Quantum logic for genuine quantum simulators

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ABSTRACT

Recently we proved that there are two non-isomorphic models of the calculus of quantum logic corresponding to an infinite-dimensional Hilbert space representation: an orthomodular lattice and a weakly orthomodular lattice. We also discovered that there are two non-isomorphic models of the calculus of classical logic: a distributive lattice (Boolean algebra) and a weakly distributive lattice. In this work we consider implications of these results to a quantum simulator which should mimic quantum systems by giving precise instructions on how to produce input states, how to evolve them, and how to read off the final states. We analyze which conditions quantum states of a quantum computer currently obey and which they should obey in order to enable full quantum computing, i.e., proper quantum mathematics. In particular we find several new conditions which lattices of Hilbert space subspaces must satisfy.

Keywords: quantum computation, Hilbert space, Hilbert lattice, orthoarguesian property, quantum logic

1. INTRODUCTION

In this paper we consider a problem of building a quantum simulator, i.e., a general purpose quantum computer which would not be limited to particular algorithms such as Shor's or Grover's.[1] Can we dispense with algorithms which would simulate and calculate Schrödinger equation[2] and find a way to type in its Hamiltonian at a console of a quantum computer and thus simulate the system (atom, molecule, ...) in one step?

Computational instructions to a quantum computer for handling inputs to give desired outputs are lately simply called quantum logic.[3] The latter logic, however, cannot be a proper logic because, as we have recently shown,[4] it has at least two models, one of which is not the Hilbert space. This might look surprising, but as an even bigger surprise comes our another recent discovery—a century and a half after George Boole—that classical logic also has at least two models one of which is not a Boolean algebra and not even orthomodular. We present the related results in Section 2. Hence one cannot use any logic—a language of propositions—in any computer, classical or quantum, only one of its models. In quantum case it is a particular algebra underlying quantum theory.

The problem with making such an algebra a general machine language capable of solving and simulating any given Hamiltonian is that input elements (states) must satisfy additional conditions which do not result from basic quantum computer operations carried out on qubits (quantum bits, two dimensional Hilbert space pure quantum systems), as, for example, qubit superposition, entanglement,[5] rotation, and phase shift control. To see this, in Section 3, we investigate an algebra underlying the Hilbert space of an arbitrary quantum system, called the Hilbert lattice. We find a sequence of equations and additional conditions which must hold in a Hilbert space describing quantum systems and discuss their possible implementation into a quantum simulator.

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2. LOGICS AND THEIR MODELS

Let us introduce quantum and classical logics together.

Logical propositions are based on elementary propositions $p_0, p_1, p_2, ...$ and the following connectives: $\neg$ (negation), $\rightarrow$ (implication), and $\wedge$ (conjunction). The set of propositions Q° is defined formally as follows:

p_j is a proposition for $j = 0, 1, 2, ...$

$\neg A$ is a proposition iff A is a proposition.

$A \rightarrow B$ is a proposition iff A and B are propositions.

$A \wedge B$ is a proposition iff A and B are propositions.

The disjunction is introduced by the following definition: $A \vee B \stackrel{\text{def}}{=} \neg(\neg A \wedge \neg B)$.

Our metalanguage consists of axiom schemata from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: $\sim$ (*not*), & (*and*), $\veebar$ (*or*), $\Rightarrow$ (*if..., then*), and $\Leftrightarrow$ (*iff*), with the usual *classical* meaning.

The bi–implication is defined as: $A \leftrightarrow B \stackrel{\text{def}}{=} (A \rightarrow B) \wedge (B \rightarrow A)$.

We define an axiom system underlying both quantum and classical logics by means of axioms and rules of inference given below.

The sign $\vdash$ may be interpreted as "*it is asserted in* UQL." Connective $\neg$ binds stronger and $\rightarrow$ weaker than $\vee$ and $\wedge$, and we shall occasionally omit brackets under the usual convention.

Axioms.

A1. $\vdash A \rightarrow A$

A2. $\vdash A \rightarrow \neg\neg A$

A3. $\vdash \neg\neg A \rightarrow A$

A4. $\vdash A \wedge B \rightarrow A$

A5. $\vdash A \wedge B \rightarrow B$

A6. $\vdash A \wedge \neg A \rightarrow B$

Rules of Inference.

R1. $\vdash A \rightarrow B \quad \& \quad \vdash B \rightarrow C \quad \Rightarrow \quad \vdash A \rightarrow C$

R2. $\vdash A \rightarrow B \quad \Rightarrow \quad \vdash \neg B \rightarrow \neg A$

R3. $\vdash A \rightarrow B \quad \& \quad \vdash A \rightarrow C \quad \Rightarrow \quad \vdash A \rightarrow A \wedge C$

R4. $\vdash A \vee \neg A \rightarrow B \quad \Rightarrow \quad \vdash B$

We define quantum logic, QL as the above axiomatic system in which the operation of implication is defined as $A \rightarrow B = A \rightarrow_{ql} B \stackrel{\text{def}}{=} \neg A \vee (A \wedge B)$ and to which the following "orthomodularity" axiom is added:

A7. $\vdash A \vee (\neg A \wedge (A \vee B) \ \leftrightarrow_{ql} \ (A \vee B)$

We define classical logic, CL as the above axiomatic system in which the operation of implication is defined as $A \rightarrow B = A \rightarrow_{cl} B \stackrel{\text{def}}{=} \neg A \vee (A \wedge B)$ and to which the following ("distributivity") axiom is added:

A8. $\vdash A \vee (B \wedge C) \ \leftrightarrow_{cl} \ (A \vee B) \wedge (A \vee C)$

The afore defined quantum and classical logics are equivalent to any textbook definition.[4,6]

Let us now look at possible models for the above logics. Closed subspaces of Hilbert space form an algebra called Hilbert lattice. A Hilbert lattice is a kind of orthomodular lattice which we, in this section, introduce starting with an ortholattice. In any Hilbert lattice the operation *meet*, $a \cap b$, corresponds to set intersection, $\mathcal{H}_a \bigcap \mathcal{H}_b$, of subspaces $\mathcal{H}_a, \mathcal{H}_b$ of Hilbert space $\mathcal{H}$, the ordering relation $a \le b$ corresponds to $\mathcal{H}_a \subseteq \mathcal{H}_b$, the operation *join*, $a \cup b$, corresponds to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_a \bigcup \mathcal{H}_b$, and $a^\perp$ corresponds to $\mathcal{H}_a^\perp$.

An ortholattice is algebra $\mathrm{OL} = \langle L^\circ, ^\perp, \cap \rangle$ in which the following conditions are satisfied for any $a, b, c \in L^\circ$:

L1. $a \le a^{\perp\perp} \quad \& \quad a^{\perp\perp} \le a$

L2. $a \cap b \le a \quad \& \quad a \cap b \le b$

L3. $a \le b \quad \& \quad b \le a \quad \Rightarrow \quad a = b$

L4. $a \le 1$

L5. $a \le b \quad \Rightarrow \quad b^\perp \le a^\perp$

L6. $a \le b \quad \& \quad b \le c \quad \Rightarrow \quad a \le c$

L7. $a \le b \quad \& \quad a \le c \quad \Rightarrow \quad a \le b \cap c$

where $a \le b \overset{\mathrm{def}}{=} a \cap b = a, \qquad 0 \overset{\mathrm{def}}{=} a \cap a^\perp.$
Also

$$a \cup b \overset{\mathrm{def}}{=} (a^\perp \cap b^\perp)^\perp, \qquad 1 \overset{\mathrm{def}}{=} a \cup a^\perp.$$

The above definition of ortholattice is equivalent to the other formulations in the textbooks.[7]

An ortholattice is weakly orthomodular, WOML if the following conditions are satisfied for any $a, b \in L^\circ$.[6]

L8. $a \cup (b \cap (a^\perp \cup b^\perp)) \leftrightarrow_{ql} a \cup b = 1$

and an ortholattice is weakly distributive, WDL if the following condition is satisfied for any $a, b \in L^\circ$.[4]

L9. $a \cup (b \cap c) \leftrightarrow_{cl} (a \cup b) \cap (a \cup c) = 1$

An ortholattice is orthomodular, OML if the following conditions are satisfied for any $a, b \in L^\circ$.[8]

L8a. $a \leftrightarrow_{ql} b \ = \ 1 \quad \Longrightarrow \quad a = b$

and an ortholattice is distributive, DL, Boolean algebra, if the following condition is satisfied for any $a, b \in L^\circ$.[9]

L9a. $a \leftrightarrow_{cl} b \ = \ 1 \quad \Longrightarrow \quad a = b$

where the implications $a \to b$ are defined as follows

$$a \to_{cl} b \ \overset{\mathrm{def}}{=} \ a^\perp \cup b \qquad \text{(classical)}$$

$$a \to_{ql} b \ \overset{\mathrm{def}}{=} \ a^\perp \cup (a \cap b) \qquad \text{(quantum)}$$

Now, to prove that a logic has a model, means, in effect, that one must prove all the axioms and rules of inference of the logic in the corresponding lattice (soundness) and that one must prove all the lattice conditions by means of equivalence classes of propositions from the logic (completeness).

As for the soundness, logic contains the connectives $\to$, $\leftrightarrow$, $\equiv$, $\vee$, $\wedge$, and $\neg$ which we represent with their lattice counterparts: $\to$, $\leftrightarrow$, $\equiv$, $\cup$, $\cap$, and $^\perp$. It is trivial to show that:

$$a \le b \quad \Rightarrow \quad a \to b = 1 \qquad (1)$$

and therefore all the axioms of form $\vdash A \to B$ can be represented by $a \to b = 1$. Also, it is straightforward to prove that the set of formulas from the logic is closed under the rules of inference. Therefore, the soundness for quantum logic can be proved by both, WOML and OML, and for classical logic by both, WDL and DL.

So, the clue for the existence of two models for each logic was obviously hidden in the completeness proof (for 150 years for classical logic and for 65 years for quantum logic). The completeness proof means that we have to form classes of equivalence of propositions through operations in the logic: $\neg/\approx$, $\vee/\approx$, and $\wedge/\approx$ interpreted as $^\perp$, $\cup$, and $\cap$, respectively. In this way we obtain an algebra $\langle \mathcal{F}^\circ/\approx, \neg/\approx, \vee/\approx, \wedge/\approx\rangle$ which is called the Lindenbaum algebra, $\mathcal{A}$. For so many years mathematicians were convinced that a possible definition of a class of equivalence on a logic is exhausted by the following one:

$$A \approx B \stackrel{\text{def}}{=} \vdash A \leftrightarrow B \tag{2}$$

This meant that:

$$a = b \qquad \Leftrightarrow \qquad |A| = |B| \qquad \Leftrightarrow \qquad \vdash A \leftrightarrow B. \tag{3}$$

where $|A|, |B|$ are classes of equivalence. But this, by **L8a**, for $\leftrightarrow_{ql}$ and **L9a**, for $\leftrightarrow_{cl}$ means that we obtain the orthomodularity and the distributivity as a direct consequence of the definition of the classes of equivalence and not due to the "ortomodularity" and "distributivity" axioms, **A7** and **A8**, respectively. More over, **A7** turns out to be redundant: it can be inferred from **A1-6**.[4] It is $a = b \Leftrightarrow \vdash A \leftrightarrow_{ql} B$ what turns **A7** into $a \cup (b \cap (a^\perp \cup b^\perp)) = a \cup b$.

Can we define classes of equivalence in another way? Yes, we can. The clue is to prevent turning **A7** into $a \cup (b \cap (a^\perp \cup b^\perp)) = a \cup b$. And this is what the lattice O6 shown bellow does. Any orthomodular equation fails in it (and any equation that fails in it is orthomodular).[10]

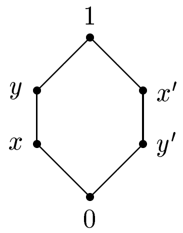

Figure 1. Ortholattice O6

Thus, relation $\approx$ defined as

$$A \approx B \stackrel{\text{def}}{=} \vdash A \leftrightarrow B \;\&\; (\forall o \in \mathcal{O}6)[(\forall C)(o(C) = 1) \Rightarrow o(A) = o(B)], \tag{4}$$

where $\mathcal{O}6$ is the set of all mappings $o : \mathcal{F}^\circ \longrightarrow \text{O6}$ such that for $A, B \in \mathcal{F}^\circ$, $o(\neg A) = o(A)'$ and $o(A \vee B) = o(A) \cup o(B)$, does the task.[4] With its help we can prove that WOML, which is not orthomodular, models quantum logic and that WDL, which is neither distributive nor orthomodular, models classical logic.

But how is it possible that no one has found the other models for so many years? The answer is simple. Neither classical computers, nor the standard evaluation of logical propositions make any use of the syntax of logic—they only use its *numerical* model: the Boolean algebra. Had they ever used the logic proper and its proper syntax they simply would not have worked—at least not in the way we are used to.

3. UNIVERSAL ALGEBRA FOR QUANTUM COMPUTERS

In the standard quantum mechanics, the phase space of a particle is the vector space of all wave functions, and wave functions are suitable functions from the space $\mathbb{R}^3$ to $\mathbb{R}$ representing (up to a multiplicative factor) the probability density for the particle. This space is always infinite dimensional and if we want it to be finite-dimensional, we must suppose that the entire space has only a finite number of points. That would mean that we should reformulate the whole quantum theory because in the finite dimensional case we would not have continuous functions and integrals any more. Therefore we here investigate only the infinite dimensional case. Recall that the orthomodularity from the previous section corresponds to the infinite dimensional Hilbert space (the finite dimensional one is modular).

A general quantum algebra underlying Hilbert space does exist. It is the Hilbert lattice we shall elaborate below. However, its present axiomatic definition by means of universal and existential quantifiers and infinite dimensionality does not allow us to feed a quantum computer with it. What we would need is an equational formulation of the Hilbert lattice. Let us first review the Hilbert lattice.

DEFINITION 3.1. *An* OML *which satisfies the following conditions is called a Hilbert lattice,* HL.

1. Completeness: *The meet and join of any subset of* HL *always exist.*
2. Atomic: *Every non-zero element in* HL *is greater than or equal to an atom. (An atom a is a non-zero lattice element with $0 < b \le a$ only if $b = a$. An atom corresponds to a pure state.)*
3. Superposition Principle: *(The atom c is a superposition of the atoms a and b if $c \ne a$, $c \ne b$, and $c \le a \cup b$.)*
   (a) *Given two different atoms a and b, there is at least one other atom c, $c \ne a$ and $c \ne b$, that is, a superposition of a and b.*
   (b) *If atom c is a superposition of distinct atoms a and b, then atom a is a superposition of atoms b and c.*
4. Minimal length: *The lattice contains at least three elements a, b, c satisfying:* $0 < a < b < c < 1$.

The above conditions suffice to establish isomorphism between HL and the closed subspaces of any Hilbert space, $\mathcal{C}(\mathcal{H})$, through the following well-known theorem.[11]

THEOREM 3.2. *For every Hilbert lattice $(\mathcal{L}_\mathcal{H}, \le, ')$ there exists a *-field $\mathcal{K}$ with an involution $.^* : \mathcal{K} \to \mathcal{K}$ and a Hilbert space $(\mathcal{H}, \langle .|. \rangle)$ over $\mathcal{K}$, such that $(\mathcal{C}(\mathcal{H}), \subseteq, \perp)$ is ortho-isomorphic to $(\mathcal{L}_\mathcal{H}, \le, ')$.*

*Conversely, let $(\mathcal{H}, \langle .|. \rangle)$ be an infinite-dimensional Hilbert space over a *-field $\mathcal{K}$ and let*

$$\mathcal{C}(\mathcal{H}) \stackrel{\mathrm{def}}{=} \{\mathcal{X} \ \subset \mathcal{H} \mid \mathcal{X}^{\perp\perp} = \mathcal{X}\} \tag{5}$$

be the set of all biorthogonal closed subspaces of $\mathcal{H}$. Then $(\mathcal{C}(\mathcal{H}), \subseteq, \perp)$ is a Hilbert lattice relative to:

$$a \cap b \ = \ \mathcal{X}_a \cap \mathcal{X}_b \qquad \text{and} \qquad a \cup b \ = \ (\mathcal{X}_a + \mathcal{X}_b)^{\perp\perp}. \tag{6}$$

In order to determine the *-field over which Hilbert space in Theorem 3.2 is defined we make use of the following theorem.

THEOREM 3.3. [Solèr-Mayet-Holland] *Hilbert space $\mathcal{H}$ from Theorem 3.2 is an infinite-dimensional one defined over a complex field $\mathbb{C}$ if the following conditions are met:*

5. Infinite orthogonality: HL *contains a countably infinite sequence of orthogonal elements.*

6. Unitary orthoautomorphism: *For any two orthogonal atoms a and b there is an automorphism $\mathcal{U}$ such that $\mathcal{U}(a) = b$, which satisfies $\mathcal{U}(a') = \mathcal{U}(a)'$, i.e., it is an* orthoautomorphism*, and whose mapping into $\mathcal{H}$ is a unitary operator U and therefore we also call it* unitary.

7. $\mathbb{C}$ characterization: *There are pairwise orthogonal elements $a, b, c \in \mathrm{L}$ such that $(\exists d, e \in \mathrm{L})(0 < d < a \ \& \ 0 < e < b)$ and there is an automorphism $\mathcal{V}$ in* L *such that $(\mathcal{V}(c) < c)$, $(\forall f \in \mathrm{L} : f \le a)(\mathcal{V}(f) = f)$, $(\forall g \in \mathrm{L} : g \le b)(\mathcal{V}(g) = g)$ and $(\exists h \in \mathrm{L})(0 \le h \le a \cup b \ \& \ \mathcal{V}(\mathcal{V}(h)) \ne h)$.*

Of the above conditions, only conditions 4, 5, and 7 seem to be a problem. We wanted to estimate whether one could somehow substitute the condition 4 and 5, i.e., the existence of propositions strictly between 0 and 1 and the infinite dimensionality, by a sequence of equations which might eventually serve to approximate infinite dimensionality. Until 15 years ago, no such sequence was known. The only equation known to hold in Hilbert space was the so-called orthoarguesian equation (with six variables, which we have shown to be reducible to four variables). The reason for that is that it is extremely difficult to deal with such equations.

Since already equations with 4 variables contain at least about 30 terms which one cannot further simplify, a proper tool for finding and handling the equations is indispensable. As a great help came Greechie lattices in which such equations must either fail or hold (as in O6 above). E.g., to find that two equations cannot be inferred from each other it suffices to find two Greechie lattices which the equations interchangeably pass and fail.

The first attempt at automated generation of Greechie lattices was made in the early eighties by G. Beuttenmüller a former student of G. Kalmbach.[10] According to the data, a generation of Greechie lattices with 13 blocks on the best computer of the time would take at least several centuries. On today's fastest PC it would take about 30 years, so, we rewrote it in C: it would take about 27 days. Since that would mean about a year for 14 blocks and almost half a century for 15 blocks we looked for another approach.

The technique of *isomorph-free exhaustive generation*[12] of Greechie lattices gave us not only a tremendous speed gain—48 seconds, 6 minutes, 51 minutes, 8 hours and 122 hours for 13–17 blocks, respectively (for a PC running at 800 MHz)—but also essentially new results when combined with a computer program for verifying equations on lattices.[13,14] One of the most important results is the following n-variable generalized orhtoarguesian equation and its consequences.[14]

DEFINITION 3.4. *We define an operation* $\overset{(n)}{\equiv}$ *on* n *variables* $a_1,\dots,a_n$ $(n\ge 3)$ *as follows:*

$$\begin{aligned}
a_1\overset{(3)}{\equiv}a_2 &\overset{\mathrm{def}}{=} a_1\overset{a_3}{\equiv}a_2 = ((a_1\to_1 a_3)\cap(a_2\to_1 a_3))\cup((a_1'\to_1 a_3)\cap(a_2'\to_1 a_3))\\
a_1\overset{(4)}{\equiv}a_2 &\overset{\mathrm{def}}{=} a_1\overset{a_4,a_3}{\equiv}a_2 = (a_1\overset{(3)}{\equiv}a_2)\cup((a_1\overset{(3)}{\equiv}a_4)\cap(a_2\overset{(3)}{\equiv}a_4))\\
a_1\overset{(5)}{\equiv}a_2 &\overset{\mathrm{def}}{=} (a_1\overset{(4)}{\equiv}a_2)\cup((a_1\overset{(4)}{\equiv}a_5)\cap(a_2\overset{(4)}{\equiv}a_5))\\
a_1\overset{(n)}{\equiv}a_2 &\overset{\mathrm{def}}{=} (a_1\overset{(n-1)}{\equiv}a_2)\cup((a_1\overset{(n-1)}{\equiv}a_n)\cap(a_2\overset{(n-1)}{\equiv}a_n))\ .
\end{aligned}$$

THEOREM 3.5. *In any Hilbert lattice the following equation holds:*

$$(a_1\to_1 a_3)\cap(a_1\overset{(n)}{\equiv}a_2)\le a_2\to_1 a_3\,. \tag{7}$$

An OL *in which this equation holds we call* nOA. *Every* nOA *is an* OML. 4OA *is equivalent to the standard*[10] *orthoarguesian equation.*

THEOREM 3.6. *In any* nOA *we have:*

$$a_1\overset{(n)}{\equiv}a_2=1 \qquad\Leftrightarrow\qquad a_1\to_1 a_3=a_2\to_1 a_3 \tag{8}$$

This also means that $a_1\overset{(n)}{\equiv}a_2$ *being equal to one is a relation of equivalence.*

Fifteen years ago Godowski found another sequence of equations.[15,16]

THEOREM 3.7.

$$(a_1\to_{ql} a_2)\cap(a_2\to_{ql} a_3)\cdots\cap(a_{i-1}\to_{ql} a_i)\cap(a_i\to_{ql} a_1)\le a_1\to_{ql} a_i,\qquad i=1,2,3,\dots \tag{9}$$

Godowski's and orthoarguesian equations are indenpendent, i.e., cannot be reduced to each other. Whether this two infinite classes of equations exhaust the equations in the infinite dimensional Hilbert space, i.e., whether they form a recursively enumerable set is not known.

4. CONCLUSION

We find that quantum logic is, in addition to an orthomodular lattice, also modeled by a weakly orthomodular lattice and that classical logic is, in addition to a Boolean algebra, also modeled by a weakly distributive lattice. Both new models turn out to be non-orthomodular. We prove the soundness and completeness of the calculuses for the models. This rules out a possibility to deal with a logic of elementary propositions within either quantum or classical computer.

It turns out that one can design a quantum simulator—a general purpose quantum computer—only by taking into account an algebra underlying the quantum theory. We investigated the standard description of quantum systems by means of infinite dimensional Hilbert space which is required if we wanted to have continuous probability density for quantum systems. An axiomatic system for the algebra has been developed recently and we review it in Section 3 indicating that it might be possible to substitute sequence of equations on the algebra for some or even all axioms. Algebraic equations could be directly implemented into a quantum computer by means of quantum gate design. To this end we investigated the orthoarguesian equation and found a new infinite class of generalized orthoarguesian equations. Further investigation on the extent to which infinite sets of equations in the Hilbert space can substitute its axioms is under way.

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