# A NEW AXIOMATIZATION OF UNIFIED QUANTUM LOGIC

**M. Pavičić**

*Department of Mathematics, University of Zagreb,*
*Kačićeva 26, Pošt. pret. 217, CRO–41001 Zagreb, Croatia.*

**Summary.** Unified quantum logic which is a propositional logic underlying quantum formalism is given a new much simplified axiomatization. A statistical basis for this propositional logical system is given so as to interpret unified quantum logic as a system of deduction. The soundness and completeness of algebraic semantics are proved. Kripkean and probabilistic semantics are discussed.

PACS numbers: 03.65.Bz — *Quantum logic, quantum measurements.*

## 1. INTRODUCTION

The question as to whether quantum logic can be considered a theory of deduction underlying quantum mechanics has been given many apparently contradictory answers to date. It has been argued that quantum logic is necessarily an empirical logic and that, therefore, it cannot be a theory of *a priori* valid inferences.[18,*Ch.8.6*] At the same time, many axiomatic deductive calculuses, all of which have an equational class of orthomodular lattices as their model, have actually been formulated. [For a review see 29]

The two approaches are only apparently contradictory since they are actually yet another expression of the individual versus the statistical interpretation of quantum mechanics.[18] A difference between the two approaches formulated *within* the Hilbert space frame of quantum mechanics is given in Refs. [30,32] where we investigate quantum probability equal to unity which characterizes *statistically repeatable* and *predictable* measurements of the first kind. In this case probability equal to unity ascribes a unique value of a measured observable to the ensemble of individual systems measured. It does not necessarily ascribe a unique experimental value of the measured observable to *all* individual systems as it does to an ensemble of them. Whether it does or not, depends on a particular function. The function would exhibit a jump at one end point of the closed probability interval [0,1] if the probability equal to one ascribed the value to *all* the individual systems. If not, it would stay continuous. We review the result in Sec. 2.

*International Journal of Theoretical Physics*, **31**, 1753–1766 (1992)

While a difference in experimental values of a function defined on the semi–closed interval [0,1) as opposed to the closed interval [0,1] cannot be expected to be measurable, an important physical contribution of this result is that the assumptions of the individual repeatability and "non–repeatability" (of *statistically repeatable* quantum measurements) correspond to properties of a well–defined function and that the only way to ascribe an individual system a proper value registered by a statistically repeatable measurement is to *postulate* this.

Thus it seems that quantum logic is first of all an *a priori* calculus which is surely weakly confirmable by quantum mechanics in the same way in which classical logic is confirmable by classical mechanics. Classical logic has a distributive lattice as its model which enables us to build up the phase space for classical logic, and quantum logic has an orthomodular lattice as its model which enables us to build up the Hilbert space for quantum mechanics. It may be that (like classical logic) quantum logic is strongly empirically confirmable as well, since quantum formalism is in agreement with both possibilities. However, such an additional feature of individual quantum measurements can hardly be proved by experiments conceivable today.

Whether or not quantum logic can be considered an *a priori* axiomatic calculus underlying quantum mechanics has met with scepticism for yet other reasons. First, the objection has been raised that quantum logic does not satisfy many axioms and rules valid in classical logic some of which have traditionally been taken to be indispensable to a "proper" logic. [18] Such an objection has gradually been dropped since many quantum logics were actually axiomatized "in a manner completely analogous to classical propositional logic." [14] Secondly, a problem has been raised about the fact that quantum logics using different operations of implications can apparently not satisfy a common axiomatic system.[8,13,15,35] The latter objection was met in [29] where exactly such a system is formulated and named unified quantum logic. In Sec. 3 we present a new and essentially simpler axiomatization of unified quantum logic.

Finally, quantum logic lacks simple non–algebraic semantics which are apparently needed as a clue to certain important unresolved problems, e.g., as to whether quantum logic has a finite model property or whether it is decidable.[11] Several such semantics have been formulated (Kripkean semantics by Goldblatt [10] and Dalla Chiara [4], and probabilistic semantics by Bodiou [2], Morgan [24] and Pavičić [27]) but none of them have proved successful in solving these problems. Most probabilistic semantics show that a probability function needed to prove the completeness theorem for the semantics is not guaranteed existence so far as quantum logic proper is concerned. It seems, however, that by adding particular new axioms, thus obtaining a logic between orthomodualar logic and modular logic, we can assure the existence of such a function.[21,22] An analogous conclusion can be conjectured for the reflexive and symmetric Kripkean accessibility relation used by Goldblatt [11] to prove that there are no first–order conditions imposable on such a relation in order to give a proper semantics. In Sec. 4 we, therefore, indicate a possibility of using another relation of accessiblity and discuss some problems of Kripkean and probabilistic semantics

## 2. RELATIVE–FREQUENCY APPROACH TO REPEATABLE MEASUREMENTS

Our aim is to formulate an expression [given by Eq. (1)] which is a function of the relative frequency of the measured data as well as of the corresponding theoretical (Hilbert space) probability and which has a well–defined physical meaning.

When individual quantum systems are subjected to YES–NO measurements of a discrete observable, unrestricted by any conservation law, the eigenvalue of the measured observable (projector) corresponds to a particular property of the *ensemble* of the individual systems. For repeated YES–NO measurements of a discrete observable a YES–event occurs almost certainly, i.e. with probability equal to unity, and from a statistical point of view such measurements *are* repeatable. However, in looking at individual events we face the following dilemma.

We can take the view that a YES–event with probability one always occurs. In this case a measurement is considered repeatable in both senses: statistical and individual. An individual system is then considered to possess a particular property strictly.

The other possibility is to assume that a YES–event with probability one occurs *almost* always. In this case the individual repeatability is not admitted. An individual system is then considered to possess no particular property strictly.

The afore–mentioned expression [given by Eq. (1)] takes two different values for each of the two possibilities and is therefore a "measure" for individual repeatability.

Let us consider spin preparation–detection measurements for spin–$s$ particles. Quantum systems are prepared, one by one, by a preparation device (a Stern–Gerlach device) and detected, one by one, by a detection device (another Stern–Gerlach device) deflected at an angle $\alpha$ relative to the preparation device. In effect, we carry out quantum YES–NO measurements. Quantum mechanics then predicts that the relative frequency $N_+/N$ of the number $N_+$ of detections of the prepared property (spin projection $m$ prepared in the *statistical* sense of the word) on the systems among the total number $N$ of the prepared systems approaches probability $p = p^{(s)}_{mm}(\alpha) = [d^s_{mm}(\alpha)]^2$ (where $d^s_{mm}(\alpha)$ is a diagonal element of the rotation matrix).

The first basic feature of any quantum YES–NO measurement of the first kind is that particular individual events are completely independent. The second basic feature of such measurements is that trials form an exchangeable sequence. Taken together, the trials are Bernoulli trials, i.e. they form Bernoulli sequences. Thus we can estimate ideal quantum frequencies, i.e. frequencies of an infinite number of individual YES–NO experiments, by means of quantum theoretical probabilities as elaborated below.

A direct consequence of the law of large numbers for Bernoulli trials is

$P(\lim_{N\to\infty} \frac{N_+}{N} = p) = 1, \qquad \text{where } p = \langle \frac{N_+}{N} \rangle.$

Starting from this expression and the following lemmas (proved in [30]),

**Lemma 2.1.** $\lim_{N\to\infty} P(\frac{N_+}{N} = p) = 0.$

**Lemma 2.2.** *$\lim_{N\to\infty} P(p-\eta\Delta p \le \frac{N_+}{N} \le p+\eta\Delta p) = \frac{1}{2\pi}\int_{-\eta}^{\eta} e^{-x^2/2}dx$, where $\Delta p = \sqrt{\frac{p(1-p)}{N}}$, and $0<|\eta(N)|<\infty$.*

we are able to prove (in Ref. [32]) the following theorems.

**Theorem 2.1.** *The function*

$$G(p) \stackrel{\text{def}}{=} L^{-1} \lim_{N\to\infty} \big[|\alpha(\frac{N_+}{N}) - \alpha(p)|N^{1/2}\big], \tag{1}$$

*where $L = \lim_{N\to\infty}|\eta(N)|$ is a bounded random (stochastic) variable, is well–defined and continuous on the open interval $(p_1,1)$, where $p_1 = [d^s_{mm}(\alpha_1)]^2$, where $\alpha_1$ is such (always existing) that $p = [d^s_{mm}(\alpha)]^2$ is a continuous monotonic decreasing function defined on $[0,\alpha_1]$ and differentiable on $(0,\alpha_1)$.*

**Theorem 2.2.** *If*

$$H(p) \stackrel{\text{def}}{=} \sqrt{p(1-p)}\,\Big|\frac{d\,p}{d\alpha}\Big|^{-1} = \frac{1}{2}\sqrt{1-(d^s_{mm})^2}\,\Big|\frac{d\,(d^s_{mm})}{d\alpha}\Big|^{-1}, \qquad 0\le p\le 1,$$

*then $G(p) = H(p)$ on $(p_1,1)$.*

**Theorem 2.3.**

$$\lim_{p\to 1} G(p) = \lim_{p\to 1} H(p) = \lim_{\alpha\to 0} H\big[p(\alpha)\big] = [2(s^2+s-m)]^{-1/2}, \tag{2}$$

*where $m = -s,\dots,+s$.*

Turning our attention to the probability equal to one we see from the definition of $H(p)$ given in Theorem 2 that $H$ is not defined for the probability equal to one: $H(1) = \frac{0}{0}$. However, its limit exists and is given by expression (2). Thus a continuous extension of H to $(p_1,1]$ exists and is given by $\tilde H(p)$ for $p\in(p_1,1)$ and $\tilde H(1)$ is equal to the right–hand side of Eq. (2).

The function $G$, on the other hand, cannot be approached in the same way because we do not know whether $G(1)$ is defined at all and if it is we do not know which values it should be ascribed.

We do not know whether $G(1)$ is defined or not because the strong law of large numbers, which alone establishes the link between the probability and the relative frequency in question, is simply not valid for the end points of the closed interval [0,1]. It is valid only on the open interval $0<p<1$.

And we do not know which values it should be ascribed if it is defined because the quantum formalism does not say anything on the relative frequency corresponding to $p$=1 either.

Thus we are left with the following three possibilities.

1. $G(p)$ is continuous at 1. A necessary and sufficient condition for this is $G(1) = \lim_{p\to 1} G(p)$. In this case we cannot strictly have $N_+ = N$ since then $G(1) = 0 \neq \lim_{p\to 1} G(p)$ obtains a contradiction.
2. $G(1)$ is undefined. In this case we also cannot have $N_+ = N$ since the latter equation makes $G(1)$ defined, i.e. equal to zero.
3. $G(1) = 0$. In this case we must have $N_+ = N$. And *vice versa*: if the latter equation holds we get $G(1) = 0$.

Hence, a measurement of a discrete spin observable $s$ can be considered repeatable with respect to individual measured systems if and only if $G\big(p^{(s)}_{mm}\big)$ exhibits a jump–discontinuity for $p^{(s)}_{mm}=1$ in the sense of point 3 above.

The interpretative differences between the points are as follows.

1 & 2 admit only the statistical interpretation of the quantum formalism and banish the repeatable measurements on individual systems from quantum mechanics altogether. Possibility 1 seems to be more plausible than possibility 2 because the assumed continuity of $G$ makes it approach its classical value for large spins. Notably, for a classical probability we have $\lim_{p\to 1} G_{\rm cl}(p) = 0$ and from the expression (2) we get $\lim_{s\to\infty} \lim_{p\to 1} G(p) = 0$.

Point 3 admits the individual interpretation of quantum formalism and assumes that the repeatability in the statistical sense implies the repeatability in the individual sense. By adopting this interpretation we cannot but assume that nature differentiate open intervals from closed ones, i.e. distinguishes between two infinitely close points.

The result obtained supports the view that the logic underlying quantum formalism is based on the statistics of individual quantum measurements and not on the individual quantum measurements themselves.

## 3. THE NEW AXIOMATIZATION OF UNIFIED QUANTUM LOGIC

Quantum logic is usually not considered a proper logical system because of the lack of a proper operation of implication and because of the lack of a proper Kripkean semantics. Thus quantum logics which have an orthomodular lattice as their Lindenbaum–Tarski algebra, i.e. as their model, either use a relation of implication [7,10,25] or one of the five possible operations of implication.[1,5,8,16,19,34] It has been conjectured that no common axiomatization exists for the latter systems. In Ref. [30] this conjecture is disproved by constructing a common system named unified quantum logic which merges all five operations of implication. In this section we present a much simplified axiomatization of unified quantum logic. (See Theorem 1 given below when comparing the axiomatization with the one presented in Ref. [30].)

The propositions are based on elementary propositions $p_0, p_1, p_2, \ldots$ and the following connectives: $\neg$ (negation), $\to$ (implication), and $\vee$ (disjunction).

The set of propositions $Q^0$ is defined formally as follows:

$p_j$ is a proposition for $j = 0, 1, 2, ...$
$\neg A$ is a proposition iff $A$ is a proposition.
$A \to B$ is a proposition iff $A$ and $B$ are propositions.
$A \vee B$ is a proposition iff $A$ and $B$ are propositions.

The conjuction is introduced by the following definition: $A \wedge B \stackrel{\text{def}}{=} \neg(\neg A \vee \neg B)$.

Our metalanguage consists of axiom schemata from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: & (*and*), $\sim$ (*not*), $\Rightarrow$ (*if…, then*), and $\Leftrightarrow$ (*iff*), with the usual *classical* meaning.

We define unified quantum logic UQL as the axiom system given below. The sign $\vdash$ may be interpreted as "*it is asserted in* UQL." Connective $\neg$ binds stronger and $\to$ weaker than $\vee$ and $\wedge$, and we shall occasionally omit brackets under the usual convention. To avoid a clumsy statement of the rule of substitution, we use axiom schemata instead of axioms and from now on whenever we mention axioms we mean axiom schemata.

**Axiom Schemata.**

A1. $\vdash A \to A$

A2. $\vdash A \leftrightarrow \neg\neg A$

A3. $\vdash A \to A \vee B$

A4. $\vdash B \to A \vee B$

A5. $\vdash B \to A \vee \neg A$

**Rules of Inference.**

R1. $\vdash A \to B \quad \& \quad \vdash B \to C \quad \Rightarrow \quad \vdash A \to C$

R2. $\vdash A \to B \quad \Rightarrow \quad \vdash \neg B \to \neg A$

R3. $\vdash A \to C \quad \& \quad \vdash B \to C \quad \Rightarrow \quad \vdash A \vee B \to C$

R4. $\vdash (B \vee \neg B) \to A \quad \Leftrightarrow \quad \vdash A$

The operation of implication $A \to B$ is one of the following:

$$A \to_1 B \stackrel{\text{def}}{=} \neg A \vee (A \wedge B) \qquad \text{(Mittelstaedt)}$$

$$A \to_2 B \stackrel{\text{def}}{=} \neg B \to_1 \neg A \qquad \text{(Dishkant)}$$

$$A \to_3 B \stackrel{\text{def}}{=} (\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee \big((\neg A \vee B) \wedge A\big) \qquad \text{(Kalmbach)}$$

$$A \to_4 B \stackrel{\text{def}}{=} \neg B \to_3 \neg A \qquad \text{(non–tollens)}$$

$$A \to_5 B \stackrel{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B) \qquad \text{(relevance)}$$

We prove that UQL is really quantum logic by proving that UQL has an orthomodular lattice as a model.

**Definition 3.1.** *We call* $\mathcal{L} = < L, h >$ *a model of the set* $Q^0$ *if* $L$ *is an orthomodular lattice and if* $h\colon \mathrm{UQL} \mapsto L$ *is a morphism in* $L$ *preserving the operations* $\neg$, $\vee$, *and* $\rightarrow$ *while turning them into* $^\perp$, $\cup$, *and* $\supset_i$ $(i = 1,\ldots,5)$,† *and satisfying* $h(A) = 1$ *for any* $A \in Q^0$ *for which* $\vdash A$ *holds.*

**Definition 3.2.** *We call a proposition* $A \in Q^0$ *true in the the model* $\mathcal{L}$ *if for any morphism* $h\colon \mathrm{UQL} \mapsto L$, $h(A)=1$ *holds.*

We prove the soundness of UQL for valid formulas from $L$ by means of the following theorem.

**Theorem 3.1.** $\vdash A \quad \Longrightarrow \quad$ *A is true in any orthomodular model of* UQL.

*Proof.* By analogy with the binary formulation of quantum logic, [10,28] it is obvious that A1–A5 hold true in any $\mathcal{L}$, and that the statement is preserved by applications of R1–R3. Verification of R4 is also straightforward and we omit it. ■

Let us now prove some simple theorems for subsequent usage and for the sake of completeness.

**Theorem 3.2.**

| | | | |
|---|---|---|---|
| T1. | $\vdash A \rightarrow B$ | $\Longrightarrow$ | $\vdash A \wedge C \rightarrow B \wedge C$ |
| R5. | $\vdash A \leftrightarrow B$ | $\Longrightarrow$ | $\vdash (C \rightarrow A) \leftrightarrow (C \rightarrow A)$ |
| R6. | $\vdash A \leftrightarrow B$ | $\Longrightarrow$ | $\vdash (A \rightarrow C) \leftrightarrow (B \rightarrow C)$ |

*Proof.* T1 is trivially satisfied in any ortholattice.

The derivation of R5 and R6 is straightforward but tedious since it involves an explicit handling of all five afore–given implications in turn. Therefore we shall only illustrate it by deriving R6 for $\rightarrow_1$ and leave the rest to the reader.

Let us first consider $A \rightarrow B$.

$$\vdash A \rightarrow B \quad \Rightarrow \quad [\text{T1, A4, \& R1}] \quad \Rightarrow \ \vdash A \wedge C \rightarrow \neg B \vee (B \wedge C). \tag{3}$$

On the other hand,

$$\vdash B \rightarrow A \quad \Rightarrow \quad [\text{R2, A3, \& R1}] \ \Rightarrow \ \vdash \neg A \rightarrow \neg B \vee (B \wedge C) \tag{4}$$

Combining (3) and (4) and using R3 we obtain

$$\vdash A \leftrightarrow B \quad \Rightarrow \ \vdash \neg A \vee (A \wedge C) \rightarrow \neg B \vee (B \wedge C) \quad \Longleftrightarrow \ \vdash (A \rightarrow_1 C) \rightarrow (B \rightarrow_1 C).$$

Similarly we obtain

$$\vdash A \leftrightarrow B \quad \Rightarrow \ \vdash (B \rightarrow_1 C) \rightarrow (A \rightarrow_1 C).$$

Hence R6 for $\rightarrow_1$.

In an analogous way we prove R5 as well as R6 for the other implications. ■

---

† Defined in a lattice by analogy with the definitions given above. See [28,29] for details.

**Theorem 3.3.** *Let* $\mathrm{UQL}_i$ *denotes* UQL *with* $\to\, = \,\to_i$, $i = 1,\ldots,5$. *Then in any* $\mathrm{UQL}_i$ *we can infer* A1–A5 *and* R1–R4 *for any* $\to_j$, $j = 1,\ldots,5$.

*Proof.* Straightforward. ■

**Theorem 3.4.** UQL *with* $A \to B \;=\; A \rightharpoonup B \overset{\mathrm{def}}{=} \neg A \vee B$ *is classical logic.*

*Proof.* Straightforward. ■

To prove the completeness of UQL for the class of valid formulas of $L$, we first define relation $\equiv$ and prove some related lemmas.

**Definition 3.3.** $\qquad A \equiv B \overset{\mathrm{def}}{=} \;\vdash A \leftrightarrow B.$

where $\;\vdash A \leftrightarrow B\;$ means $\;\vdash A \to B \;\;\&\;\; \vdash B \to A.$

**Lemma 3.1.** *The relation* $\equiv$ *is a congruence relation on the algebra of propositions* $\mathcal{A} = < Q^0, \neg, \vee, \to >$.

*Proof.* The proof for $\neg$ and $\vee$ is obvious. The proof for $\to$ is (with the help of R5 and R6 from Th. 2) straightforward and we omit it. ■

**Lemma 3.2.** *The Lindenbaum–Tarski algebra* $\mathcal{A}/\!\equiv$ *is an orthomodular lattice, i. e., the conditions defining the lattice are true for* $\neg/\equiv$, $\vee/\equiv$, *and* $\to /\equiv$ *turning into* $^\perp$, $\cup$, *and* $\supset_i$ *by means of natural isomorphism* $k{:}\mathcal{A} \mapsto \mathcal{A}/\!\equiv$ *which is induced by the congruence relation* $\equiv$ *and which satisfies* $k(\neg A) = [k(A)]^\perp$, $k(A \vee B) = k(A) \cup k(B)$, *and* $k(A \to B) = k(A) \supset_i k(B)$.

*Proof.* On account of formal analogy with the binary formulation of quantum logic, we consider the proofs of the conditions for an ortholattice to be well–known and we omit them. As for the orthomodularity we shall prove

$$a \supset_i b = 1, \quad i = 1,\ldots,5 \qquad \Longleftrightarrow \qquad a \cup b = b \tag{5}$$

which is yet another way to express it as shown in [29].

Let us assume $\vdash A \to B$. By A1 and R3 we obtain $\vdash A \vee B \to B$ and A6 gives $\vdash B \to A \vee B$. Therefore $\vdash A \vee B \leftrightarrow B$.

On the other hand, the assumption can, with the help of R4 and A5, be expressed as $\vdash (A \to B) \leftrightarrow (C \vee \neg C)$.

Taken together, we obtain the following metaequivalence:

$\vdash (A \to B) \leftrightarrow (C \vee \neg C) \quad \Leftrightarrow \;\vdash (A \vee B) \leftrightarrow B.$

Thus we get $\;k(A) \supset_i k(B) = 1 \quad \Leftrightarrow \quad k(A) \cup k(B) = k(B)\,$. Hence (5) holds. ■

**Corrolary.** $\quad \mathcal{A}/\!\equiv$ *is a model of theses of* UQL.

**Lemma 3.3.** $k(A) = 1 \quad \Longrightarrow \quad \vdash A.$

*Proof.* Since $k(B \vee \neg B) = 1$, we have $k(B \vee \neg B) = k(A)$, i.e., $(B \vee \neg B) \equiv A$, and we obtain the statement by R4. ■

Thus we have proved the completeness of UQL for valid formulas of $L$, that is, the following theorem.

**Theorem 3.5.** *If $A$ is true in any model of* UQL, *then* $\vdash A$.

Taken together, UQL is a proper quantum–logical deductive system so far as its algebraic semantics is concerned.

## 4. KRIPKEAN AND PROBABILISTIC SEMANTICS FOR QUANTUM LOGIC

Instead of a conclusion, in this section we shall review some results and open problems of the semantical approach to quantum logic.

Another sense in which an axiomatic system can be considered a proper logic is given by the possibility of finding a particular relation of accessibility which characterizes the system, thus equipping it with a modal, i.e. Kripkean semantics.

Once found, the relation of accessibility may offer a canonical model which would falsify all non–theorems, i.e., establish decidability and possibly even the finite model property. For quantum logic such a relation has not been found as yet. What has been achieved is a way of imposing a particular restriction on a frame characterizing a weaker, so called orthologic or minimal quantum logic thus obtaining a Kripkean "quasi–semantics" for quantum logic.[10,25] The relation of accessibility used for this purpose is a reflexive and symmetric one. It determines the orthoframe which characterizes minimal quantum logic. Whether a class of orthoframes characterizes quantum logic proper is not known. What is known is that even if it does the frames cannot be defined by first–order conditions on a reflexive and symmetric relation of accessibility as proved by Goldblatt.[11] He proved this using a correspondence with the Hilbert space where a negation of the orthogonality relation can play the role of a reflexive and symmetric relation of accessibility.

However, if it were possible to find a relation of accessibility for quantum logic which is not reflexive and symmetric, then the possibility of imposing first–order conditions on such a relation in order to characterize the logic would still be open. For, although the irreflexive† and symmetric orthogonality relation obviously plays a crucial role in an algebraico–logical representation of the Hilbert space quantum formalism such a relation does not necessarily characterize the propositional logic underlying the formalism.

---

† The orthogonality relation cannot be used directly to characterize an axiom in modal quantum logic since the irreflexiveness corresponds to no axiom. Thus the negation of the orthogonality relation has been used to give the accessibility relation.

Besides, even if the orthomodularity itself is not characterized by any first–order conditions such investigations might help to find a possible characterization of propositional logic underlying Hilbertian quantum formalism which may be characterized by first–order conditions despite the fact that it contains the orthomodualrity axiom.

What we have in mind is a possible parallel with the following results in modal logic.

The system KM,† where

M: $\quad \Box\Diamond A \rightharpoonup \Diamond\Box A,$

is not first–order definable while KT4M (S4+M) as well as K4M are. [17]

In quantum formalism the propositional logic underlying the Hilbert space is not a *bare* quantum logic since it possesses other properties, such as the Desarguesian one, as well.[9] Thus *proper quantum* propositional logic is stronger than quantum logic *proper*. Whether this logic is characterized by a reflexive and symmetric relation of accessibility is an open question and therefore it makes sense to investigate other possibilities.

One way to find a relation of accessibility for quantum logic is to embed it in a modal system characterized by the relation.

Originally Goldblatt [10] and Dalla Chiara [6] embedded minimal quantum logic into the Brouwerian KTB system‡ and Dishkant [6] embedded quantum logic into an extension of KTB.

As of a relation which is neither reflexive nor symmetric in Ref. [29] we carried out an embedding into modal system $\mathrm{Br}^-$ characterized by the following conditions (containing reflexivity and symmetry as a special case) on the relation of accessibility R:

$$\forall w_1 \exists w_2 \big[ w_1 R w_2 \ \ \& \ \ \forall w_3 (w_2 R w_3 \ \Rightarrow \ w_1 R w_3)\big],$$

$$\forall w_1 \forall w_2 \big\{ w_1 R w_2 \ \Rightarrow \ \exists w_3 \big[ w_2 R w_3 \ \ \& \ \ \forall w_4 (w_3 R w_4 \ \Rightarrow \ w_1 R w_4)\big]\big\} \ .$$

The embedding is carried out by a translation which differs from the one used in [10] and [4].

In Ref. [31] we carried out an embedding into a modal system which is $\mathrm{Br}^-$ extended by an axiom using the same translation as in [10] and [4].

Yet another semantical approach to quantum logic can be achieved by means of probabilistic semantics.

Probabilistic semantics lacks *possible worlds* and *frames* but it proves useful when the relation of accessibility cannot be characterized by first–order conditions. E.g., it has been proved that there is a probabilistic semantics for every extension of classical sentence logic. [23]

---

† To designate modal systems we mostly adopt the classification from [3].

‡ T: $\Box A \rightharpoonup A$ is characterized by a reflexive and B: $A \rightharpoonup \Box\Diamond A$ by a symmetric relation of accessibility.

Probabilistic semantics have been formulated for quantum logic but they are still far from being satisfactory. Most probabilistic semantics use probability functions which are actually *states* , i.e., have the strong orthogonality property and/or the Jauch–Piron property.[2,20,24,27] And it is well–known that there are Lindenbaum–Tarski algebras for quantum logic, i.e., orthomodular lattices, which do not admit a state on them. [12] Thus a probability function needed to prove the completeness theorem for the semantics is not guaranteed existence so far as quantum logic proper is concerned. Our conjecture is (in the same way as for Kripkean semantics and the relation of accessibility problem) that by adding particular new axioms, thus obtaining a logic between quantum logic and a modular logic, we can assure the existence of such a function. [21,22] There is, however, an alternative probabilistic semantics formulated by Morgan [24] whose probability function has its existence assured. It would be interesting to know whether there is a parallel between such alternative probabilistic semantics and the afore–mentioned alternative relations of accessibility.

## REFERENCES


[1] **Abbott, J. C. (1976),** Orthoimplication algebras, *Studia Logica,* **35,** 173–177.

[2] **Bodiou, G. (1957),** Probabilité sur un treillis non modulaire, *Publ. Inst. Statist. Univ. Paris,* **6,** 11–25.

[3] **Chellas, B. F. (1980),** *Modal logic: an introduction,* Cambridge University Press, Cambridge.

[4] **Dalla Chiara, M. L. (1986),** Quantum logic, in Gabbay, D. and F. Guenthner (eds.), *Handbook of philosophical logic, Vol. III,* D. Reidel, Dordrecht–Holland (1986), pp. 427–469.

[5] **Dishkant, H. (1974),** The first order predicate calculus based on the minimal logic of quantum mechanics, *Rep. Math. Logic,* **3,** 9–18.

[6] **Dishkant, H. (1977),** Imbedding of the quantum logic in the modal system of Brower, *J. Symb. Logic,* **42,** 321–328.

[7] **Finch, P. D. (1970),** Quantum logic as an implication algebra, *Bull. Austral. Math. Soc.* **2,** 101–106.

[8] **Georgacarakos, G. N. (1980),** Equationally definable implication algebras for orthomodular lattices, *Studia Logica,* **39,** 5–18.

[9] **Godowski, R. and R. Greechie (1984),** Some equations related to the states on orthomodular lattices, *Demonstratio Math.* **17,** 241–250.

[10] **Goldblatt, R. I. (1974),** Semantic analysis of orthologic, *J. Phil. Logic,* **3,** 19–35.

[11] **Goldblatt, R. I. (1984),** Orthomodularity is not elementary, *J. Symb. Logic,* **49,** 401–404.

[12] **Greechie, R. J. (1971),** Orthomodular lattices admitting no states, *J. Comb. Theory,* **10 A,** 119–132.

[13] **Hardegree, G. M. (1975),** Stalnaker conditionals and quantum logics, *J. Phil. Logic,* **4,** 399–421.

[14] **Hardegree, G. M. (1979),** The conditional in abstract and concrete quantum logic, in Hooker, C. A. (ed.), *The logico-algebraic approach to quantum mechanics,* Volume II. *Contemporary consolidation,* D. Reidel, Dordrecht-Holland (1979), pp. 49–108.

[15] **Hardegree, G. M. (1981),** Material implication in orthomodular (and Boolean) lattices, *Notre Dame J. Formal Logic,* **22,** 163–183.

[16] **Hardegree, G. M. (1981a),** An axiomatic system for orthomodular quantum logic, *Studia Logica,* **40,** 1–12.

[17] **Hughes, G. E. and M. J. Cresswell (1984),** *A companion to modal logic,* Methuen, New York, p. 47.

[18] **Jammer, M. (1974),** *The philosophy of quantum mechanics. The interpretations of quantum mechanics in historical perspective,* John Wiley & Sons, New York.

[19] **Kalmbach, G. (1974),** Orthomodular logic, *Z. Math. Logik Grundl. Math.* **20,** 395–406.

[20] **Mączyński, M. (1973),** The orthogonality postulate in axiomatic quantum mechanics, *Int. J. Theor. Phys.* **8,** 353–360.

[21] **Mayet, R. (1985),** Varieties of orthomodular lattices related to states, *Algebra Universalis,* **20,** 368–396.

[22] **Mayet, R. (1986),** Equational basis for some varieties of orthomodular lattices related to states, *Algebra Universalis,* **23,** 167–195.

[23] **Morgan, C. G. (1982),** There is a probabilistic semantics for every extension of classical sentence logic, *J. Phil. Logic,* **11,** 431–442.

[24] **Morgan, C. G. (1983),** Probabilistic semantics for orthologic and quantum logic, *Logique et Analyse,* **26,** *(103–104)* 323–339.

[25] **Nishimura, H. (1980),** Sequential method in quantum logic, *J. Symb. Logic,* **45,** 339–352.

[26] **Pavičić, M. (1983),** The other way round: Quantum logic as metalogic, in Weingartner, P. and J. Czermak (eds.), *Epistemology and philosophy of science,* [Proceedings of the 7th International Wittgenstein symposium, Kirchberg am Wechsel, Austria, August 22–29, 1982], D. Reidel / Hölder–Pichler–Tempsky, Dordrecht–Holland / Wien (1983), pp. 402–407.

[27] **Pavičić, M. (1987),** Probabilistic semantics for quantum logic, in Rabinovich, V. L. (1987) (ed.), *Abstracts of the 8th International Congress on Logic, Methodology, and Philosophy of Science,* [Moscow, August, 1987], Academy of sciences of the USSR, Moscow, **Vol. 2,** pp. 105–107.

[28] **Pavičić, M. (1987a),** Minimal quantum logic with merged implications, *Int. J. Theor. Phys.* **26,** 845–952.

[29] **Pavičić, M. (1989),** Unified quantum logic, *Found. Phys.* **19,** 999–1016.

[30] **Pavičić, M. (1990),** A relative frequency criterion for the repeatability of quantum measurements, *Nuovo Cim.* **105 B,** 1103–1112. Errata, *Ibid.* **106 B,** 105–106.

[31] **Pavičić, M. (1990a),** A theory of deduction for quantum mechanics, *Nuova Critica,* **I–II,** *(Nuova Serie), Quaderno 13–14,* 109–129.

[32] **Pavičić, M. (1990b),** There is a formal difference between the Copenhagen and the statistical interpretation of quantum mechanics, in Mizerski, J., A. Posiewnik, J. Pykacz, and M. Żukowski (Eds.), *Problems in Quantum Physics; Gdańsk '89,* World Scientific, Singapore – New Jersey (1990), pp. 440–452.

[33] **Piron, C. (1976),** *Foundations of quantum physics,* W. A. Benjamin, Reading, Mass..

[34] **Piziak, R. (1974),** Orthomodular lattices as implication algebras, *J. Phil. Logic,* **3,** 413–438.

[35] **Zeman, J. J. (1978),** Generalized normal logic, *J. Phil. Logic,* **7,** 225–243.