen-Specker Set of Quantum Tests

Vincenzo D'Ambrosio, ¹ Isabelle Herbauts, ² Elias Amselem, ² Eleonora Nagali, ¹ Mohamed Bourennane, ² Fabio Sciarrino, ^{1,3} and Adán Cabello ^{4,2} ¹ Dipartimento di Fisica, "Sapienza" Università di Roma, I-00185 Roma, Italy ² Department of Physics, Stockholm University, S-10691 Stockholm, Sweden ³ Istituto Nazionale di Ottica (INO-CNR), Largo E. Fermi 6, I-50125 Firenze, Italy ⁴ Departamento de Física Aplicada II, Universidad de Sevilla, E-41012 Sevilla, Spain (Received 20 September 2012; published 14 February 2013)

The Definition of the Kochen-Specker Sets

Definition 1. Every KS set is a set of vectors in a Hilbert space \mathcal{H}^n , $n \geq 3$ to which it is impossible to assign 1's and 0's in such a way that:

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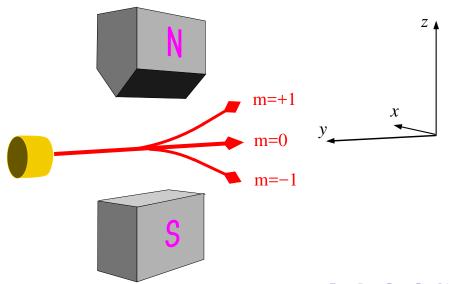
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- 1. No two orthogonal vectors are both assigned the value 1;
- 2. Not all of any mutually orthogonal vectors are assigned the value 0.

Stern-Gerlach (SG) Experiment



Orthogonality and Nonlinearity; 4-Dim Example

$$\mathbf{a}_{A} \cdot \mathbf{a}_{B} = a_{A1}a_{B1} + a_{A2}a_{B2} + a_{A3}a_{B3} + a_{A4}a_{B4} = 0,$$
 $\mathbf{a}_{A} \cdot \mathbf{a}_{C} = a_{A1}a_{C1} + a_{A2}a_{C2} + a_{A3}a_{C3} + a_{A4}a_{C4} = 0,$
 $\mathbf{a}_{A} \cdot \mathbf{a}_{D} = a_{A1}a_{D1} + a_{A2}a_{D2} + a_{A3}a_{D3} + a_{A4}a_{D4} = 0,$
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Brute Force — Mission Impossible



McKay-Megill-Pavičić (MMP) Hypergraphs

Fortunately, I realised that these equations can be reduced to a generation and then filtering of hypergraphs, in particular McKay-Megill-Pavičić (MMP) hypergraphs, which Brendan D. McKay, Norman D. Megill, and I defined previously for another purpose.

Definition 2. We define MMP hypergraphs as follows

- (i) Every vertex belongs to at least one edge;
- (ii) Every edge contains at least 3 vertices;
- (iii) Edges that intersect each other in n-2 vertices contain at least n vertices.

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This and many other subsequently developed algorithms developed by Brendan D. McKay, Norman D. Megill, Jean-Pierre Merlet, P.K. Aravind, Mordecai Waegell, and Mladen Pavičić (2005-2016) enabled us to generate KS sets exhaustively (in principle).

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MMP Formalism

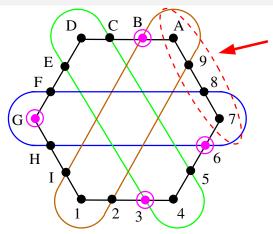
We encode MMP hypergraphs by means of alphanumeric and other printable ASCII characters. Each vertex is represented by one of the following characters: 123456789ABCDEFGHIJKLMNOPQRSTUVWXYZ1" #\$%&'()*-/:; <=>? @[\]^_`{|}~, and then again all these characters prefixed by '+', then prefixed by '++', etc.

MMP Formalism

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Each edge is represented by a string of characters that represent vertices within a single line. Edges are separated by commas. The line must end with a full stop. Skipping of characters is allowed. A line forms a representation of a hypergraph. The order of the edges is irrelevant. The numbers of vertices and edges are unlimited. We often present MMP hypergraphs starting with edges forming the biggest loop to facilitate their possible drawing.

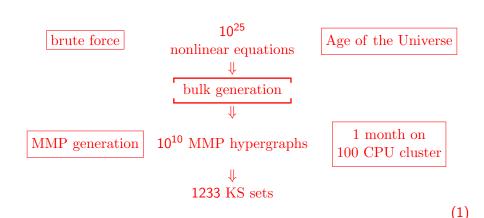
Contextuality Visualisation



1234,4567,789A,ABCD,DEFG,GHI1,29BI,35CE,68FH.

$$1 = \{0,0,0,1\}, \ldots A = \{0,1,1,0\}, \ldots C = \{1,1,-1,-1\}, \ldots I = \{0,1,0,0\}.$$

Algorithms Are Statistically Polynomially Complex but Nevertheless Demanding



M. Pavičić, J-P. Merlet, B.D. McKay & N.D. Megill (2005)

\	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	total
18	1																1
19		1															1
20		1	5	1													7
21			2	11	4	1											18
22			1	9	36	23	12	3	1								85
23				2	19	76	79	58	27	11	3	1					276
24				1	6	39	137	187	188	136	83	41	18	6	2	1	845
total	1	2	8	24	65	139	228	248	216	147	86	42	18	6	2	1	1233

Table: KS sets for systems with 4 degrees of freedom with up to 24 vectors with component values from $\{-1,0,1\}$.

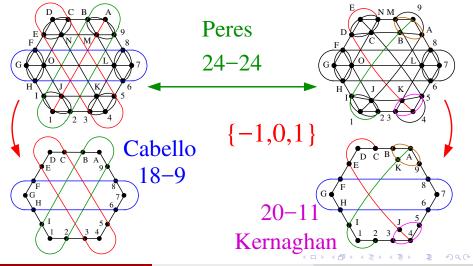
M. Pavičić, J-P. Merlet, B.D. McKay & N.D. Megill (2005)

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18	1																1
19		1															1
20		1	5	1													7
21			2	11	4	1											18
22			1	9	36	23	12	3	1								85
23				2	19	76	79	58	27	11	3	1					276
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We also found 37 new KS sets with 22 through 24 vectors with component values from other sets (not from $\{-1,0,1\}$)

Asher Peres 24-24 (1991), M. Kernaghan 20-11 (1994), Adán Cabello, J. Estebaranz, & G. García-Alcaine 18-9 (1996)





Stripping Peres' 24-24 KS set gives us the same 1233 KS sets.



isomorphic filtering | all 1233 KS sets

< 1 min on 1 CPU)



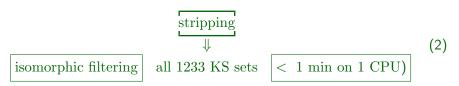
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Now, let us consider experimental implementation.



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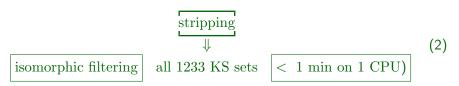


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Experimentally distinguishable are only *critical KS sets*, i.e., those KS sets that do not properly contain any KS subset.



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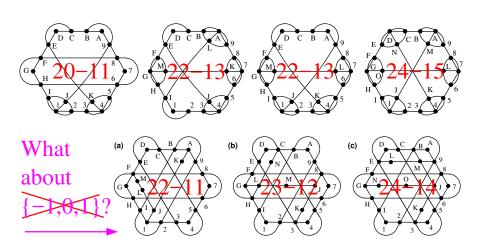


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There are altogether six critical subsets of Peres' 24-24 set. These are Cabello's 18-9, Kernagahn's 20-11, and the following 4 of ours

Critical KS Sets with Components from $\{-1,0,1\}$ and Other KS Sets



60-74 Vector Class: Billions of Critical KS Sets

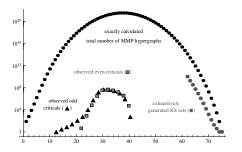
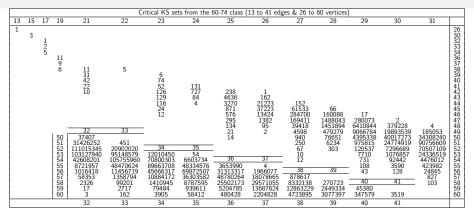


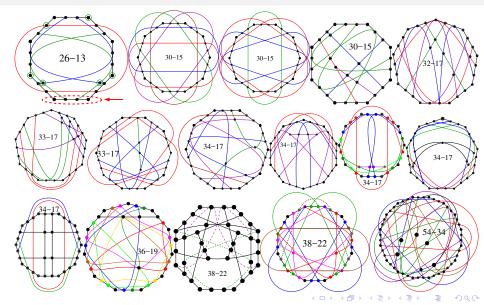
Figure: Statistics calculated for subsets of 60-74 given on a logarithmic scale. There are more than 10^9 critical KS sets. Given numbers of critical KS sets with 13 to 27 edges (on the x-axis) are exhaustive. The number of criticals with 32 edges is the biggest; we estimate that they do not exceed 10^{10} . Given numbers of noncritical KS sets with more than 61 edges are also exhaustive.

60-74 Vector Class: Billions of Critical KS Sets



List of 1540184852 non-isomorphic KS critical sets from the 60-74 class we obtained on our cluster. We conjecture that all possible types of vertex-edge sets are given here, i.e., that an exhaustive generation would not provide us with any new type. An exhaustive generation might give up to about an order of magnitude more samples of these sets.

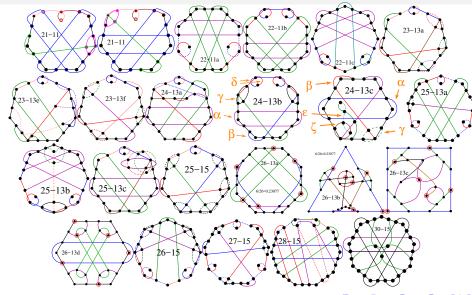
Chosen Critical Sets from the 60-74 Class



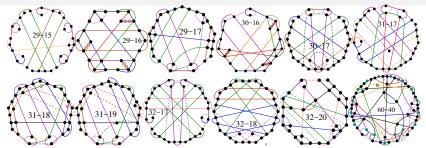
60-105 Vector Class: Millions of Critical KS Sets

									105 class	(9 to 40	edges &	18 to 60	vertices)				
0	9	11	13	15	16	17	18	19	20	21	22	23	24	25	26	27	8
00000000000	⊙	2 2 3 3	2 6 25 23 15	1 3 46 138 252 159 65	2 11	9 123 890 1812 1944 890 238	7 215 444 381 239 13 10 3	42 1173 5884 13776	1 42 3549	665							18 20 21 22 23 24 25 26 27 28 29 30 31 32 33 33 34 40 41 42 43 44
	28	29	30	. ⊙		1	10	16501 11606	11005 13459	8289 40535	$^{1}_{411}$	27					36
44 45 46 47 48 49 50 51 52 53 54 55 57 57 58 59 60	84 3404 15374 28757 30880 22962 12687 5410 1775 394 41 5 1	1 41 1957 13368 37731 63876 79656 79656 79057 22611 7582 1868 706 1	342 4877 14899 20713 17174 9890 4317 1577 457 88 18 18	31 77 2516 11260 20368 22505 18025 11225 18025 1124 497 497 497 497 1	741 5458 10815 10589 6239 2611 873 254 254 45 8	33 206 2619 7314 8179 5388 2509 864 241 66 4	3 3 34 18 1226 3970 4534 2810 1099 281 63 11 34	4059 835 12 1 1	8183 3161 532 170 35 3 0 91 872 1299 743 226 29 36	89747 120118 100316 54486 15806 66 6 0 11 399 399 495 239 49 37	9477 34244 58355 55221 30340 11214 2644 641 97 111	2162 24573 110686 280813 445033 445675 340138 162232 39997 5127 41	1 1024 11634 36354 59429 60432 42123 19078 5875 1190 182 17 1 ©	47 1648 19146 92755 265314 521889 693125 617069 353232 137593 26039 2927 3	2085 13592 32795 45112 40720 26801 12783 3977 853 117 1	46 1512 15500 60379 146333 248373 297725 244506 137570 48945 15678 1773 180	37 38 39 40 41 42 43 44 45 46 47 48 50 51 52 53

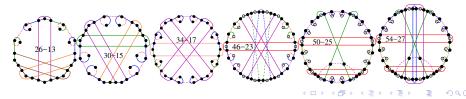
Examples from the 60-105 Operator Class



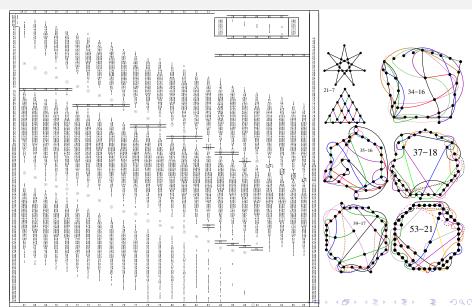
Further Examples from the 60-105 Operator Class



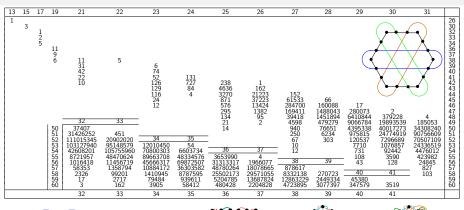
Some 60-105-class KS critical sets with no parity proofs (even number of edges) compared with some sets with parity proofs (odd number of edges).

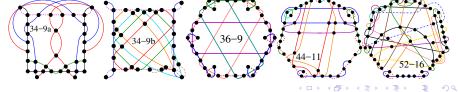


3714503 6-dim Critical KS Sets from the 236-1216 Class



1802014 8-dim Critical KS Sets from the 120-2024 Class



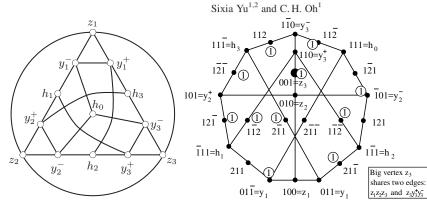


Yu-Oh 13 Vector Set

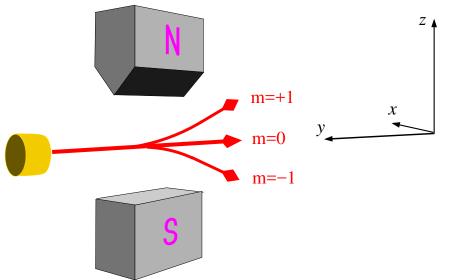
PRL 108, 030402 (2012)

PHYSICAL REVIEW LETTERS

State-Independent Proof of Kochen-Specker Theorem with 13 Rays



Stern-Gerlach (SG) Experiment



Yu and Oh claim that their 13 vector set proves the KS theorem since vertices $h_0 - h_3$ cannot be assigned more than one 1 simultaneously.

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The probability of assigning '1', when any of 25 tests is chosen at random, is between 11/25 = 0.32 and 9/25 = 0.48 giving '1' on average with the probability of 2/5 = 0.4 and since it is greater than quantum 1/3 = 0.33, there is no contextuality.

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The noncontextuality for $h_0 - h_3$ means that they must be assigned '1', together with all other vertices among 25, with the probability of 2/5 which sets $4\times2/5=1.6$ and this is >4/3=1.33. Hence, the set cannot be a proof of the KS theorem by definition since the upper bound for $h_0 - h_3$ values is 1.6 and not 1 as Yu and Oh claim.

...and Consequences—The experiment is Not about the Contextuality

Consequently, the following recent experiment on this Yu-Oh result does not prove the KS theorem either.

Selected for a Viewpoint in Physics

PHYSICAL REVIEW LETTERS

week ending 12 OCTOBER 2012

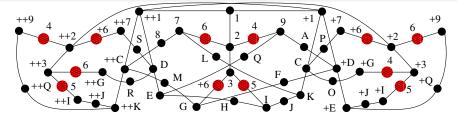


State-Independent Experimental Test of Quantum Contextuality in an Indivisible System

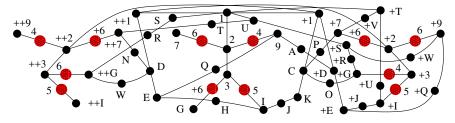
C. Zu, Y.-X. Wang, D.-L. Deng, Z. X.-Y. Chang, K. Liu, P.-Y. Hou, H.-X. Yang, and L.-M. Duan Center for Quantum Information, IIIS, Tsinghua University, Beijing, China
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA

PRL 109, 150401 (2012)

The smallest 3-Dim KS Sets



Bub's 3-dim KS critical set with 49 vertices and 36 edges.



Conway-Kochen 3-dim KS critical set with 51 (not 31 as usually claimed) vertices and 37 edges.

Acknowledgements 😊



The speaker's attendance at this conference was sponsored by the Alexander von Humboldt Foundation.

http://www.humboldt-foundation.de

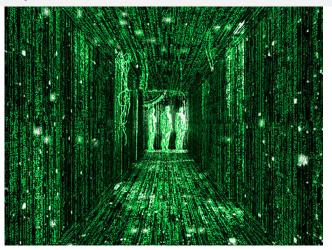


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Computational support was provided by the cluster *Isabella* of the *University Computing Centre* of the *University of Zagreb* and by the *Croatian National Grid Infrastructure*.

Travelling expenses were covered through the project IP-2014-09-7515.

Thanks for your attention 😊



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