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On a formal difference between the individual and statistical interpretation of quantum theory

M. Pavičić [1]

Laboratoire de Méchanique Quantique, University of Reims, F-51062 Reims Cedex, France

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A discussion is given of a recently formulated function which exhibits a jump for the end points of a closed interval in case of the individual interpretation of quantum measurements of the first kind and stays continuous on the whole interval for their statistical interpretation.

Recently I formulated [1] a function which differentiates quantum events which occur with probability equal to unity from the ones which factually occur. The result served me to establish [2] a formal difference between the individual (Copenhagen [3, p. 440], orthodox [4]) and statistical [5] interpretation of quantum theory *within its formalism* taking the function to represent von Neumann's *property* [6]. This *formal* difference between the two interpretations is not measurable but nevertheless provides a theoretical distinction between them. Since such a difference was taken to be "highly significant for the interpretation of quantum theory" [4] and was at the same time criticized [4] I consider it helpful to discuss it in some detail, which is the purpose of this Letter.

The only way in which quantum theory – without regard to the collapse postulate – connects the "elements of the physical reality" (i.e. what we observe) with their "counterparts in the theory" [7] is by means of the Born formula which gives us the probability that the outcome of an experiment will confirm an observable or a property of an *ensemble* of systems [6, p. 439].

In case of discrete observables we are able to prepare a property and we can, by an appropriate detection (determination, measurement), verify the property with certainty – i.e. with probability one/equal to unity ([7, p. 777, 7th line from below], [6, pp. 250, 439]), i.e. almost certainly, almost sure or "except on a null-event" [8, p. 20]. This means that for repeatable measurements we only know that a property will be verified with certainty (with probability one) – that is for *an ensemble*. Whether the property will be verified for *each* individual system thus prepared we can only guess. For there is no "counterpart in the theory" of an individual detection even if it is carried out "with certainty": The Born probabilistic formula – which is the only link between the theory and measurements – refers only to ensembles. However, as shown below, we can consistently *postulate* whether a measurement of the first order verifies a prepared repeatable property for *each* system or not.

The approach I took rests on combining the Malus angle expressed by probability with that expressed by relative frequency. For connecting probability $0<p<1$ with the corresponding relative frequency I used the strong law of large numbers for the infinite number of Bernoulli trials which – being independent and exchangeable – perfectly represent quantum measurements on individual quantum systems. These properties of the individual quantum measurements I used to reduce their repeatability to suc-

[1] Permanent address: Department of Mathematics, University of Zagreb, Kačićeva 26, Pošt. pret. 217, CRO-41001 Zagreb, Croatia.

cessive measurements (as noticed in ref. [4]) but that has no influence on the whole argumentation which rests exclusively on the fact that finitely many experiments out of infinitely many of them may be assumed to fail and nevertheless to build up to probability one.

The argument supporting the statistical interpretation is that probability one of e.g. electrons passing perfectly aligned Stern–Gerlach devices *does imply* that the relative frequency N_+/N of the number N_+ of detections of the prepared property (e.g. *spin-up*) for the systems among the total number N of the prepared systems approaches probability $p=\langle N_+/N\rangle = 1$ almost certainly:

$$P\left(\lim_{N\to\infty}\frac{N_+}{N}=1\right)=1\,, \qquad (1)$$

but *does not imply* that N_+ analytically equals N, i.e. it does not necessarily follow that the analytical equation $N_+=N$ should be satisfied.

We therefore must postulate what we want: either $N_+=N$ *and* (1) or $N_+\neq N$ *and* (1). (Since already the central limit theorem itself, which served us to infer (1), holds only on the open interval $0<p<1$, it would be inconsistent to try to *prove* one or the other possibility and I therefore cannot agree with Home and Whitaker when they say: "Even though the intrinsic probability of an event may be unity, [Pavičić] attempts to show that the relative frequency of occurrence cannot be." [4].)

Of course, the possibility $N_+\neq N$ does not seem very plausible by itself and we therefore used the Malus law to construct the function which reflects the two possibilities and proved a theorem which directly supports another difference between the probability and frequency treatment of individual quantum measurements.

As for the theorem we proved that

$$\lim_{N\to\infty} P\left(\frac{N_+}{N}=p\right)=0, \quad 0<p<1\,, \qquad (2)$$

which expresses randomness of individual results as clustering only around p (*almost* never strictly at p). (As I learned recently, a related result was achieved by Mugur-Schächter [9] in a different informational content.) Thus it does not take two theories [3, p. 441] for a distinction between probabilities and frequencies. It suffices to ascribe frequencies to individual systems and related probabilities to their ensembles in order to obtain the difference expressed by eq. (2).

As for the function which reflects the two above stated possibilities I will just briefly sketch it here. The reader can find all the relevant theorems and proofs in ref. [1], a generalization to the spin-s case in ref. [2], and a discussion with possible implications for the algebra structure underlying quantum theory in ref. [10]. The function refers to the quantum Malus law and reads

$$G(p) \stackrel{\text{def}}{=} L^{-1}\lim_{N\to\infty}\left[\,|\alpha(N_+/N)-\alpha(p)|N^{1/2}\right]\,,$$

where α is the angle at which the detection device (a Stern–Gerlach device for spin-s particles, an analyzer for photons) is deflected with regard to the preparation device (another Stern–Gerlach device, polarizer) and where L is a bounded random (stochastic) variable: $0<L<\infty$. The function is well defined and continuous (or piecewise continuous) on the open interval (0, 1). In general it does not correspond to an operator but it does represent a *property* in the sense of von Neumann [6]. For electrons, e.g., it is equal [1]:

$$G(p)=H(p)\stackrel{\text{def}}{=}H[p(\alpha)]=\frac{\sin\alpha}{\sin\alpha}\,.$$

Turning our attention to the probability equal to one we see [1] from the definition of $H(p)$ that H is not defined for the probability equal to one: $H(1)=\frac{0}{0}$. However, its limit exists and equals 1. Thus a continuous extension $\tilde{H}$ of H to [0,1] exists and is given by $\tilde{H}(p)=1$ for $p\in(0,1)$ and $\tilde{H}(1)=1$.

We now assume (completely agreeing with ref. [4] that one cannot *prove* this but that is exactly the point of postulating one or the other possibility) that L is bounded and positive not only for $0<p<1$ but for $0\leqslant p\leqslant 1$ as well. L is a stochastic, random variable defined so as to match random oscillation of the angle α expressed by means of frequency as opposed to one expressed by means of probability on the basis of the theorems proved in refs. [1] and [2] for the open interval (0, 1) (or its subsections). Dr. Whitaker, in a discussion which we had a year ago in Cesena, put forward a possibility that L can in principle become infinite for $p=1$. For $p=1$ and $N_+=N$ this is possible but the result remains the same: See the *interpretative difference* of point (iii) below (in parentheses). For $p=1$ and $N_+\neq N$ (see points (i), (ii) below) one cannot accept such a possibility for the following reasons.

L is defined as $L=\lim_{N\to\infty}|\chi(N)|$ where $\chi(N)$ matches stochastical fluctuation of N_+/N so as to make it equal to $p+\chi(N)\Delta p$, where Δp is the standard deviation from p of the Bernoulli distribution and therefore $\Delta p=\sqrt{p(1-p)/N}$ [1,2]. For $p=1$ we obviously cannot use $\chi(N)\Delta p$ for *measuring the difference* between p and N_+/N because in this case we strictly have $\Delta p=0$ independently of how large N is. However, we can reelaborate the whole problem from the "*inverted*" side using the Bayes (beta) distribution which is a binomial distribution whose variable is not p as for the Bernoulli distribution but N_+/N [1]. (For N approaching infinity, i.e. for the limit case, switching from one distribution to the other does not cause any problem.)

The mean value of the Bernoulli distribution lies at p and the maximum value approaches it as N approaches infinity. The means value of the Bayes distribution, on the other hand, lies at N_+/N. The standard deviation for the Bayes distribution is

$$\Delta f=\sqrt{(N_+/N^2)(1-N_+/N)}$$

and we can interpret $\chi(N)$ so as to make p equal to $N_+/N+\chi(N)\Delta f$. In this case $p=1$ does not cause a problem for L because then Δf must not be strictly zero (but only in the limit) in order *not* to make *strictly* $N_+/N=p=1$. Hence $L<\infty$ since $\chi(N)$ must stay finite in the limit in the same way in which it must stay so in eq. (13) of ref. [1] or in eq. (6) of ref. [2]. However, we again have to stress that one can *prove* the central limit theorem only for the open interval $0<p<1$.

Thus we are left with the following three possibilities for G (which hold for an arbitrary spin s too [2]).

(i) $G(p)$ is continuous at 1. A necessary and sufficient condition for this is $G(1)=\lim_{p\to 1}G(p)$. In this case we cannot strictly have $N_+=N$ since then $G(1)=0\neq\lim_{p\to 1}G(p)$ gives a contradiction.

(ii) $G(1)$ is undefined. In this case we also cannot have $N_+=N$ since the latter equation makes $G(1)$ defined, i.e. equal to zero.

(iii) $G(1)=0$. In this case we must have $N_+=N$. And *vice versa*: if the latter equation holds we get $G(1)=0$.

Hence, under the given assumptions a measurement of a discrete observable can be considered repeatable with respect to individual measured systems if and only if $G(p)$ exhibits a jump-discontinuity for $p=1$ in the sense of point (iii) above.

The interpretative differences between the points are as follows.

Points (i), (ii) admit only the statistical interpretation of the quantum formalism and banish the repeatable measurements on individual systems from quantum mechanics altogether. Of course, the repeatability in the statistical sense remains untouched. Possibility (i) seems to be more plausible than possibility (ii) because the assumed continuity of G makes it approach its classical value for large spins [2]. Notably, for a classical probability we have $\lim_{p\to 1}G_{cl}(p)=0$ and for "large spins" we get $\lim_{s\to\infty}\lim_{p\to 1}G(p)=0$.

Point (iii) admits the individual interpretation of quantum formalism and assumes that the repeatability in the statistical sense implies the repeatability in the individual sense. By adopting this interpretation we cannot but assume that nature differentiates open intervals from closed ones, i.e. distinguishes between two infinitely close points. (The same conclusion about nature we would have to draw if we assumed a sudden *jump* in definition of the random function L leaving $G(1)$ undefined.)

At first sight the statistical interpretation, i.e. points (i) and (ii) and their implicit appeal to L, seems hard to support since this apparently invokes a demand for a Gaussian distribution of N_+/N to be centered at 0 or 1 which would clearly be impossible. However, we should bear in mind that the Gaussian distribution of N_+/N, which is exact when N tends to infinity, is but an approximation of the proper binomial distribution of N_+/N for only arbitrary large N's. Furthermore, this approximation is less appropriate, the closer the corresponding probability p is to zero or to one. And the binomial distribution is not symmetric as the Gaussian one but skewed to the left and right on the right and left half of the (0, 1) interval, respectively. Namely, it is easy to calculate that the skewness, defined as $s_3:=\langle(N_+/N-p)^3\rangle/$

$(\Delta p)^3$, is given by $s_3 = (q-p)(pqN)^{-1/2}$. Therefore, since p, being the mean of N_+/N, cannot tend to 0 faster than $1/N$ and cannot tend to 1 faster than $(N-1)/N$ (see the argument concerning the *beta* distribution in the discussion of ref. [1]), the distribution of N_+/N is skewed more to the left so as to obey $s_3 > -1$ or to the right so as to obey $s_3 < +1$, the closer p is to 1 or 0, respectively. Taking into account that for N approaching infinity an *exact* approximation of the binomial distribution holds *only* on the open interval $0 < p < 1$, the distribution of N_+/N for $p = 0 = \lim_{N\to\infty}[N_+(0, N)/N]$ and for $p = 1 = \lim_{N\to\infty}[N_+(1, N)/N]$ could on no better ground be assumed symmetry than asymmetric. Besides, as I stressed above we can reelaborate the whole approach so as to use the Bayes (beta) distribution instead of the Bernoulli one and then we do not face such a problem since the Bayes distribution has its mean value at N_+/N.

In order to show that the above differences obtained for the discrete observables suffice for a conclusion on all observables and the interpretation of quantum theory in general we turn to the problem of repeatability within the theory of measurement. It was shown within the theory that both continuous observables [11] and discrete observables which do not commute with conserved quantities [12] cannot strictly satisfy either the repeatability hypothesis or the collapse postulate but at best only approximately. This means that for such observables no property can be prepared with certainty and our difference then enables us to postulate the exclusion (or not) of the repeatability for individual events – the probabilistic repeatability of course remains intact – even for the discrete observables which undergo measurements of the first kind. (In this case eq. (2) holds even for probability one.) The quantum formalism thus allows and supports Ozawa's conjecture: "The nonexistence of repeatable measuring processes of continuous observables suggests that we should investigate the approximately repeatable measuring processes as models of measurements in quantum mechanics. Moreover, this direction of investigation is appropriate not only for continuous observables [but also for] discrete observables... The author believes that, in future investigation on really existing approximately repeatable measurements, our framework of measuring processes will provide a nice basis." [11, p. 80]. And the present elaboration shows that if such a programme demanded a complete exclusion of probability equal to unity from the theory of measurement that would not be in contradiction with real measurements of individual events and their statistics but that would demand essential changes in the definition of the quantum probability function.

Let us therefore go back to the standard quantum formalism to see how far we can go with the discrete observables strictly within this formalism.

For discrete observables on which a measurement of the first kind was carried out the collapse postulate reads: "The measurement transforms [the observed system] from the state ϕ into one of the states ϕ_n, $n = 1, 2, \ldots$ the probabilities for which are respectively $p = |(\phi, \phi_n)^2|$, $n = 1, 2, \ldots$." [6, pp. 439].

States in general as well as eigenstates (together with the corresponding eigenvalues) in particular are only probabilistic concepts – probability amplitudes which in our case have to give probability equal to unity – and the "observed system" in the afore-mentioned von Neumann's definition of the collapse postulate is nothing else but an *ensemble*. Relative frequency is on the other hand a purely statistical concept. So *spin-up* prepared individual electrons all correspond to the *spin-up* eigenvalue. We can postulate that within an infinitely long run a finite number of electrons can "*go down*" but this has nothing to do with the *up-eigenstate* and *up-eigenvalue* to which *all* up-prepared electrons (i.e. the ensemble) belong.

Relative frequencies in general refer to ideal experiments carried out on individual systems while probabilities refer *either* to the ensemble *or* to each individual system belonging to the ensemble if we only *postulate* either one way or the other.

The odd problem as to whether an individual quantum system can be considered completely described by the standard formalism or not is thus given a new aspect: We are forced to make up our mind: either to consider the standard formalism a complete description of an individual quantum system or to understand it as a completely statistical theory. Completely statistical in the sense that eq. (2) is always satisfied and that probability equals the corresponding relative frequency only approximately. On the other hand, a classical statistical theory based on classical mechanics – excluding *chaos* – never satisfies eq. (2) since its probabilities are basically geometrical. Such an opposite behavior of quantum versus classical probability stems from the kind of their probability functions: quantum functions which are typically trigonometric polynomials have real numbers as their values (as opposed to relative frequencies) while classical functions which are typically geometric ratios have rational numbers as their values (concordant with relative frequencies). The former is obviously a direct consequence of the main feature of the individual quantum events that they form Bernoulli trials, i.e. that they are completely independent (in general we say the individual detections are unpredictable).

What do we therefore achieve by adopting one or the other interpretation? The individual interpretation means *the* completeness. "In that perfect world, nothing happens." [13]. The statistical interpretation, on the other hand, supports the view that the logic underlying quantum formalism might be based on the *statistics* of individual quantum measurements which might in turn be traced theoretically by investigating possible extensions of algebraic quantum structures. If the tracing brought us to new observables and a new theory, such a theory could not possibly turn quantum mechanics "wrong" in the same way in which the theory of relativity did so with Newton's mechanics but would simply have quantum mechanics as a restriction to standard quantum observables. Thus quantum theory might be interpreted as a "randomizer" of some subquantum observables (cf. refs. [13,14]) but, of course, it cannot be interpreted in such a way by means of "preassigned values" [15] either on a factual [16] or on a counterfactual level (the standard Bell's result).

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