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Quantum Implication Algebras

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Quantum implication algebras without complementation are formulated with the same axioms for all five quantum implications. Previous formulations of orthoimplication, orthomodular implication, and quasi-implication algebras are analyzed and put in perspective to each other and our results.

KEY WORDS: implication algebras; quantum implication algebras; semiorthomodular lattices; orthomodular lattices; orthomodular implication algebras; join semilattices.

1. INTRODUCTION

It is well-known that there are five operations of implication in an orthomodular lattice which all reduce to the classical implication in a distributive lattice (Kalmbach, 1983). It was therefore believed that implication algebras for these implications must all be different and such different algebras have explicitly been defined in the literature (Abbott, 1976; Chajda *et al.*, 2001; Clark, 1973; Georgacarakos, 1980; Hardegree, 1981a; Piziak, 1974.)

In a previous paper (Pavičić and Megill, 1998), we have shown that one can formulate quantum implication algebras with "negation" [(ortho)complementation] with the same axioms for all five quantum implications. We arrived at such a formulation of implication algebras by using a novel possibility, given in (Megill and Pavičić (2001) and Megill and Pavičić (2002), of defining different quantum operations by each other. Implicitly, the latter possibility provides us a direct way of formulating quantum algebras without complementation and in this paper we give it.

To do so, we were prompted by a recent formulation of an implication algebra (Chajda *et al.*, 2001). The authors formulate an algebra based on the Dishkant implication previously considered by Kimble (1969), Abbott (1976),

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and Georgacarakos (1980); and cited by Hardegree (1981a) and Pavičić and Megill (1998). There are also other quantum implication algebras given by Finch (1970), Clark (1973), Piziak (1974), Hardegree (1981a,b), Georgacarakos (1980), Pavičić and Megill (1998), and others. In this paper we show how are all these algebras interrelated.

2. PRELIMINARIES

Let us first repeat a definition of an orthomodular lattice. (Megill and Pavičić, 2002).

Definition 2.1. An orthomodular lattice (OML) is an algebraic structure $\langle L, \cup, ^\perp \rangle$ in which the following conditions are satisfied for any $a, b, c \in L$:

L1. $a \le a^{\perp\perp} \quad \& \quad a^{\perp\perp} \le a$
L2. $a \le a \cup b \quad \& \quad b \le a \cup b$
L3. $a \le b \quad \& \quad b \le a \Rightarrow a = b$
L4. $a \le 1$
L5. $a \le b \Rightarrow b^\perp \le a^\perp$
L6. $a \le b \quad \& \quad b \le c \Rightarrow a \le c$
L7. $a \le c \quad \& \quad b \le c \quad \Rightarrow \quad a \cup b \le c$
L8. $a \to_i b = 1 \quad \Rightarrow \quad a \le b \qquad (i = 1, \ldots, 5)$

where $a \le b \overset{\text{def}}{\Leftrightarrow} a \cup b = b$, $1 \overset{\text{def}}{=} a \cup a^\perp$. Also

$$a \cap b \overset{\text{def}}{=} (a^\perp \cup b^\perp)^\perp, \qquad 0 \overset{\text{def}}{=} a \cap a^\perp.$$

and the implications $a \to_i b$ $(i = 1, \ldots, 5)$ are defined as follows

$a \to_1 b \overset{\text{def}}{=} a^\perp \cup (a \cap b)$ (Sasaki)
$a \to_2 b \overset{\text{def}}{=} b \cup (a^\perp \cap b^\perp)$ (Dishkant)
$a \to_3 b \overset{\text{def}}{=} ((a^\perp \cap b) \cup (a^\perp \cap b^\perp)) \cup (a \cap (a^\perp \cup b))$ (Kalmbach)
$a \to_4 b \overset{\text{def}}{=} ((a \cap b) \cup (a^\perp \cap b)) \cup ((a^\perp \cap b) \cap b^\perp))$ (nontollens)
$a \to_5 b \overset{\text{def}}{=} ((a \cap b) \cup (a^\perp \cap b)) \cup (a^\perp \cap b^\perp)$ (relevance)

The following theorem is well-known.

Theorem 2.1. *The equation $a^\perp = a \to_i 0$ is true in all orthomodular lattices for $i = 1, \ldots, 5$.*

Proof: The proof is straightforward and we omit it. □

There are six Boolean-equivalent expressions for implication in an OML. In addition to the five *quantum* implications above, which are distinguished by satisfying L8 (also known as the Birkhoff-von Neumann requirement), we have the *classical* implication that does not satisfy L8 in every OML:

$$a \to_0 b \stackrel{\text{def}}{=} a^\perp \cup b \qquad \text{(classical)}$$

3. IMPLICATION ALGEBRAS BASED ON THE DISHKANT IMPLICATION

Two kinds of implicational algebras based on the Dishkant implication $\to_2$ have been proposed in the literature: *orthoimplication algebras* (Abbott, 1976) and *orthomodular implication algebras* (Chajda *et al.*, 2001). In this section we summarize the two systems and some of their principal results, which are proved in their respective articles. As much as is practical we attempt to use the terminology of the authors of those articles.

Definition 3.1. (*Abbott, 1976*). An orthoimplication algebra (OIA) is an algebraic structure $\langle \mathcal{A}, \cdot \rangle$ with a single binary operation that satisfies:

OI1 $(ab)a = a$
OI2 $(ab)b = (ba)a$
OI3 $a((ba)c) = ac$

Definition 3.2. (*Chajda et al., 2001*). An orthomodular implication algebra (OMIA) is an algebraic structure $\langle \mathcal{A}, \cdot, 1 \rangle$ with binary operation $\cdot$ and constant 1 that satisfy:

O1 $aa = 1$
O2 $a(ba) = 1$
O3 $(ab)a = a$
O4 $(ab)b = (ba)a$
O5 $(((ab)b)c)(ac) = 1$
O6 $(((((((((ab)b)c)c)c)a)a)c)a)a = (((ab)b)c)c$

We note that the theorem $aa = bb$ holds in both systems, and it can be proved under OMIA without invoking axiom O1. Thus we may treat the constant 1 of OMIA as a defined term $1 =^{\text{def}} aa$ (making axiom O1 redundant), or we may extend OIA with a constant 1 (and add an axiom $aa = 1$ for it). For ease of comparing the two systems, we choose the first approach and henceforth shall consider 1 to be a defined term in OMIA.

Both OIA and OMIA are *sound* for the Dishkant implication in the sense that if the binary operation $\cdot$ is replaced throughout by $\to_2$, each axiom becomes an equation that holds in all OMLs. Thus each of these systems corresponds to a (not necessarily complete) Dishkant implicational fragment of OML theory.

A *join semilattice* is a partially-ordered set that is bounded above and in which every pair of elements has a least upper bound. Both OIA and OMIA induce join semilattices $\langle \mathcal{A}, \cup, 1\rangle$ under the definitions $a \cup b =^{\text{def}} (ab)b$ and $1 =^{\text{def}} aa$, with the partial order defined by $a \le b \Leftrightarrow^{\text{def}} a \cup b = b \Leftrightarrow ab = 1$.

The algebras OIA and OMIA also induce, respectively, more specialized associated structures called *semiorthomodular lattices and orthomodular join semilattices*. These are defined as follows.

Definition 3.3. (Chajda *et al.*, 2001). An orthomodular join semilattice (OJS) is an algebraic structure $\langle \mathcal{A}, \cup, 1, \langle {}^{\perp}_{x}; x \in \mathcal{A}\rangle\rangle$ where $\langle \mathcal{A}, \cup, 1\rangle$ is a join semilattice and $\langle {}^{\perp}_{x}; x \in \mathcal{A}\rangle$ is a sequence of unary operations, one for each member x of $\mathcal{A}$, such that the structure $\langle F_x, \cup, {}^{\perp}_{x}\rangle$ is an orthomodular lattice, where $F_x =^{\text{def}} \{y \mid x \le y\}$ the principal filter of $\mathcal{A}$ generated by x.

Definition 3.4. (Abbott, 1976). A semiorthomodular lattice (SOL) is an OJS with the further requirement

$$\mathbf{C} \quad a \le b \le c \ \Rightarrow\ c^{\perp}_{b} = c^{\perp}_{a} \cup b.$$

Theorem 3.1. (Abbott, 1976). *(i) Every OIA induces an SOL under the definition* $a^{\perp}_{b} =^{\text{def}} ab$ *for* $a \in F_b$. *(ii) Every SOL induces an OIA under the definition* $ab =^{\text{def}} (a \cup b)^{\perp}_{b}$.

Theorem 3.2. (Chajda *et al.*, 2001). *(i) Every OMIA induces an OJS under the definition* $a^{\perp}_{b} =^{\text{def}} ab$ *for* $a \in F_b$. *(ii) Every OJS induces an OMIA under the definition* $ab =^{\text{def}} (a \cup b)^{\perp}_{b}$.

4. RELATIONSHIP BETWEEN ALGEBRAS OIA AND OMIA

In this section we show that the axioms of OMIA can be derived from the axioms of OIA but not vice-versa.

Theorem 4.1. *Every OIA is an OMIA.*

Proof: To show this, we derive the axioms of OMIA from the axioms of OIA.

O1 is Lemma 1(i) of Abbott (1976).
O2 is Lemma 1(v) of Abbott (1976).
O3 is the same as OI1.
O4 is the same as OI2.

O5 can be expressed as $(a \cup b)c \leq ac$. From Th. 2 of Abbott (1976), $a \leq a \cup b$. Therefore from Th. 1 of Abbott (1976), $(a \cup b)c \leq ac$.

We can now assume that Lemma 4 of Chajda *et al.* (2001), which makes use of O1–O5 only, holds in OIA.

The associative law $a \cup (b \cup c) = (a \cup b) \cup c$ is derived as follows. Relations OL1–OL5 of Megill and Pavičić (2002) correspond to (v)–(viii) and (x) of Lemma 4 of Chajda *et al.* (2001). In Megill and Pavičić (2002) the associative law L2a is proved using OL1–OL5 only, so it also holds in OIA. The associative law allows us to omit parentheses and (with the help of OI2) disregard the order of joins in what follows.

O6 can be expressed as $((((a \cup b \cup c)c) \cup a)c) \cup a = a \cup b \cup c$. The OM4 part of Th. 4 of Abbott (1976) contains a proof of

$$x \leq y \quad \& \quad y \leq z \quad \Rightarrow \quad y \cup ((y \cup (zx))x) = z$$

or using OI2 and rewriting,

$$x \leq y \quad \& \quad y \leq z \quad \Rightarrow \quad (((zx) \cup y)x) \cup y = z$$

We substitute c for x, $a \cup c$ for y, and $a \cup b \cup c$ for z:

$$c \leq a \cup c \quad \& \quad a \cup c \leq a \cup b \cup c \quad \Rightarrow$$
$$((((a \cup b \cup c)c) \cup a \cup c)c) \cup a \cup c = a \cup b \cup c$$

The hypotheses are satisfied by Th. 2 of Abbott (1976), so we have

$$((((a \cup b \cup c)c) \cup a \cup c)c) \cup a \cup c = a \cup b \cup c$$

From (v), (viii), and (x) of Lemma 4 of Chajda *et al.* (2001) we have $a \leq b \Rightarrow a \cup b = b$. By Lemma 1(v) of Abbott (1976), $c \leq xc$ so $(xc) \cup c = xc$. Applying this twice, the above becomes

$$((((a \cup b \cup c)c) \cup a)c) \cup a = a \cup b \cup c$$

which is O6. □

On the other hand, it turns out that not every OMIA is an OIA.

Theorem 4.2. *There exist OMIAs that are not OIAs.*

Proof: Table I specifies an OMIA, i.e., any assignment to the variables in the OMIA axioms will result in an equality using the operation values in this table. On the other hand, this OMIA is not an OIA. To see this, choose $a = 5$, $b = 2$, and $c = 0$ in Axiom OI3. Then $a((ba)c) = 5((2 \cdot 5)0) = 5(3 \cdot 0) = 5 \cdot 2 = 10$ but $ac = 5 \cdot 0 = 4$. □

Table I. Example of an Orthomodular Implication Algebra (OMIA), With Operation ab, That is Not an Orthoimplication Algebra (OIA)

a \ b	0	1	2	3	4	5	6	7	8	9	10	11
0	**1**	1	1	1	1	1	1	1	1	1	1	1
1	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
2	**3**	1	**1**	3	1	3	1	3	1	3	1	3
3	**2**	1	2	**1**	4	8	6	**10**	8	**4**	10	**6**
4	**5**	1	**6**	3	**1**	5	6	7	8	3	10	11
5	**4**	1	10	1	4	1	6	10	1	4	10	6
6	**7**	1	**4**	3	4	5	**1**	7	8	9	10	3
7	**6**	1	8	1	4	8	6	**1**	8	4	1	6
8	**9**	1	**10**	3	4	**3**	6	7	**1**	9	10	11
9	**8**	1	**6**	1	1	8	6	10	8	**1**	10	6
10	**11**	1	**8**	3	4	5	6	**3**	8	9	**1**	11
11	**10**	1	4	1	4	8	1	10	8	4	10	**1**

Note. The bold entries specify the partial functions $a_b^\perp$ for the OJS of Fig. 1.

Theorem 4.2 tells us that the axioms of OIA cannot be derived from the axioms of OMIA. In particular, this proves that the axioms of OMIA are incomplete. In other words there exist equational theorems of OML, expressible purely in terms of the Dishkant implication, that cannot be proved from the axioms of OMIA. Axiom OI3 of OIA is one such example. Another example that does not hold in all OMIAs is the "implication version of the orthomodular law" of Abbott (1976):

$$a \le b \le c \quad \text{implies} \quad c = (ca)b. \tag{1}$$

The OMIA of Table I violates this law as can be seen by choosing $a = 0$, $b = 2$, $c = 4$.

Similarly, not all OJSs are SOLs. The join semilattice of Fig. 1 along with the $a_b^\perp$ operations specified by Table I (see table footnote) define an OJS. However, this OJS violates condition C of Definition 3.4, as can be seen by choosing $a = 0$, $b = 2$, $c = 4$. [Although this example also happens to be a lattice, we remind the reader that in general join semilattices are not bounded below.]

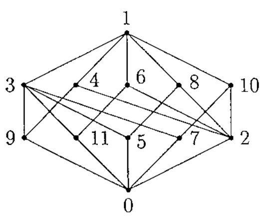

Fig. 1. Join semilattice induced by the OMIA of Table I. When combined with the partial functions $a_b^\perp$ of Table I (see table footnote), it provides an example of an orthomodular join semilattice (OJS) that is not a semiorthomodular lattice (SOL).

In conclusion, we have shown that the axioms of OMIA are not complete, since in particular they are strictly weaker than the axioms of OIA. On the other hand, the completeness of the axioms for OIA is apparently not known (Hardegree, 1981a). Future work towards seeking a complete Dishkant implicational fragment of OML theory might prove more fruitful by investigating OIA, rather than OMIA, as a starting point.

5. IMPLICATION ALGEBRA BASED ON THE SASAKI IMPLICATION

Apparently the only other pure implicational fragment of OML theory that has been studied are "quasi-implicational algebras" based on the Sasaki implication $\to_1$ (Hardegree, 1981a,b).

Definition 5.1. (Hardegree, 198la). A quasi-implication algebra (QSIA) is an algebraic structure $\langle \mathcal{A}, \circ \rangle$ with a single binary operation that satisfies:

QS1 $(a \circ b) \circ a = a$
QS2 $(a \circ b) \circ (a \circ c) = (b \circ a) \circ (b \circ c)$
QS3 $((a \circ b) \circ (b \circ a)) \circ a = ((b \circ a) \circ (a \circ b)) \circ b$

QSIA is sound for the Sasaki implication in the sense that if the binary operation $\circ$ is replaced throughout by $\to_1$, each axiom becomes an equation that holds in all OMLs.

An important result is that QSIA is also *complete* in the sense that when $\circ$ is interpreted as $\to_1$, its theorems are precisely those equational theorems of OML theory where each side of an equation is expressible purely in terms of polynomials built from $\to_1$ (Hardegree, 1981b).

A simple observation also shows that every QSIA induces an OIA (and an OMIA by Theorem 4.1).

Theorem 5.1. *Every QSIA induces an OIA under the definition* $ab \overset{\text{def}}{=} (b \circ a) \circ (a \circ b)$.

Proof: In any OML, $a \to_2 b = (b \to_1 a) \to_1 (a \to_1 b)$. Since OIA is sound for $\to_2$ in OML, we can replace $\to_2$ for $\cdot$ throughout the axioms of OIA, then express them in terms of $\to_1$ per this equation, to obtain equations built from $\to_1$ that hold in all OMLs. By the completeness of QSIA, each of these equations is provable under QSIA after substituting $\circ$ for $\to_1$. □

The converse, that every OIA induces a QSIA, is not obtainable with a simple substitutional definition since it is impossible to express $\to_1$ in terms of a polynomial built from $\to_2$. Thus there is a sense in which QSIA is "richer" than

OIA. Whether there exists a more indirect isomorphism between OIA and QSIA is unknown.

6. THE RELATIONSHIPS AMONG THE VARIOUS IMPLICATIONS

From the observation in the previous section that $\to_2$ can be expressed in terms of $\to_1$, we were led to investigate the other ways of expressing one implication in terms of another.

With the assistance of the computer programs `beran.c` and `bercomb.c` (obtainable from the authors), we exhausted the possibilities and obtained the results in Table II, where we show shortest expressions for each implication that can express other ones. For completeness we also include the classical implication $\to_0$.

Any OML polynomial with two generators (variables) corresponds to one of 96 possible expressions (Beran expressions). For brevity, we label Beran expressions with the numbers assigned in Beran (1985, p. 82). The Beran numbers for implications $a \to_i b$ are 94, 78, 46, 30, 62, and 14 for $i = 0, \ldots, 5$ respectively. We refer the reader to Beran (1985, p. 82) for the expressions corresponding to any Beran numbers we do not show explicitly.

Polynomials built from the $\to_2$ operation generate only six of the 96 possible expressions: a (with Beran number 22), $b \to_2 a$ (29), b (39), $a \to_2 b$ (46), $a \cup b$ (92), and 1 (96).

The other quantum implications $\to_1$, $\to_3$, $\to_4$, and $\to_5$ generate respectively 28, 18, 22, and 36 Beran expressions. In Table III we show their Beran numbers. In particular, we note from this table that the intersection of the sets of Beran numbers for all quantum implications is the same as the set of Beran numbers for $\to_2$, and the union of them is the same as the set of Beran numbers for $\to_5$.

Table II. The Shortest Expressions of the Implications in Terms of Others[a]

$a \to_i b$	$a \to_i b$ expressed in terms of other implications
$a \to_0 b =$	$((b \to_1 a) \to_1 a) \to_1 b$, $a \to_3 (a \to_3 b)$, $((a \to_4 b) \to_4 b) \to_4 b$, $(b \to_5 a) \to_5 (a \to_5 b)$, $a \to_5 ((b \to_5 a) \to_5 b)$
$a \to_1 b =$	$a \to_5 (a \to_5 b)$
$a \to_2 b =$	$(b \to_1 a) \to_1 (a \to_1 b)$, $(b \to_3 a) \to_3 (a \to_3 b)$, $((a \to_3 b) \to_3 a) \to_3 b$, $a \to_4 (a \to_4 b)$, $((a \to_5 b) \to_5 b) \to_5 b$, $((b \to_5 a) \to_5 a) \to_5 b$, $((a \to_5 b) \to_5 a) \to_5 b$
$a \to_3 b =$	$(a \to_1 (b \to_1 a)) \to_1 ((b \to_1 a) \to_1 (a \to_1 b))$, $(a \to_5 (b \to_5 a)) \to_5 (a \to_5 b)$
$a \to_4 b =$	$((b \to_1 a) \to_1 a) \to_1 (a \to_1 b)$, $(((b \to_5 a) \to_5 b) \to_5 b) \to_5 (a \to_5 b)$
$a \to_5 b =$	[none other than $a \to_5 b$ itself]

[a] When there are more than one shortest, all are shown.

Table III. The Beran Numbers for All Possible Polynomials With Two Generators Built From Implications $\to_i$

$\to_i$	Beran numbers for $\to_i$ polynomials with two generators
$\to_0$	22 28 39 44 93 94 96
$\to_1$	22 23 28 29 30 32 38 39 44 45 46 48 54 60 61 62 64 71 76 77 78 80 78 80 86 87 92 93 94 96
$\to_2$	22 29 39 46 92 96
$\to_3$	22 23 28 29 30 32 38 39 44 45 46 48 86 87 92 93 94 96
$\to_4$	22 28 29 32 39 44 46 48 55 60 62 64 70 76 77 80 86 87 92 93 94 96
$\to_5$	6 7 12 13 14 16 22 23 28 29 30 32 38 39 44 45 46 48 54 55 60 61 62 64 70 71 76 77 78 80 86 87 92 93 94 96

Thus $\to_5$ is the "richest" and $\to_2$ the "poorest" generator. In particular, $\to_5$ can generate all other implications, and all quantum implications can generate $\to_2$.

7. QUANTUM IMPLICATION ALGEBRA

In Pavičić and Megill (1998), we showed that a single, structurally identical expression, that holds when its operation is any one of quantum implications, can represent the join operation:

$$a \cup b = (a \to_i b) \to_i (((a \to_i b) \to_i (b \to_i a)) \to_i a) \tag{2}$$

holds in any OML for $i = 1, \ldots, 5$. This observation allowed us to construct, by adding a constant 0, an OML-equivalent algebra with an (unspecified) quantum implication as its only binary operation. Prompted by this result, we investigated the possibility of a purely implicational system having a single quantum implication as its sole operation.

In the previous section we observed that the $\to_2$ implication is unique in that it can be generated by any one of the other quantum implications. It turns out that there exists a *single* expression with an operation which, if replaced throughout by any one of the quantum implications $\to_i$, $i = 1, \ldots, 5$, will evaluate to $\to_2$.

Theorem 7.1. *The equation*

$$a \to_2 b = (b \to_i (b \to_i a)) \to_i (((a \to_i b) \to_i a) \to_i b) \tag{3}$$

holds in any OML, for all $i \in \{1, 2, 3, 4, 5\}$.

Proof: The verification is straightforward. $\square$

This allows us to define an implicational algebra that works when the binary operation is interpreted as any quantum implication.

Definition 7.1. A quantum implication algebra (QIA) is an algebraic structure $\langle \mathcal{A}, \bullet \rangle$ with a single binary operation that satisfies:

Q1 $(a \star b) \star a = a$
Q2 $(a \star b) \star b = (b \star a) \star a$
Q3 $a \star ((b \star a) \star c) = a \star c$

where $a \star b \stackrel{\text{def}}{=} (b \bullet (b \bullet a)) \bullet (((a \bullet b) \bullet a) \bullet b)$

Theorem 7.2. *QIA is sound for any quantum implication $\to_i$, $i = 1, \ldots, 5$ in the sense that if the binary operation $\bullet$ is replaced throughout by $\to_i$, each axiom becomes an equation that holds in all OMLs.*

Proof: The axioms of QIA are the same as the axioms of OIA with $\star$ substituted for $\cdot$. Soundness follows from Theorem 7.1 and the soundness of OIA. □

Theorem 7.3. *Every* QIA *induces an* OIA *under the definition $ab =^{\text{def}} a \star b$.*

Proof: The axioms of QIA become the axioms of OIA when $\cdot$ is substituted for $\star$. □

As a corollary, every QIA induces a semiorthomodular lattice (SOL), following the proof of Abbott (1976). Conversely, every SOL induces a QIA by Theorem 7.5(ii) below.

Lemma 7.4. *The following equation holds in every* OIA *(and every* OMIA*):*

$$ab = (b(ba))(((ab)a)b) \tag{4}$$

Proof: We show this equation holds in OMIA, and that it holds in OIA follows from Theorem 4.1. (i) $b(ba) = ((ba)b)(ba) = ba$ using O3 twice. (ii) $((ab)a)b = ab$ using O3. (iii) $ab \le (ba)(ab)$ using O2. (iv) $a \le ba$ using O2, so $(ba)(ab) \le a(ab) = ab$ by Lemma 4(ix) of Chajda *et al.* (2001) and O3. (v) From (iii) and (iv), we have $ab = (ba)(ab)$ by Lemma 4(vi) of Chajda *et al.* (2001). Substituting (i) and (ii) into this we obtain the result. □

Theorem 7.5. *(i) Every OIA induces a QIA under the definition $a \bullet b =^{\text{def}} ab$. (ii) Every SOL induces a QIA under the definition $a \bullet b =^{\text{def}} (a \cup b)^{\perp}_{b}$.*

Proof: (i) We convert each axiom of OIA by simultaneously expanding each occurrence of $\cdot$ into the right-hand side of Eq. (4). Substituting $\bullet$ for $\cdot$ throughout, we obtain the axioms of QIA. (ii) Immediate from (i) and Theorem 3.1 (ii). □

The system QIA that we have given is not complete. For example, the equation $a \bullet (a \bullet a) = a \bullet a$ is not a theorem of QIA (by virtue of the structure of Axioms Q1–Q3) even though it is sound for all quantum implications. QIA was devised for our purposes to be sufficient to induce an OIA, and nothing more. What such a complete axiomatization would look like, and even whether it can be finitely axiomatized, remain open problems.

8. UNIFIED QUANTUM IMPLICATION ALGEBRAS

In the previous section we have shown how one can construct an implication algebra with the same axioms for all five possible implications. If we are interested in specific implications, we can construct more specialized algebras with somewhat shorter axioms if we—in Definition 7.1—chose $a \star b \stackrel{\text{def}}{=} a \bullet b$ (for $\to_2$), or $(b \bullet a) \bullet (a \bullet b)$ (for $\to_1$ and $\to_3$), or $((a \bullet b) \bullet a) \bullet b$ (for $\to_3$ and $\to_5$), or $a \bullet (a \bullet b)$ (for $\to_4$). Another possible choice is $a \star b \stackrel{\text{def}}{=} (a \sqcup b) \bullet b$ where $a \sqcup b$ is defined as in Definition 8.1. None of these algebras is proven to be complete (and therefore "maximal") in the sense of QSIA (see Section 5).

On the other hand, one can take a more direct approach of finding implication algebras which would comply with the following objectives:

1. proving that the algebras are partially ordered sets bounded from above;
2. proving that the algebras induce join semilattices in which every principal order filter generates an orthomodular lattice;
3. proving that the algebras, when they contain a smallest element 0, can induce orthomodular lattices.

While QIA satisfies these objectives, its axioms are very long. Systems designed specifically with these objectives as their goal can have shorter axioms that are easier to work with. Here we give examples of such systems.

Definition 8.1. A unified quantum implication algebras UQIAi are algebraic structures $\langle \mathcal{A}, \bullet \rangle$ with single binary operations that satisty:

UQ1 $a \bullet a = b \bullet b$
UQ2 $a \bullet (a \sqcup b) = 1$
UQ3 $b \bullet (a \sqcup b) = 1$
UQ4 $a \bullet 1 = 1$
UQ5 $a \bullet b = 1 \ \ \& \ \ b \bullet a = 1 \ \ \Leftrightarrow \ \ a = b$
UQ6 $a \bullet b = 1 \ \ \& \ \ b \bullet c = 1 \ \ \Rightarrow \ \ a \bullet c = 1$
UQ7 $a \bullet c = 1 \ \ \& \ \ b \bullet c = 1 \ \ \Rightarrow \ \ (a \sqcup b) \bullet c = 1$
UQ8 $b \bullet a = 1 \ \ \Rightarrow \ \ a \sqcup (a \bullet b) = 1$
UQ9 $b \bullet a = 1 \ \ \Rightarrow \ \ ((a \bullet b) \bullet b) \bullet a = 1$

UQ10 $b \bullet a = 1 \quad \Rightarrow \quad a \bullet ((a \bullet b) \bullet b) = 1$
UQ11 $b \bullet a = 1 \quad \& \quad c \bullet a = 1 \quad \& \quad c \bullet b = 1 \quad \Rightarrow \quad (a \bullet c) \bullet (b \bullet c) = 1$
UQ12 $c \bullet a = 1 \quad \& \quad c \bullet b = 1 \quad \& \quad a \bullet b = 1 \quad \& \quad a \sqcup (b \bullet c) = 1$
$\qquad \Rightarrow \quad b \bullet a = 1$

where $1 \overset{\text{def}}{\Leftrightarrow} a \bullet a$ and $a \sqcup b$ means either $(a \bullet b) \bullet b$ (for either $\to_2$ or $\to_5$), or $((a \bullet b) \bullet (b \bullet a)) \bullet a$ (for either $\to_1$ or $\to_3$), or $(a \bullet (a \bullet b)) \bullet b$ (for $\to_4$), or $((((a \bullet b) \bullet (b \bullet a)) \bullet a) \bullet b) \bullet b$ (For $\to_i$, $i = 1, \ldots, 5$).

The above nonunique ways of expressing $a \sqcup b$ is a consequence of the fact that in an OML one cannot express $a \sqcup b$ in unique ways by using nothing but implications. (By "unique" we mean that an expression, in an OML, evaluates to $a \cup b$ for only one of the five implications and no others.) In an OML one can use implications *and* complements in, e.g., the following way:

1. $a \cup b = b^\perp \to_1 ((b^\perp \to_1 a^\perp)^\perp \to_1 (b \to_1 a^\perp)^\perp)^\perp$
2. $a \cup b = b^\perp \to_2 (b^\perp \to_2 (b^\perp \to_2 a^\perp)^\perp)^\perp \to_2 a$
3. $a \cup b = b^\perp \to_3 (b^\perp \to_3 a)$
4. $a \cup b = a^\perp \to_4 (b^\perp \to_4 a)$
5. $a \cup b = (a \to_5 b^\perp) \to_5 (b^\perp \to_5 a)$

Here, e.g., no one of $\to_i$, $i = 1, \ldots, 5$ except $\to_3$ would satisfy the 3rd line. However, one can again express implications by each other, so that, in the end, ambiguous expressions are equally proper as these ones.

Like QIA, algebras UQIA(i) are fragments of "maximal" algebras for their respective implications or sets of implications. However, they are sufficiently strong to accomplish our objectives above. Among other possibilities, they could be useful starting points in a search for maximal algebras (which are currently open problems for all cases except the $\to_1$ of QSIA).

Theorem 8.1. *Every unified quantum implication algebra* UQIA $= \langle \mathcal{A}, \bullet \rangle$ *determines an associated partially ordered set with an upper bound under:*

$$a \le b \overset{\text{def}}{\Leftrightarrow} ab = 1 \tag{5}$$

Proof: We have to prove

$$(1)\ a \le a$$
$$(2)\ a \le b \quad \& \quad b \le a \quad \Rightarrow \quad a = b$$
$$(3)\ a \le b \quad \& \quad b \le c \quad \Rightarrow \quad a \le c$$
$$(4)\ a \le 1$$

(1) follows from the definition of 1 and Eq. (5).
(2) follows from **UQ5** and Eq. (5).

(3) follows from **UQ6** and Eq. (5).
(4) follows from **UQ4** and Eq. (5). □

Theorem 8.2. *$\langle \mathcal{A}, \leq, \cup, 1\rangle$ in which one defines: $a \cup b \overset{\text{def}}{\Leftrightarrow} a \sqcup b$, is a join semilattice.*

Proof: We have to prove that $a \cup b = \sup\{a, b\}$, i.e., that the following conditions are satisfied:

$$(1)\ a \leq a \cup b$$

$$(2)\ b \leq a \cup b$$

$$(3)\ a \leq c \quad \& \quad b \leq c \quad \Rightarrow \quad a \cup b \leq c$$

(1) follows from **UQ2**
(2) follows from **UQ3**
(3) follows from **UQ7** □

Theorem 8.3. *If $m \in A$ is a fixed element and one defines:*

$$a_m^{\perp} \overset{\text{def}}{\Leftrightarrow} am, \tag{6}$$

and

$$a \cap b \overset{\text{def}}{\Leftrightarrow} ((am) \cup (bm))m \qquad \textit{for}\ \ a, b \in \mathfrak{I}_m \tag{7}$$

then $\langle \mathfrak{I}_m, \cup, \cap, m, 1, a_m^{\perp}\rangle$, where $\mathfrak{I}_m = \{a \in \mathcal{A} \mid m \leq a\}$ is the principal order filter generated by m, is an orthomodular lattice.

Proof: Proof We have to prove that the following conditions for the above ($m \leq a$) are satisfied:

$$(1)\ a \cup a_m^{\perp} = 1 \tag{8}$$

$$(2)\ a_{mm}^{\perp\perp} = a \tag{9}$$

$$(3)\ a \leq b \quad \Rightarrow \quad b_m^{\perp} \leq a_m^{\perp} \tag{10}$$

(1) follows from **UQ8** since $ma = 1$ holds for any a.
(2) follows from **UQ9, UQ10**, and **UQ5**.
(3) follows from **UQ11** by taking $c = m$ since $ma = 1$ and $mb = 1$ hold for any a and b.

Then we have to prove that $a \cap b = \inf\{a, b\}$, i.e., that the following conditions are satisfied:

$$(1)\ a \cap b \leq a$$

$$(2)\ a \cap b \leq b$$

$$(3)\ a \leq b \quad \& \quad a \leq c \quad \Rightarrow \quad a \leq b \cap c$$

(1) follows from **UQ2** and Eq. (10).
(2) follows from **UQ3** and Eq. (10).
(3) follows from **UQ7** and Eqs. (10) and (9).

In the end we have to prove the orthomodularity. By taking $c = m$, we get $ma = 1$ and $mb = 1$, i.e., $m \leq a$ and $m \leq b$ for any a and b so that **UQ12** gives us the orthomodularity:

$$a \leq b \quad \& \quad a \cup b_m^{\perp} = 1 \quad \Rightarrow \quad b \leq a \qquad \square$$

Corollary 8.4. *A UQIA with a smallest element* 0*, i.e. satisfying the axiom* $0 \bullet a = 1$*, induces an* OML *under the definitions* $a \cup b \stackrel{\text{def}}{=} a \sqcup b$ *and* $a' \stackrel{\text{def}}{=} a \bullet 0$*. A* QIA *with a smallest element* 0 *induces an* OML *under the definitions* $a \cup b \stackrel{\text{def}}{=} (a \star b) \star b$ *and* $a' \stackrel{\text{def}}{=} a \star 0$.

Proof: Straightforward. $\square$

9. CONCLUSION

We have investigated implication algebras for orthomodular lattices. We have first compared the systems previously given by Abbott (1976) (OIA, orthoimplication algebra), Chajda *et al.* (2001) (OMIA, orthomodular implication algebra), and Hardegree (1981a,b) (QSIA, quasi-implication algebra).

In Section 4, we proved that the axioms of OMIA can be derived from the axioms of OIA but not vice-versa. In other words, we have shown that the axioms of OMIA are not complete. In particular, the implication version of the orthomodular law does not hold in OMIA contrary to its name (*orthomodular* implication algebra). Whether OIA is complete in the sense of Hardegree's QSIA remains an open problem. For, QSIAs theorems are precisely those equational theorems of the OML theory where each side of an equation is expressible purely in terms of polynomials built from the corresponding OML (Sasaki) implication. If one wanted to attack the completeness problem along the way taken by Hardegree, we conjecture that the relevance implication ($i = 5$) would be the most promising with respect to Table II. Also, we would like to point out that the first axiom of both OIA and QSIA is the OML property $a \cup b = b \cup a$ expressed by means of implications. Their second axiom is the OML property $a = a$, where the left a is given as its shortest implication presentation involving two variables (Megill and Pavičić, 2002).

In Section 6, we investigate the other ways of expressing one implication in terms of another and in Section 7, we combined the obtained results to show how one can formulate quantum implication algebras, QIAs which keep the same form for all five possible implications from OML thus capturing an essential properties that are common to all quantum implications.

In Section 8, we formulated unified quantum implication algebras (UQIAs) for all implications. They are so weak that they do not yield a single axiom of either OIA or QSIA. Still, their join semilattices with 0 induce orthomodular lattices.

An open problem is devising a maximal extensions of QIA and UQIA that are complete, in the sense that its theorems are precisely those equational theorems of OML theory that hold regardless of which quantum implication $\rightarrow_i$, $i = 1, \ldots, 5$ we substitute for $\bullet$. A complete axiomatization of QIA and UQIA would be interesting because it would provide a general way to explore properties that are common to all quantum implications. It would also provide a way around philosophical debates about which quantum implication is the "proper" or "true" implication for quantum logic, since any of its results immediately apply to whichever one we prefer. And, finally, it might reduce concerns about being led astray by "toy" systems (Urquhart, 1983) since we would not be focusing on the specialized properties of any one implication in particular.

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