Identity Rule for Classical and Quantum Theories

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It is shown that the identity rule—a rule of inference which has the form of modus ponens but with the operation of identity substituted for the operation of implication—turns any ortholattice into either an orthomodular lattice (a model of a quantum theory) or a distributive lattice (a model of classical theory). It is also shown that—as opposed to the implication algebras—one cannot construct an identity algebra although the identity rule contains the operation of identity as the only operation.

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1 INTRODUCTION

(Pavičić & Megill, 1998a) have shown that binary orthologic becomes either quantum or classical logic when nothing but modus ponens rule is added to it, depending on the kind of the operation of implication used. In lattice notation (lattice being a model of a logic and therefore of a corresponding theory) the result reads as follows:

Theorem 1.1. *An ortholattice in which the following modus ponens rule*

$$a \le b \qquad \& \qquad a \to b \le c \to d \qquad \Rightarrow \qquad c \le d$$

holds is an orthomodular lattice for '$\to$' being one of the five quantum implications and a distributive lattice for '$\to$' being the classical implication.

An obvious disadvantage of this rule for inferring theorems in varieties of quantum logic is that one does not know which of the five implications one should choose. Therefore it would be interesting to find a unique operation for a rule analogous to the one from the Theorem 1.1. To arrive at such a rule let us examine the proof of the above theorem. We obtained it by connecting the operation of implication and the relation of implication in the following way.

Theorem 1.2. *(Pavičić, 1987) An ortholattice in which*

$$a \to b \;=\; 1 \qquad \Leftrightarrow \qquad a \le b$$

holds is a distributive lattice for '$\to$' being the classical implication and an orthomodular lattice for '$\to$' being one of the five quantum implications.

Since the following theorem holds

Theorem 1.3. *(Pavičić, 1993) An ortholattice in which*

QL $$a \equiv b \;=\; 1 \qquad \Leftrightarrow \qquad a = b$$

holds is an orthomodular lattice for the identity being defined as

Definition 1.3.1. $a \equiv b \stackrel{\rm def}{=} (a \cap b) \cup (a^\perp \cap b^\perp)$

Therefore we could conjecture that the identity and equality can be substituted for the implication and inequality, respectively, in the Theorem 1.1. In the next section we prove the conjecture.

2 THE IDENTITY RULE

Our conjecture on the existence of the identity rule meets an apparent counter argument in the Theorem 3.2 of (Pavičić, 1993). We shall therefore first show that the theorem is only partially correct by proving the following theorem.

Theorem 2.1. *An ortholattice in which*

$$\textbf{CL} \qquad a \equiv_c b \ = \ 1 \qquad \Leftrightarrow \qquad a = b$$

holds is a distributive lattice for the identity being defined as

Definition 2.1.1. $a \equiv_c b \stackrel{\text{def}}{=} (a^\perp \cup \ b) \cap (a \cup b^\perp)$

Proof. In (Pavičić, 1993) we proved that an ortholattice to which **CL** is added is orthomodular. Since we are able to prove that it also violates the orthomodular lattice OM_6 (Fig. 1) the lattice is distributive (Abbott, 1976). To see that the orthomodular lattice OM_6 (Fig. 1) is violated (Abbott, 1976) one can easily check that $a = b$ does not follow from $(a^\perp \cup b) \cap (a \cup b^\perp) = 1$

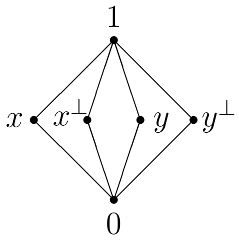

Figure 1: Orthomodular lattice OM_6

□

Our main theorems now read as follows:

Theorem 2.2. *An ortholattice in which the following identity rule*

$$a = b \qquad \& \qquad a \equiv b = c \equiv d \qquad \Rightarrow \qquad c = d$$

holds is an orthomodular lattice and vice versa.

Proof. To prove the first part of the theorem we choose b which is equal to a and obtain $a \equiv b = 1$. This yields

$$c \equiv d = 1 \qquad \Rightarrow \qquad c = d$$

and by Theorem 1.3, the orthomodularity.

To prove the vice versa part we start with the premises $a = b$ and $a \equiv b = c \equiv d$. But by Theorem 1.3 $a = b$ gives $a \equiv b = 1$ therefore $c \equiv d = 1$. Thus, again by Theorem 1.3, we obtain the conclusion, $c = d$. □

Theorem 2.3. *An ortholattice in which the following identity rule*

$$a = b \qquad \& \qquad a \equiv_c b = c \equiv_c d \qquad \Rightarrow \qquad c = d$$

holds is a distributive lattice and vice versa.

Proof. Using Theorem 2.1 we prove this theorem in complete analogy to the the proof of the previous theorem. □

3 IDENTITY ALGEBRA CANNOT BE FORMULATED

(Pavičić & Megill, 1998b) have shown that join and negation in an orthomodular lattice can be obtained in the same for the classical and all five quantum implications. Starting from this result, they formulated a quantum implication algebra with a single primitive—and in this sense unique—implication. A natural question which in the light of these results springs from the above theorems is whether one can express join and negation by means of the two above-defined operation of identity—which are by definition unique, i.e., whether an "identity algebra" can be formulated. By the following theorem we are going to answer to this question in the negative.

Theorem 3.1. *Orthocomplementation in an orthomodular lattice can be expressed as* $a^\perp = a \equiv 1 = a \equiv_c 1$. *Join, meet, and implications in an orthomodular lattice cannot be expressed by means of either quantum or classical identity.*

Proof. Free orthomodular lattices with two generators (expressions with two elements) can be represented by the direct product $\mathrm{OM}_6 \times 2^4$ (Beran, 1985). Denoting the elements of the Bollean algebra 2^4 by $b_1 = (0,0,0,0)$, $b_1 = (1,0,0,0)$,…,$b_{16} = (1,1,1,1)$, we can write down all 96 elements of the lattice in the form (a_i, b_j), $i = 1,\ldots,6, j = 1,\ldots,16$, where a_i are the elements of the orthomodular lattice OM_6 (Fig. 1). We can easily check that (a_i, b_{12}) through $(a_i, b_{15}), i = 1, \ldots, 6$, are exactly all six implications (five quantum ones and the classical one) and join, while (a_i, b_2) through (a_i, b_5) are their negations (which include meet).

We prove the theorem by checking that $b_k \;\equiv\!= (b_k \cap b_l) \cup (b_k^\perp \cap b_l^\perp), k = 1, 6, \ldots, 11, 16; l = 1, 6, \ldots, 11, 16$, cannot yield any one of $b_2, \ldots, b_5, b_{12}, \ldots, b_{15}$. For, we should start with either $a \equiv b = (0, b_8) = (0, 1, 0, 0, 1)$ or $a^\perp \equiv b = (0, b_9) = (0, 0, 1, 1, 0)$, etc., and arrive at either $a \cup b = (0, b_{12}) = (1, 1, 1, 1, 0)$ or $a \cap b = (0, b_2) = (0, 1, 0, 0, 0)$, etc. However, that would mean that starting with b's containing two unities we should—applying $(b_k \cap b_l) \cup (b_k^\perp \cap b_l^\perp)$ on them—arrive at a b containing either one or three units, which is clearly impossible. (For example, the first meet can give one unity and then the second meet of the orthocomplemented elements also gives one unity and the final join yields two units. Looking at all other combinations, we always arrive at an even number of unities and zeros.) $\square$

4 CONCLUSION

It is well known that quantum logic lacks a suitable rule of inference mainly because it does not have a unique operation of implication. In Section 2, we therefore considered unique operations of identity and found that one can formulate identity rules which are analogues to the lattice to lattice-theoretic modus ponens rules given by Theorem1.1. The rules turn an ortholattice into either an orthomodular or a distributive lattice as shown by Theorems 2.2 and 2.3.

It is interesting that two such rules which at first sight might be taken for ortholattice theorems can be substituted for the orthomodularity and the distributivity axioms. Therefore, in Section 3, we investigated whether it would be possible to arrive at an "identity algebra" which would use the identity as the only primitive operation and answered this question in the negative.

The main contribution of our recent (Pavičić & Megill, 1998a, 1998b) and present results to understanding of principles of modeling quantum and classical theories by means of lattices and other algebraic structures and logics is that the latter are recognized as systems handling operations and relations between the propositions and primarily the propositions themselves.

In (Pavičić & Megill, 1998a, 1998b) we have shown that one can model both quantum and classical theories by practically nothing but operations and relation of implication and without dealing directly with elementary propositions and attaching values to them. For quantum theories and their logics, which cannot ascribe values to all propositions, the appropriateness of such an approach is obvious. For classical theories and their logics this is an interesting alternative to the standard approach. In the present paper we have shown that even higher axioms (orthomodularity and distributivity) in such models do not require implication and ordering: identifying propositions which can be considered equal suffices for the axiomatization.

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