International Journal of Theoretical Physics, Vol. 32, No. 10, 1993

Probabilistic Forcing in Quantum Logics

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Received April 30, 1993

It is shown that an orthomodular lattice can be axiomatized as an ortholattice with a *unique* operation of identity (bi-implication) instead of the operation of implication, and a corresponding algebraic unified quantum logic is formulated. A statistical YES–NO physical interpretation of the quantum logical propositions is then provided to establish a support for a novel YES–NO representation of quantum logic which prompts a conjecture about a possible completion of quantum logic by means of probabilistic forcing.

1. INTRODUCTION

In this paper we shall anwser two basic questions of quantum logics, present a novel probabilistic representation of the logic and of quantum measurements, and propose a probabilistic forcing as a possible tool for completing quantum logic.

The questions are, first, whether there is a *unique* object language operation which can take over the role of the unique classical operation of implication (conditional, set-theoretic inclusion), and, second, whether there is a relation which is more appropriate for set-theoretic representation of quantum-theoretic measurements than the usual irreflexive and symmetric orthogonality relation.

We answer the first question in the positive by substituting the bi-implication for the implication and the equality for the ordering relation, thus at the same time making the ordering within quantum sets irrelevant. This renders the usual techniques of quantum logic as a deductive *inferential* theory inappropriate and ascribes quantum deductive logic a particular *equational* meaning. The result is obtained in Section 2.

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0020-7748/93/1000-1965$07.00/0

We answer the second question by representing quantum logic with the help of an intransitive and symmetric YES–NO relation instead of the projector-stemmed irreflexive and symmetric orthogonality relation.This makes the usual modal, Kripkean, and imbedding approaches inapplicable, since an intransitive relation does not correspond to any modal formula in the corresponding systems. The representation is presented in Section 4 at the end of which a probabilistic forcing is defined.

In Section 3 we provided a physical interpretation of quantum logic based on the statistics of measurements as a bridge between Sections 2 and 4.

2. NONORDERED QUANTUM LOGIC

Quantum theory generates five different conditionals (in the orthomodular lattice and quantum logic) which reduce to the classical conditional when the propositions are *commensurable*.

We have shown in Pavičić (1987*a*) that the orthomodularity boils down to the equivalence of all five mentioned conditions with the lattice-theoretic conditions (the *relation* of implication) and we also formulated (Pavičić, 1989, 1992*a*) a unified quantum logic which gives a common and unique axiomatization for all possible conditionals.

Orthomodularity is thus reduced to a connection between object language implications and the model language ordering relation.

However, we can do more by reducing orthomodularity to a connection between a *unique* object language bi-implication and the model language equality.

Let us introduce the appropriate axiomatization of quantum logic.

Its propositions are based on elementary propositions $p_0, p_1, p_2, \ldots$, and the following connectives: $\neg$ (negation), $\leftrightarrow$ (bi-implication), and $\vee$ (disjunction).

The set of propositions Q^0 is defined formally as follows:

p_j is a proposition for $j = 0, 1, 2, \ldots$.
$\neg A$ is a proposition iff A is a proposition.
$A \leftrightarrow B$ is a proposition iff A and B are propositions.
$A \vee B$ is a proposition iff A and B are propositions.

The conjunction is introduced by the following definition:

$$A \wedge B \stackrel{\text{def}}{=} \neg(\neg A \vee \neg B)$$

Our metalanguage consists of axiom schemata from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: $\sim$ *(not)*, & *(and)*, $\veebar$ *(or)*, $\Rightarrow$ *(if . . . , then)*, and $\Leftrightarrow$ *(iff)*, with the usual *classical* meaning.

The bi-implication is defined as

$$A \leftrightarrow B \overset{\text{def}}{=} (\neg A \wedge \neg B) \vee (A \wedge B)$$

We define quantum logic as the axiom system AUQL (algebraic unified quantum logic) given below. The sign $\vdash$ may be interpreted as "*it is asserted in* AUQL." Connective $\neg$ binds stronger and $\leftrightarrow$ weaker than $\vee$ and $\wedge$, and we shall occasionally omit brackets under the usual convention. To avoid a clumsy statement of the rule of substitution, we use axiom schemata instead of axioms and from now on whenever we mention axioms we mean axiom schemata.

Axiom schemata:

AL1. $\vdash A \vee B \leftrightarrow B \vee A$

AL2. $\vdash A \leftrightarrow A \wedge (A \vee B)$

AL3. $\vdash A \leftrightarrow A \wedge (A \vee \neg B)$

AL4. $\vdash (A \vee B) \vee C \leftrightarrow \neg((\neg C \wedge \neg B) \wedge \neg A)$

Rule of inference:

RL1. $\vdash (C \vee \neg C) \leftrightarrow (A \leftrightarrow B) \quad \Rightarrow \quad \vdash A \leftrightarrow B$

It can easily be shown that in quantum logic the afore-stated definition of bi-implication coincides with the usual one:

$$A \leftrightarrow B \overset{\text{def}}{=} A \rightarrow_j B \;\&\; B \rightarrow_j A, \qquad j = 1, \ldots, 5$$

where the operation of implication $A \rightarrow_j B$ is one of the following:

$$A \rightarrow_1 B \overset{\text{def}}{=} \neg A \vee (A \wedge B) \qquad \text{(Mittelstaedt)}$$

$$A \rightarrow_2 B \overset{\text{def}}{=} B \vee (\neg A \wedge \neg B) \qquad \text{(Dishkant)}$$

$$A \rightarrow_3 B \overset{\text{def}}{=} (\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee ((\neg A \vee B) \wedge A) \qquad \text{(Kalmbach)}$$

$$A \rightarrow_4 B \overset{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge B) \vee ((\neg A \vee B) \wedge \neg B) \qquad \text{(non-tollens)}$$

$$A \rightarrow_5 B \overset{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B) \qquad \text{(relevance)}$$

To prove that AUGL is really a quantum logic, we have to prove that the Lindenbaum algebra for AUQL is an orthomodular lattice. By the orthomodular lattice we mean an algebra $L = \langle L^0, {}^\perp, \cup, \cap\rangle$ such that the following conditions are satisfied for any $a, b, c \in L^0$:

L1. $a \cup b = b \cup a$
L2. $(a \cup b) \cup c = a \cup (b \cup a)$
L3. $a^{\perp\perp} = a$
L4. $a \cup (b \cup b^\perp) = b \cup b^\perp$
L5. $a \cup (a \cap b) = a$
L6. $a \cap b = (a^\perp \cup b^\perp)^\perp$
L7. $a \supset_i b = c \cup c^\perp \quad \Rightarrow \quad a \le b \qquad (i = 1, \ldots, 5)$

where $a \le b \stackrel{\text{def}}{=} a \cup b = b$ and $a \supset_i b$ $(i = 1, \ldots, 5)$ is defined in a way which is completely analogous to the one in the logic. From now on we shall use the following notation: $a \cup a^\perp \stackrel{\text{def}}{=} 1$ and $a \cap a^\perp \stackrel{\text{def}}{=} 0$. Of course, L is also orthocomplemented, since lattices with unique orthocomplements and orthomodular lattices coincide (Rose, 1964; Fáy, 1967).

An algebra $\langle L^0, {}^\perp, \cup, \cap\rangle$ in which the conditions L1–L6 are satisfied is an ortholattice.

An algebra $\langle L^0, {}^\perp, \cup, \cap\rangle$ in which L1–L6 holds and L7 is satisfied by $a \supset b \stackrel{\text{def}}{=} a^\perp \cup b$ is a distributive lattice with 1 and 0 (Boolean algebra).

That L is really an orthomodular lattice, i.e., that L7 can be used instead of the usual orthomodularity law $a \cup b = ((a \cup b) \cap b^\perp) \cup b$, was proved in Pavičić (1987*a*, 1989).

To prove that the lattice is the Lindenbaum algebra for AUQL, we introduce the following definitions.

Definition 2.1. We call $\mathscr{L} = \langle L, h\rangle$ a model of the set Q^0 (of propositions from AUQL) if L is an orthomodular lattice and if h: AUQL $\mapsto L$ is a morphism in L preserving the operations $\neg$, $\vee$, and $\leftrightarrow$ while turning them into ${}^\perp$, $\cup$, and $\equiv$, and satisfying $h(A) = 1$ for any $A \in Q^0$ for which $\vdash A$ holds.

Definition 2.2. We call a proposition $A \in Q^0$ true in the model $\mathscr{L}$ if for any morphism h: AUQL $\mapsto L$, $h(A) = 1$ holds.

Definition 2.3. We call the expression $(a \supset_i b) \cap (b \supset_i a)$ $(i = 1, \ldots, 5)$ *identity* and denote it by $a \equiv b$. The two elements a, b satisfying $a \equiv b = 1$ we call *identical*.

Definition 2.4. We call the expression $(a \supset b) \cap (b \supset a)$ *classical identity* and denote it by $a \equiv_0 b$. The two elements a, b satisfying $a \equiv_0 b = 1$ we call *classically identical*.

Lemma 2.5. In any orthomodular lattice: $a \equiv b = (a \cap b) \cup (a^\perp \cap b^\perp)$.

Proof. We omit the easy proof. To our knowledge the lemma was first mentioned by Hardegree (1981). ■

Lemma 2.6. In any ortholattice: $a \equiv_0 b = (a^\perp \cup b) \cap (a \cup b^\perp)$.

Proof. Obvious by definition. ■

The following theorem characterizes an orthomodular lattice by means of the operation of identity and the lattice-theoretic equation instead of the operation of implication and the lattice-theoretic ordering.

Theorem 2.7. An ortholattice in which any two identical elements are equal, i.e., in which:

L7′. $a \equiv b = 1 \quad \Rightarrow \quad a = b$

holds is an orthomodular lattice and vice versa.

Proof. The vice versa part follows directly from L7 and Definition 3, since right to left metaequivalence holds in any ortholattice. So we have to prove the orthomodularity condition by means of L1–L6 and L7′. Let us take the following well-known form of the orthomodularity:

$$a \le b \quad \& \quad b^\perp \cup a = 1 \quad \Rightarrow \quad b \le a$$

The first premise can be written as $a \cup b = b$ and as $a \cap b = a$. The former equation can, by using the lattice analog for R2, be written as $b^\perp = a^\perp \cap b^\perp$. Introducing these $b^\perp$ and a into the second premise, the latter reads $(a^\perp \cap b^\perp) \cup (a \cap b) = 1$. Now L7′ gives $a = b$, which is, in effect, the wanted conclusion. ■

This extraordinary feature of orthomodular lattices and therefore of quantum logic characterizes them in a similar way in which the ordering relation versus the operation of implication characterizes distributive lattices. In other words, the identity which makes two elements both identical and equal in an ortholattice, thus making the lattice orthomodular, is unique. We prove this so as to prove that the classical identity which makes any two elements of an ortholattice both classically identical and equal does not turn the lattice into a distributive one, but makes it a lattice which is between being genuinely orthomodular and distributive. That, by doing so, we really prove the wanted uniqueness of the identity stems from the fact that there are only five implications in an orthomodular lattice which reduce to the classical one for commensurable elements. To our knowledge Hardegree (1981) was first who observed that Kotas' (1987) theorem on the existence of exactly five (plus classical itself) such implications in any modular lattice is valid for orthomodular lattices as well.

Theorem 2.8. An ortholattice in which any two classically identical elements are equal, i.e., in which

L7″. $a \equiv_0 b = 1 \quad \Leftrightarrow \quad a = b$

holds is a nongenuine orthomodular lattice which is not distributive.

Proof. As given in Pavičić (1993*a*). ■

We can now prove the soundness of AUQL for valid formulas from L by means of the following theorem.

Definition 2.9. We call a proposition $A \in Q^0$ true in the model $\mathscr{L}$ if for any morphism h: AUQL $\mapsto L$, $h(A) = 1$ holds.

Soundness Theorem 2.10. $\vdash A$ only if A is true in any orthomodular model of AUQL.

Proof. As given in Pavičić (1993*a*). ■

Lemma 2.11. The Lindenbaum–Tarski algebra $\mathscr{A}/\leftrightarrow$ is an orthomodular lattice with the natural isomorphism k: $\mathscr{A} \mapsto \mathscr{A}/\leftrightarrow$ which is induced by the congruence relation $\leftrightarrow$ and which satisfies $k(\neg A) = [k(A)^{\perp}$, $k(A \vee B) = k(A) \cup k(B)$, and $k(A \leftrightarrow B) = k(A) \equiv k(B)$.

Proof. The proof is straightforward and we omit it. ■

Completeness Theorem 2.12. If A is true in any model of AUQL, then $\vdash A$.

Proof. The proof is straightforward and we omit it. ■

3. STATISTICAL PHYSICAL INTERPRETATION OF QUANTUM LOGIC

Hultgren and Shimony (1977) showed that in building a complete Hilbert space edifice we cannot rely *only* on standard outcomes of the experiments carried out on *individual* systems. For, we cannot measure all the states we can describe with the help of the Hilbert space formalism by means of standard individual YES–NO measurements. For example, if we decide to orient the measuring device in direction $\mathbf{n}$ in order to measure the spin components of the spin operator $\mathbf{s}$ whose eigenvectors are $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$, then the state $[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}]$ can easily be shown not to be an eigenstate of the measured operator $\mathbf{n} \cdot \mathbf{s}$.

A possible remedy for such *unrepresentable* states seems to be the disputed Jauch infinite filter procedure for introducing conjunctions. For, apparently there are infinitely many atoms of the lattice of the subspaces of the Hilbert space which do not belong to the finite lattice of individual YES–NO measurements, but which can be recovered by Jauch's procedure. This is not a problem for quantum logic if we look at it as at a structure which corresponds ot the Hilbert space, because the structure (complete uniquely orthocomplemented[2] atomistic lattice satisfying the covering law) demands by itself an infinite number of atoms. (Shimony, 1971; Ivert and Sjödin, 1978). But if we looked at quantum logic as at a logic of YES–NO discrete measurements and tried to recover the Hilbert space axioms by empirically plausible assumptions, then we would obviously wish to avoid any *infinitary* procedure, which, like Jauch's, in principle simply cannot be substituted by any *arbitrary long* one.

On the other hand, Swift and Wright (1980) have shown that one can extend the standard experimental setup for measuring spins so as to employ electric fields in place of magnetic ones in order to make *every Hermitian operator acting on the Hilbert space of spin-s particle* measurable. Thus, we can deal on an equal footing with individual systems as with ensembles and represent states of the disputed kind ($[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}]$) like d'Espagnat's (1966, 1984) mixtures of the second kind.[3] These possibilities immediately address the question of approaching a preparation–detection YES–NO procedure. Are we to take the *individual* or the *ensemble* approach?

If we adopt the individual approach, then we bring the old Bohr "completeness solution" to the stage. That is, only given the whole experimental arrangement can we make an individual system determined by a discrete observable *repeatable*. This also means that we have to deal with *all* Hermitian operators in what is, as illustrated by Swift and Wright (1980), hardly feasible.

If we adopt the ensemble approach, we can apply the statistical approach to the definition of our propositions within the logic we use.

One can show that the statistical approach is not weaker than the individual approach but is rival with it (Pavičić, 1990*a–c*). Since that is often misunderstood in the literature, we shall provide some details here.

Let us take *repeatability* as "measure" of individual as opposed to statistical interpretation.

In order to verify in which state an individual observed system is, we have to measure not only its beam, but also the beams of its orthocomplement, i.e., both *statistical "properties"* in the long run. If the state were a

[2]Lattices are orthomodular *iff* they are uniquely orthocomplemented (Rose, 1964).

[3]d'Espagnat introduced the mixture of the second kind (*improper* mixtures) in order to take into account mixturelike data as well as the correlations of the separated subsystems of Bell-like systems. In our case we deal with the spin detections and the correlations with the spins prepared along some other directions. Since the correlations boil down to the same diagonal elements of the rotation matrix (Pavičić, 1990*c*), formally both approaches coincide.

mixture, such a "property" could not be encoded into individual particles. Thus we cannot speak of the repeatability of such systems. On the other hand, continuous observables (Ozawa, 1984) and discrete observables which do not commute with conserved quantities (Araki and Yanase, 1960) are both known not to satisfy the repeatability hypothesis.

Apparently, all these unrepeatable systems behave differently than the ones characterized by discrete observables.

But there is a way to treat all the observables in a common way.

We can *exclude* the "repeatability of individual events" — leaving only statistical repeatability, which turns into approximate repeatability for continuous observables. In other words, we can exclude *individual repeatability* even for discrete observables which undergo measurements of the first kind. In doing so we start with links between propositions and data.

The only way in which quantum theory connects the "elements of the physical reality" with their "counterparts in the theory" is by means of the Born formula, which gives us the probability that the outcome of an experiment will confirm an observable or a property of an *ensemble* of systems (von Neumann, 1955, p. 439). Strictly speaking, what we measure is the mean value of an operator, not the operator, not the state, not the wave function. To say that a measurement of the operator A yields the eigenvalue a or the state $|\psi_a\rangle$ only means that the measurement gives $\langle\psi_a|A|\psi_a\rangle/\langle\psi_a|\psi_a\rangle$, which is then *equal* to a.

In other words, in the case of discrete observables we say that we are able to prepare a property whenever by an appropriate measurement we can later verify the property with certainty—i.e., with probability one—that is, *on ensemble*. Whether the property will be verified on *each* so prepared individual system we can only guess. For, there is no "counterpart in the theory" of an individual detection even if it is carried out "with certainty": The Born probabilistic formula—which is the only link between the theory and measurements—refers only to ensembles. However, we can consistently *postulate* whether a measurement of the first order is verifying a prepared repeatable property on *each* system or not.

To show this we combine the Malus angle (between the preparing and the detecting Stern–Gerlach devices) expressed by probability with that expressed by relative frequency. To connect probability $0 < p < 1$ with the corresponding relative frequency we used the strong law of large numbers for the infinite number of Bernoulli trials which—being independent and exchangeable—perfectly represent quantum measurements on individual quantum systems. We use these properties of the individual quantum measurements to reduce their repeatability to successive measurements, but that has no influence on the whole argumentation, which rests exclusively on the fact that finitely many experiments out of infinitely many of them may be assumed to fail and to nevertheless build up to probability one.

When electrons pass perfectly aligned Stern–Gerlach devices with *certainty* this *does imply* that the relative frequency N_+/N of the number N_+ of detections of the prepared property (e.g., *spin up*) on the systems among the total number N of the prepared systems approaches probability $p = \langle N_+/N \rangle = 1$ almost certainly:

$$P\left(\lim_{N\to\infty} \frac{N_+}{N} = 1\right) = 1 \tag{1}$$

but *does not imply* that N_+ analytically equals N, i.e., it does not necessarily follow that the analytical equation $N_+ = N$ should be satisfied.

Hence we must *postulate*: either $N_+ = N$ *and* (1) or $N_+ \neq N$ *and* (1).

The possibility $N_+ \neq N$ does not seem very plausible by itself and we therefore proved a theorem on a difference between the probability and frequency and constructed a function which reflects the two possibilities.

As for the theorem, we proved that

$$\lim_{N\to\infty} P\left(\frac{N_+}{N} = p\right) = 0, \qquad 0 < p < 1 \tag{2}$$

which expresses randomness of individual results as clustering only *around* p (Pavičić, 1990*a*).

As for the function, we will just briefly sketch it here.[4] The function refers to the quantum Malus law and reads

$$G(p) \stackrel{\text{def}}{=} L^{-1} \lim_{N\to\infty} \left[\left| \alpha\left(\frac{N_+}{N}\right) - \alpha(p) \right| N^{1/2} \right]$$

where α is the angle at which the detection device (a Stern–Gerlach device for spin-s particles, an analyzer for photons) is deflected with regard to the preparation device (another Stern–Gerlach device, polarizer) and where L is a bounded random (stochastic) variable: $0 < L < \infty$. The function represents a *property* in the sense of von Neumann. For electrons and for

[4]The reader can find all the relevant theorems and proofs in Pavičić (1990*a*), a generalization to the spin-s case in Pavičić (1990*c*), and a discussion with possible implications on the algebraic structure underlying quantum theory in Pavičić (1990*b*, 1992*a*, 1993*c*).

projection 0 of spin 1, it is equal to (Pavičić, 1990*a*)

$$G(p) = H(p) \overset{\text{def}}{=} H[p(\alpha)] = \frac{\sin\alpha}{\sin\alpha}$$

We see (Pavičić, 1990*a*) that H is not defined for the probability equal to one: $H(1) = 0/0$. However, its limit exists and equals 1. Thus, a continuous extension $\tilde{H}$ of H to $[0, 1]$ exists and is given by $\tilde{H}(p) = 1$ for $p \in (0, 1)$ and $\tilde{H}(1) = 1$.

We are left with three possibilities [which hold for an arbitrary spin s, too (Pavičić, 1990*c*)], of which we shall here consider only the following two (that are "physical"):

1. $G(p)$ is continuous at 1. A necessary and sufficient condition for this is $G(1) = \lim_{p\to 1} G(p)$. In this case we cannot strictly have $N_+ = N$, since then $G(1) = 0 \neq \lim_{p\to 1} G(p)$ obtains a contradiction.
2. $G(1) = 0$. In this case we must have $N_+ = N$. And *vice versa*: if the latter equation holds, we get $G(1) = 0$.

Hence, under the given assumptions a measurement of a discrete observable can be considered repeatable with respect to individual measured systems if and only if $G(p)$ exhibits a jump discontinuity for $p = 1$ in the sense of point 2 above.

The interpretative differences between the points are as follows.

1. Admits only the statistical interpretation of the quantum formalism and banishes repeatable measurements on individual systems from quantum mechanics altogether. Of course, repeatability in the statistical sense remains untouched. The assumed continuity of G makes it approach its classical value for large spins. (Pavičić, 1990c). Notably, for a classical probability we have $\lim_{p\to 1} (G_{\text{cl}}(p) = 0$ and for "large spins" we get $\lim_{s\to\infty} \lim_{p\to 1} (G(p) = 0$.
2. Admits the individual interpretation of the quantum formalism and assumes that repeatability in the statistical sense implies repeatability in the individual sense. By adopting this interpretation we cannot but assume that nature differentiates open intervals from closed ones, i.e., distinguishes between two infinitely close points.

By keeping to the former possibility we introduce all the logicoalgebraic propositions of the structure (logic, lattice, . . .) underlying the Hilbertian theory of quantum measurements directly such as d'Espagnat's mixtures of the second kind and thus we avoid the aforementioned infinitary procedure, which actually boils down to postulating what we lack to reach the Hilbertian structure.

We have to stress here that by avoiding Jauch's infinitary procedure we did not get rid of any postulation. We only substituted the statistical interpretation for the individual interpretation postulate and the Jauch infinitary postulate. We did so because we feel that the former postulation is physically more plausible since it fits better into the quantum logic approach and resolves the paradoxes of Hultgren and Shimony by generating all the propositions according to a feasible experimental recipe.

The most important consequence of the obtained results and the appropriate statistical physical interpretation of quantum logic is that we can base the logic and its propositions on the statics of YES–NO quantum measurements, which is what we are going to do in the next section.

4. YES–NO VERSUS PROBABILISTIC REPRESENTATION OF QUANTUM LOGIC

Comparing the representations by means of the operations of implication and bi-implication presented in Section 2, we can easily come to a conjecture that other ordering-like quantum logic concepts can be redefined along a similar line eventually bringing us to a new modeling and proper semantics of quantum logic.

Thus, although quantum logic cannot be represented by means of the conditions of the first order imposed on the above orthogonality (which appears as the relation of accessibility in the Kripkean, i.e., modal approach) as proved by Goldblatt (1984) we can approach the whole problem from the "*equational side*," picking up another relation which is not orthogonal but, let us say, *orthogonal-like*, which closely follows the statistical interpretation of YES–NO quantum measurements outlined in the previous section. The new relation does not follow the algebra of projectors but the algebra of YES–NO linear subspaces and their orthocomplements. It is given in a set-theoretic way and it is weaker than (i.e., it follows from) MacLaren's (1965) orthogonality. We shall call it the YES–NO relation since it perfectly corresponds to YES–NO quantum experiments.

We establish our representation (semantics) by introducing the YES–NO quantum frame and the YES–NO relation for the algebraic unified quantum logic.

Definition 4.1. $\mathscr{F} = \langle X, \ominus \rangle$ is a YES–NO quantum frame iff X is a nonempty set, the carrier set of $\mathscr{F}$, and $\ominus$ is a YES–NO relation, i.e., $\ominus \subseteq X \times X$ is symmetric and intransitive.

Definition 4.2. Y is said to be a YES–NO subset iff

$$Y \subseteq Z \subset X \quad \Rightarrow \quad (\forall x \in Z)(x \in Y \veebar x \ominus Y)$$

where

$$x \ominus Y \stackrel{\text{def}}{=} (\forall y \in Y)(x \ominus y)$$

Thus, any element of a proper subset of the carrier set X either belongs to a subset of that subset or to its relative complement. To pick up a *proper* subset is important because a direct reference to X would bring us to the Boolean algebra instead of orthomodular lattice. We rely on the well-known representation of orthomodular structures, by which they can be obtained by gluing together the Boolean algebras, the representation "initiated" by Greechie.

Lemma 4.3. A YES–NO subset $Y \subseteq Z \subset X$ is YES–NO closed (in $Z \subset X$). If we denote $Y^{\ominus} = \{x: x \ominus y, y \in Y\}$, then $Y^{\ominus\ominus} = Y$.

Proof. As given in Pavičić (1993*a*). ■

To prove the soundness of our representation, we introduce a YES–NO model by the following definition.

Definition 4.4. $\mathcal{M} = \langle X, \ominus, V\rangle$ is a YES–NO quantum model on the YES–NO quantum frame $\langle X, \ominus\rangle$ iff V is a function assigning to each propositional variable p_i a YES–NO subset $V(p_i) \subset X$. The truth of a wff A at x in $\mathcal{M}$ is defined recursively as follows:

(1) $\|p_i\| = V(p_i)$
(2) $\|A \wedge B\| = \|A\| \cap \|B\|$
(3) $\|\neg A\| = \{x: x \ominus \|A\|\}$

where we denote the set $\{x \in X: x \vDash S\}$ by $\|A\|$ (or $\|A\|^{\mathcal{M}}$) ($\mathcal{M}: x \vDash A$ reads A holds at x in $\mathcal{M}$).

Lemma 4.5. If $\mathcal{M}$ is a YES–NO model, then for any A set $\|A\|^{\mathcal{M}}$ is YES–NO closed.

Proof. As given in Pavičić (1993*a*). ■

Soundness Theorem of Quantum Logic for YES–NO Representation 4.6:

$$\vdash \Gamma \to A \quad \Rightarrow \quad \mathcal{C}: \quad \Gamma \vDash A$$

where $\mathcal{C}$ is the class of all YES–NO quantum frames.

Proof. As given in Pavičić (1993*a*). ■

We are also able to prove the opposite, i.e., that the structure of which the YES–NO representation is a model is exactly quantum logic (AUQL), but for the proof we refer to Pavičić (1993*b*).

Completeness Theorem of Quantum Logic for YES–NO *Representation 4.7:*

$$\mathscr{C}: \Gamma \models A \quad \Rightarrow \quad \vdash \Gamma \rightarrow A$$

where $\mathscr{C}$ is the class of all YES–NO quantum frames.

The completeness might—under particular restrictions—be accompanied with the finite model property and decidability of quantum logic (Pavičić, 1993*b*).

Decidability boils down to the fact that there is an effective procedure to decide on every nonthesis that it really is a nonthesis and this is very important for any axiomatization because it decides on whether the axiomatization is effective in the sense that it is recursive. The reason why the obtained decidability and the finite model property of quantum logic are not so important for physical applications and the Hilbert space in the present elaboration is the following. Our completeness proof—as opposed to other completeness proofs (given for other representations and by means of the orthogonality) by MacLaren, Goldblatt, Dishkant, Morgan, Nishimura [see the references in Pavičić (1992*b*)]—might provide proof of the finite model property and decidability, but only for the finite case, i.e., for the case when there are finitely many *elementary* propositions in the logic. However, a finite propositional lattice does not have the Hilbert space as a model, so we have to expand it so as to add an infinite set of constant elementary propositions.

We conjecture that the expansion can be done along the following lines.

The above YES–NO quantum frame rests on the function V which maps quantum logical propositions into its set. It would be ideal, though, if the frame were a probabilistic one and the function simply a measure which maps propositions into the interval [0, 1]. For quantum logic proper, Greechie's counterexamples show that such measures cannot be states, but Morgan's (1983) function shows that such a measure exists. Thus, quantum logic as a propositional calculus is not *the* propositional calculus underlying its Hilbert space model and in particular its Hilbertian states do not provide its probabilistic semantics, although its proper probabilistic semantics does exist. What we can do is to try to use the properties of the propositional calculus which we obtain from the properties of the second order by *reading off* the Hilbert space structure so as to avoid unphysical "Hilbertian" properties of the second order, e.g., the ortho-Arguesian property as well as the infinite number of elementary propositions we obtain from the atomicity together with the covering property of the Jauch–Piron Hilbertian structure. We conjecture that one can proceed the other way round: the ortho-Arguesian property together with the infinite number of elementary propositions might give the Hilbertian structure. That would be much more plausible and could eventually answer, "Why the Hilbert space?"

In order to carry out such a program one can take a set of quantum states (measures) satisfying a kind of ortho-Arguesian property as a probabilistic model (semantics) of quantum logic and expand quantum logic to the infinite Hilbertian logic so as to complete it by *forcing*, proceeding roughly as follows.

Given Q^0 (of AUQL) and a set P we define $Q(P)$ as the expanded language whose propositions are those of Q^0 plus the elementary propositions in P. By an *expansion* of Q^0 we mean $Q(P)$, where P is an *infinite* set of elementary propositions. We define a *measure* Pr *on* $Q(P)$ *relative to* AUQL as function $Pr\colon \mathrm{AUQL} \mapsto [0, 1]$ for propositions from $Q(P)$ with a *finite* set of values. Measure Pr meets a number of conditions which determine it as a probabilistic model (semantics) of AUQL. The conditions are of the kind presented in Pavičić (1987*b*) or Morgan (1983) plus a kind of ortho-Arguesian property. We define Pr *forces* A *from* $Q(P)$ *relative to* AUQL, in symbols $Pr(D|C) \Vdash A$, for the function Pr and propositions A of $Q(P)$ by induction on the complexity of sentences as follows, for each C:

$$\begin{array}{lll} (1) & Pr(D|C) \Vdash p_i & \Leftrightarrow \quad Pr(D|C) \le Pr(p_1|C) \\ (2) & Pr(D|C) \Vdash A \wedge B & \Leftrightarrow \quad Pr(D|C) \Vdash A \quad \& \quad Pr(D \mid C) \Vdash B \\ (3) & Pr(D|C) \Vdash \neg A & \Leftrightarrow \quad (\forall E)[Pr(E|C) \Vdash A \Leftrightarrow Pr(D|C) \\ & & \qquad\qquad + Pr(E|C) = 1] \end{array}$$

If we now define the *forcing companion* AUQL^f as the set of all sentences forced by function Pr (whose set of values is finite), we can obtain the result that if AUQL is countable and possesses a model completion AUQL*, then AUQL* is logically equivalent to AUQL^f. That would establish a link between the experimental quantum logic and the Hilbertian quantum logic.

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