Quantum and Classical Implication Algebras with Primitive Implications

Mladen Pavičić [1] **and Norman D. Megill** [2]

Join in an orthomodular lattice is obtained in the same form for all five quantum implications. The form holds for the classical implication in a distributive lattice as well. Even more, the definition added to an ortholattice makes it orthomodular for quantum implications and distributive for the classical one. Based on this result a quantum implication algebra with a single primitive—and in this sense unique—implication is formulated. A corresponding classical implication algebra is also formulated. The algebras are shown to be special cases of a universal implication algebra.

PACS number: 03.65.Bz, 02.10.By, 02.10.Gd

Keywords: implication algebra, orthomodular lattices, quantum logic, classical logic.

[1] Atominstitute of the Austrian Universities, Schüttelstraße 115, A-1020 Wien, Austria; E-mail: pavicic@ati.ac.at; Fax: +49–30–63933990; Department of Mathematics, University of Zagreb, Gradjevinski fakultet, Kačićeva 26, HR-10001 Zagreb, Croatia; E-mail: mpavicic@faust.irb.hr; Fax: +385–1–4828050

[2] 19 Locke Lane, Lexington, MA 02173, USA; E-mail: nm@alum.mit.edu

1 INTRODUCTION

It is well-known that there are five operations of implication in an orthomodular lattice which all reduce to the classical implication in a distributive lattice. (Kalmbach, 1983) It was therefore believed that implication algebras for these implications must all be different and such different algebras have explicitly been defined in the literature. (Clark, 1973; Piziak, 1974; Abbott, 1976; Georgacarakos, 1980; Hardegree, 1981b) The systems were restrictive and uniquely determining a particular implication in an orthomodular lattice model.

In this paper, in Sec. 3 we show that one can formulate an implication algebra which can be modeled by either bare orthomodular or distributive lattice only *after* choosing the representation of implication by means of the lattice operations. We arrive at such a formulation of implication algebra by using a novel possibility, obtained in Sec. 2, of turning an ortholattice into either an orthomodular lattice or a distributive lattice by defining the join in the the same way by means of either quantum or classical implications, respectively.

2 UNIFIED JOINS IN LATTICES

Definition 2.1. *An ortholattice is algebra* $\mathrm{OL} = \langle L^\circ, ^\perp, \cup\rangle$ *in which the following conditions are satisfied for any* $a, b, c \in L^\circ$*:*

L1. $a \le a^{\perp\perp} \quad \& \quad a^{\perp\perp} \le a$

L2. $a \le a \cup b \quad \& \quad b \le a \cup b$

L3. $a \le b \quad \& \quad b \le a \quad \Rightarrow \quad a = b$

L4. $a \le 1$

L5. $a \le b \quad \Rightarrow \quad b^\perp \le a^\perp$

L6. $a \le b \quad \& \quad b \le c \quad \Rightarrow \quad a \le c$

L7. $a \le c \quad \& \quad b \le c \quad \Rightarrow \quad a \cup b \le c$

where $a \le b \overset{\mathrm{def}}{=} a \cup b = b$, $1 \overset{\mathrm{def}}{=} a \cup a^\perp$. *Also*

$$a \cap b \overset{\mathrm{def}}{=} (a^\perp \cup b^\perp)^\perp, \qquad 0 \overset{\mathrm{def}}{=} a \cap a^\perp.$$

*An ortholattice is orthomodular (*OML*) if the following conditions are satisfied for any* $a, b \in L^\circ$.

L8. $a \to_i b = 1 \quad \Longrightarrow \quad a \le b \qquad (i = 1,...,5)$

*and an ortholattice is distributive (*DL*) if the following condition is satisfied for any* $a, b \in L^\circ$ *(Pavičić, 1987)*

L9. $a \to_0 b = 1 \quad \Longrightarrow \quad a \le b$

where the implications $a \to_i b$ $(i = 0, \ldots, 5)$ *are defined as follows*

$a \to_0 b \stackrel{\text{def}}{=} a^\perp \cup b$ (classical)

$a \to_1 b \stackrel{\text{def}}{=} a^\perp \cup (a \cap b)$ (Sasaki)

$a \to_2 b \stackrel{\text{def}}{=} b \cup (a^\perp \cap b^\perp)$ (Dishkant)

$a \to_3 b \stackrel{\text{def}}{=} ((a^\perp \cap b) \cup (a^\perp \cap b^\perp)) \cup (a \cap (a^\perp \cup b))$ (Kalmbach)

$a \to_4 b \stackrel{\text{def}}{=} ((a \cap b) \cup (a^\perp \cap b)) \cup ((a^\perp \cup b) \cap b^\perp)$ (non-tollens)

$a \to_5 b \stackrel{\text{def}}{=} ((a \cap b) \cup (a^\perp \cap b)) \cup (a^\perp \cap b^\perp)$ (relevance)

Theorem 2.1. *(i) The equation*

UJ(i)**.** $a \cup b = (a \to_i b) \to_i (((a \to_i b) \to_i (b \to_i a)) \to_i a)$

is true in all orthomodular lattices for $i = 1, \ldots, 5$ *and in all distributive lattices for* $i = 0$*; (ii) an ortholattice in which* UJ(i) *holds is an orthomodular lattice for* $i = 1, \ldots, 5$ *and a distributive lattice for* $i = 0$.

Proof. (i) For $i = 1, \ldots, 5$, the proofs are tedious but straightforward expansions of the definitions, using the Foulis-Holland theorems extensively throughout. The following equations summarize the important intermediate results.

UJ(3).1a. $(a \to_3 b)^\perp \cap (b \to_3 a)^\perp = 0$

UJ(i).1a. $(a \to_i b) \cap (b \to_i a) = (a \cap b) \cup (a^\perp \cap b^\perp)$, $i = 4, 5$

UJ(i).1b. $(a \to_i b)^\perp \cap (b \to_i a) = a \cap b^\perp$, $i = 3, 4, 5$

UJ(i).1c′. $(a \to_i b)^\perp \cup (b \to_i a) = a \cup b^\perp$, $i = 3, 4$

UJ(3).1c. $(a \to_3 b) \cap ((a \to_3 b)^\perp \cup (b \to_3 a)) = (a^\perp \cap b^\perp) \cup (a \cap (a^\perp \cup b))$

UJ(4).1c. $((a \to_4 b)^\perp \cup (b \to_4 a)) \cap (b \to_4 a)^\perp = (a^\perp \cup b^\perp) \cap (a^\perp \cup b) \cap a$

UJ(5).1c. $(a \to_5 b)^\perp \cap (b \to_5 a)^\perp = (a \cup b) \cap (a \cup b^\perp) \cap (a^\perp \cup b) \cap (a^\perp \cup b^\perp)$

UJ(i).1. $(a \to_i b) \to_i (b \to_i a) = a \cup (a^\perp \cap b^\perp)$, $i = 1, 2, 3, 4$

UJ(5).1. $(a \to_5 b) \to_5 (b \to_5 a) = a \cup b^\perp$

UJ(i).2. $(a \cup (a^\perp \cap b^\perp)) \to_i a = a \cup b$, $i = 1,2,3,4$

UJ(5).2. $(a \cup b^\perp) \to_5 a = a \cup (a^\perp \cap b)$

UJ(3).3a. $(a \to_3 b)^\perp \cap (a \cup b)^\perp = 0$

UJ(5).3a. $(a \to_5 b) \cap (a \cup (a^\perp \cap b)) = (a \cap b) \cup (a^\perp \cap b)$

UJ(i).3b. $(a \to_i b)^\perp \cap (a \cup b) = (a \to_i b)^\perp$, $i = 3,4$

UJ(5).3b. $(a \to_5 b)^\perp \cap (a \cup (a^\perp \cap b)) = a \cap (a^\perp \cup b^\perp)$

UJ(i).3c′. $(a \to_i b)^\perp \cup (a \cup b) = a \cup b$, $i = 3,4$

UJ(3).3c. $(a \to_3 b) \cap ((a \to_3 b)^\perp \cup (a \cup b)) = (a^\perp \cap b) \cup (a \cap (a^\perp \cup b))$

UJ(5).3c. $(a \to_5 b)^\perp \cap (a \cup (a^\perp \cap b))^\perp = (a \cup b) \cap (a \cup b^\perp) \cap a^\perp$

UJ(i).3. $(a \to_i b) \to_i (a \cup b) = a \cup b$, $i = 1,2,3,4$

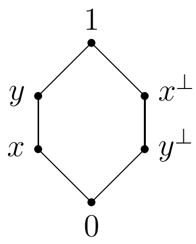
UJ(5).3. $(a \to_5 b) \to_5 (a \cup (a^\perp \cap b)) = a \cup b$

For $i = 3,4$: UJ(i).1c follows from UJ(i).1c′. For $i = 3,4,5$: UJ(i).1 follows from UJ(i).1a,b,c and the definition of $\to_i$. For $i = 3$: UJ(3).3c follows from UJ(3).3c′. For $i = 3,5$: UJ(i).3 follows from UJ(i).3a,b,c and the definition of $\to_i$. For $i = 4$: UJ(4).3 follows from UJ(4).3b,c′ and the definition of $\to_4$. For $i = 1,\ldots,5$: UJ(i) follows from UJ(i).1,2,3.

For $i = 0$: The proof follows in a trivial way from the distributivity property and we omit it.

(ii) The non-orthomodular ortholattice O6 [Fig. 1(a)] is violated by UJ(i) for $i = 1,\ldots,5$. Each UJ(i) is therefore equivalent to the orthomodular law by Theorem 2(iii) of (Kalmbach, 1983, p. 22).

Non-distributive OM_6 [Fig. 1(b)] (Abbott, 1976) is violated by UJ(0). One easily shows that UJ(0) is equivalent to the distributive law.



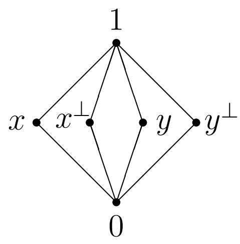


Figure 1: (a) Ortholattice O6 (b) Orthomodular lattice OM_6

□

Theorem 2.2. *The equation*

UO(i). $a^\perp = a \to_i 0$

is true in all orthomodular lattices for $i = 1, \ldots, 5$ *and in all distributive lattices for* $i = 0$.

Proof. The proof is straightforward and we omit it. □

3 QUANTUM AND CLASSICAL IMPLICATION ALGEBRAS

Many authors have given various systems of implication algebras so far. For example, systems for the Sasaki implication were formulated by (Clark, 1973; Piziak, 1974; Hardegree, 1981b, 1981a), for Dishkant's by (Abbott, 1976; Georgacarakos, 1980), for relevance implication by (Georgacarakos, 1980). In this section we give a system which is formulated by means of a primitive implication as the only operation. In doing so, while formulating definitions and theorems, we shall mostly follow (Piziak, 1974), (Abbott, 1976), and (Georgacarakos, 1980).

Definition 3.1. *Let P be a nonempty set. We designate an element in P and call it* 0. *Let* $\to : P \times P \mapsto P$. *Let* $1 \overset{\text{def}}{=} 0 \to 0$. *Then the triple* $\mathcal{UA} = \langle P, 0, \to \rangle$ *is called* universal implication algebra *provided the following axioms and rules of inference* R1–R4 *are satisfied for all* $a, b, c \in P$. *The triple* $\mathcal{QA} = \langle P, 0, \to \rangle$ *is called* quantum implication algebra *provided the following axioms and rules of inference* R1–R5q *are satisfied for all* $a, b, c \in P$ *and* classical implication algebra $\mathcal{CA} = \langle P, 0, \to \rangle$ *provided* A1–A4, R1–R4, *and* R5c *are satisfied for all* $a, b, c \in P$.

Axioms.

A1. $a \to ((a \to 0) \to 0) = 1 \quad \& \quad ((a \to 0) \to 0) \to a = 1$

A2. $a \to ((a \to b) \to (((a \to b) \to (b \to a)) \to a)) = 1$

A3. $b \to ((a \to b) \to (((a \to b) \to (b \to a)) \to a)) = 1$

A4. $a \to 1 = 1 \quad \&$

$$(a \to (a \to 0)) \to (((a \to (a \to 0)) \to ((a \to 0) \to a)) \to a) = 1$$

Rules of Inference.

R1. $a \to b = 1 \quad \Rightarrow \quad (b \to 0) \to (a \to 0) = 1$

R2. $a \to b = 1 \quad \& \quad b \to c = 1 \quad \Rightarrow \quad a \to c = 1$

R3. $a \to c = 1 \quad \& \quad b \to c = 1$
$\Rightarrow \quad ((a \to b) \to (((a \to b) \to (b \to a)) \to a)) \to c = 1$

R4. $a \to b = 1 \quad \& \quad b \to a = 1 \quad \Leftrightarrow \quad a = b$

R5q. $(a \to (b \to 0)) \to (((a \to (b \to 0)) \to ((b \to 0) \to a)) \to a) = 1$
$\& \quad a \to b = 1 \quad \Rightarrow \quad b \to a = 1$

R5c. $(a \to (b \to 0)) \to (((a \to (b \to 0)) \to ((b \to 0) \to a)) \to a) = 1$
$\Rightarrow \quad b \to a = 1$

Theorem 3.1. *Let* $\mathcal{L} = \langle P, ^\perp, \cup \rangle$ *be an orthomodular lattice* OML*. Define in* $\mathcal{L}$ *the following operations:*

D1. $1 \stackrel{\rm def}{=} a \cup a^\perp$

D2. $0 \stackrel{\rm def}{=} 1^\perp$

D3. $a \to b \stackrel{\rm def}{=} a \to_i b$ *where* $i =$ *either* 1 *or* 2 ... *or* 5.

Then the system $\mathcal{L}^{\mathcal{QA}} = \langle P, 0, \to \rangle$ *is a quantum implication algebra* $\mathcal{QA}$*. The same is valid for a universal implication algebra* $\mathcal{UA}$*.*

Proof. The proof follows straightforwardly from Definition 2.1 and Theorem 2.1 and the following property of an orthomodular lattice: (Pavičić, 1987)

L10. $a \cup b^\perp = 1 \quad \& \quad a \le b \quad \Rightarrow \quad b \le a.$ □

Theorem 3.2. *Let* $\mathcal{L} = \langle P, ^\perp, \cup \rangle$ *be a distributive lattice. Define in* $\mathcal{L}$ *the following operations:*

D1. $1 \stackrel{\rm def}{=} a \cup a^\perp$

D2. $0 \stackrel{\rm def}{=} 1^\perp$

D3. $a \to b \stackrel{\rm def}{=} a \to_0 b.$

Then the system $\mathcal{L}^{\mathcal{CA}} = \langle P, 0, \to \rangle$ *is a classical implication algebra* $\mathcal{CA}$*. The same is valid for a universal implication algebra* $\mathcal{UA}$*.*

Proof. The proof again follows straightforwardly from Definition 2.1 and Theorem 2.1 and the following well-known property of a distributive lattice:

L11. $a \cup b^\perp = 1 \quad \Rightarrow \quad b \le a.$ □

Theorem 3.3. *If* $\mathcal{L} = \langle P, 0, \to \rangle$ *is a quantum implication algebra* $\mathcal{QA}$*, then*

$$\mathcal{L}^{\mathcal{QA}^*} = \langle P, ^\perp, \cup \rangle$$

is an orthomodular lattice OML*, where* $^\perp, \cup$ *are defined in* $\mathcal{L}$ *as follows*

D1. $a^\perp \overset{\rm def}{=} a \to 0$

D2. $a \cup b \overset{\rm def}{=} (a \to b) \to (((a \to b) \to (b \to a)) \to a).$

Moreover $a \to b$ *is determined as one of* $a \to_i b,\ i = 1, \dots, 5.$

Proof. We first write all expressions of the form $a \to 0$ as $a^\perp$. Then we recognize all the $a \cup b$ expressions. Next, write down all $a \to b = 1$ expressions as $a \le b$ (which easily follows from R4). We are left with L1–L7 and L10, i.e., we obtain an orthomodular lattice. Hence, $a \to b$ from R4 must be one of $a \to_i b,\ i = 1, \dots, 5$ as given by L8. □

Theorem 3.4. *If* $\mathcal{L} = \langle P, 0, \to \rangle$ *is a classical implication algebra* $\mathcal{CA}$*, then*

$$\mathcal{L}^{\mathcal{CA}^*} = \langle P, ^\perp, \cup \rangle$$

is a distributive lattice DL*, where* $^\perp, \cup$ *are defined in* $\mathcal{L}$ *as in Theorem 3.3. Moreover* $a \to b$ *is determined as* $a \to_0 b$*.*

Proof. We first write all expressions of the form $a \to 0$ as $a^\perp$. Then we recognize all the $a \cup b$ expressions. Next, we use R4 to write down all $a \to b = 1$ expressions as $a \le b$. We are left with L1–L7 and L11, i.e., we obtain a distributive lattice. Hence, $a \to b$ from R4 must be $a \to_0 b$ as given by L9. □

Theorem 3.5. *If* $\mathcal{L} = \langle P, 0, \to \rangle$ *is a universal implication algebra* $\mathcal{UA}$*, then*

$$\mathcal{L}^{\mathcal{UA}^*} = \langle P, ^\perp, \cup \rangle$$

is a(n) distributive (orthomodular) lattice, where $^\perp, \cup$ *are defined in* $\mathcal{L}$ *as follows*

D1. $a^\perp \stackrel{\rm def}{=} a \to 0$

D2. $a \cup b \stackrel{\rm def}{=} (a \to b) \to (((a \to b) \to (b \to a)) \to a)$.

and where $\to$ *maps into* $\to_i$ *in the following way:*
$a \to_i b = a \to b\,;$ $i = 0$*—distributive* ($i = 1, \dots, 5$*—orthomodular).*

Proof. We first write all expressions of the form $a \to 0$ as $a^\perp$. Then we recognize all the $a \cup b$ expressions. Next, we use R4 to write down all $a \to b = 1$ expressions as $a \le b$. We are left with L1–L7, i.e., we obtain an ortholattice. Then, depending on whether we choose $a \to b$ from R4 to be either $a \to_0 b$ or $a \to_i b$; $i = 1, \dots, 5$ we obtain either a distributive or an orthomodular lattice. □

4 CONCLUSION

In Sec. 2 we show that an ortholattice, when the axiom UJ(i) defined in Theorem 2.1 is added to it, turns into an orthomodular lattice for $i = 1, \dots, 5$—i.e., for the so-called quantum implications—and into a distributive one for $i = 0$—i.e., for the classical implication. The axiom UJ expresses join by means of all these implications in a formally identical way.

In Sec. 3, in Definition 3.1 we employ the so expressed join to formulate quantum and classical implication algebras, $\mathcal{QA}$ and $\mathcal{CA}$, respectively. We have chosen the axiomatization from (Pavičić, 1987)—because each expression of its axioms and rules of inference contains $\cup$ and $^\perp$ at most once—and introduced join $\cup$ and orthocomplementation $^\perp$ expressed by means of a primitive implication following UJ and UO, from Theorems 2.1 and 2.2, respectively. In that way we obtain a quantum implication algebra which uses a single primitive implication. Hence, the main difference between our and all previous quantum implication algebras is that each of the latter ones corresponds to a particular quantum implication (one of five) while our does not. Besides, to our knowledge, only for three of five implications, implication algebras have been formulated so far.) As soon as we define the lattice operations on the algebra—by means of D1–D3 from Theorem 3.3—we open the orthomodular lattice possibility to express the primitive implication on the algebra by means of any of the five quantum implications but this is exactly the sense in which the obtained quantum implication algebra $\mathcal{QA}$

generalizes not only orthomodular lattices but also all previously obtained quantum implication algebras.

As also shown in Sec. 3, the two generalized quantum and classical implication algebras can themselves be generalized by dropping the rules R5 from Definition 3.1. So obtained universal implication algebra $\mathcal{UA}$—see Theorem 3.5—contains algebras $\mathcal{QA}$ and $\mathcal{CA}$, depending on how we choose to represent the implication. It would be interesting to investigate what other systems the algebra $\mathcal{UA}$ can give under different definitions of the lattice operations.

NOTE ADDED IN PROOFS

A step by step proof of Theorem 2.1 the reader can find at the following address: www1.shore.net/˜ndm/java/auql/mmexplorer.html

ACKNOWLEDGMENTS

M. P. acknowledges supports of the Ministry of Science of Croatia and the Technical University of Vienna, Austria. He would also like to thank Johann Summhammer, Atominstitut der Österreichischen Universitäten, Vienna for his hospitality.

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