A Relative Frequency Criterion for the Repeatability of Quantum Measurements.

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Summary. — A relative frequency property is employed to show that quantum YES-NO measurements of a discrete spin observable can be considered repeatable, if and only if a particular physically meaningful function, defined in the paper, exhibits a jump-discontinuity for end points of a closed interval.

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1. – Introduction.

It has recently been shown that continuous observables do not satisfy the repeatability hypothesis [1]. It has also been shown that, in the presence of a conservation law, even discrete observables are only approximately measurable [2]. Thus, both kinds of observables allow a value from their spectra to be a result of a measurement, but the value cannot be ascribed to a particular property of the measured individual system in a particular state. As opposed to this situation, when individual systems are subjected to YES-NO measurements of a discrete observable, unrestricted by any conservation law, the eigenvalue of the measured observable (projector) can always be taken to correspond to a particular property of the ensemble of individual systems. Thus, for repeated YES-NO measurements of an unrestricted discrete observable a YES-event occurs with certainty, *i.e.* with probability equal to unity, and a NO-event occurs with probability zero. In other words, from a statistical point of view, such measurements can always be considered as repeatable. However, in looking at individual events, we face the following dilemma.

We can take a view that a NO-event with probability zero never occurs. In this case, a measurement is considered repeatable in both senses: statistical and individual, as well as taken to ascribe an individual system a particular property.

The other possibility is to assume that a NO-event with probability zero can

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nevertheless occur. In this case the repeatability is not admitted so far as individual events are concerned.

The latter interpretation is attractive because it suggests a unification of descriptions of quantum measurements in the sense that the approximately repeatable measuring process might be a model of measurements in quantum mechanics, not only for continuous and restricted discrete observables, but also for unrestricted discrete observables. However, to be able to consider whether or not an event with zero probability can occur, we should first have a physically meaningful «measure» for its (non)occurrence. And the purpose of this paper is to provide such a measure.

We approach measurements of unrestricted discrete observables carried out on individual systems by means of ideal relative frequencies, *i.e.* those that apply to an infinite number of such YES-NO measurements of a particular observable. According to the lemmas provided in sect. 2, it turns out that the frequencies converge to appropriate probabilities in a way which enables defining an expression (eq. (13)) as the measure for the repeatability. The analysis carried out in sect. 3 shows that the repeatability of individual events boils down to the claim that two isolated points (0 and 1) of the probability continuum under consideration (closed interval $[0,1]$) should be ascribed different jump-discontinuous values of the expression given by eq. (13) than all other points (open interval $(0,1)$).

In order to pave the way for this conclusion, we shall first elaborate the statistics of measurements of individual systems within YES-NO experiments.

2. – Estimation of probability by relative frequency.

Let us call the event $X_i = 1$ a success on the i-th trial. The frequency with which the result 1 is obtained is

$$f = \frac{1}{N}\sum_{i=1}^{N} X_i = \frac{N_+}{N},$$

where N_+ is the number of successes in the first N trials.

The probability that the i-th trial gives the result 1 is denoted as $P(X_i = 1) \in [0,1]$. The probability that the i-th trial *and* the j-th trial give the result 1 is $P(X_i = 1, X_j = 1) \in [0,1]$. The expectation of the frequency and of its square is

$$\langle f \rangle = \frac{1}{N}\sum_{i=1}^{N} P(X_i = 1), \tag{1}$$

$$\langle f^2 \rangle = \langle ff \rangle = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} P(X_i = 1, X_j = 1). \tag{2}$$

Now, what is usually taken to be the first basic feature of any quantum YES-NO measurement is that the particular individual events are completely independent, which can be expressed as $P(X_i = 1, X_j = 1) = P(X_i = 1)\,P(X_j = 1)$, for $i \neq j$.

The second basic feature of the YES-NO measurements is that trials form an exchangeable sequence. Hence $P(X_1 = 1) = \ldots = P(X_N = 1) =: p$. Thus eq. (1) reduces to $\langle f \rangle = p$.

Since $P(X_i = 1, X_j = 1) = p$, within the sums in eq. (2), there are $N^2 - N$ of

$P(X_i = 1, X_j = 1) = p^2$, $i \neq j$ and eq. (2) reduces to $\langle f^2 \rangle = [Np + (N^2 - N)p^2]/N^2$.
The appropriate standard deviation is

$$\Delta p = [\langle f^2 \rangle - \langle f \rangle^2]^{1/2} = \sqrt{\frac{p(1-p)}{N}} = \sqrt{\frac{pq}{N}}, \tag{3}$$

where $q := 1 - p$.

By the Markov theorem [3], we obtain that the frequency $f = N_+/N$ converges to p «in probability» and, by the Lévi theorem [4], it then follows that $f = N_+/N$ converges to p almost certainly (a.c.), in symbols $N_+/N \xrightarrow{\text{a.c.}} p$, which means

$$P\left(\lim_{N\to\infty} \frac{N_+}{N} = p\right) = 1. \tag{4}$$

(Note that N_+ is actually a function of probability and of the number of trials: $N_+ = N_+(p, N)$, and that the lemmas below apply to p from the open interval $0 < p < 1$.)

However, on which minimal set and in which way does the frequency N_+/N converge to p?

We answer this question in the form of the lemmas given below. The lemmas are direct consequences of the DeMoivre-Laplace central-limit theorem. As a bridge to them, the following wording could presumably be of some help.

Lemma 1 amounts to saying that, for $(\Delta p)'$ given in the lemma, the probability of finding the frequency N_+/N within $[p - (\Delta p)', p + (\Delta p)']$ approaches 1 as N approaches infinity. Lemma 2, on the other hand, states that the probability of finding N_+/N within $[p - (\Delta p)'', p + (\Delta p)'']$, where $(\Delta p)'' = (pq)^{1/2} N^n$, $n < -1/2$, approaches 0 as N approaches infinity. Lemmas 1 and 2 taken together answer the first part of the question stating that the minimal set on which N_+/N converges to p almost certainly is given by $\{x \in \boldsymbol{R} | x \in [p - (\Delta p)', p + (\Delta p)'] \cap [p - (\Delta p)'', p + (\Delta p)'']\}$, where $(\Delta p)' - \Delta p$ and $\Delta p - (\Delta p)''$ are arbitrarily small, *i.e.* where k (see the definition of $(\Delta p)'$ in Lemma 1) and n are arbitrarily close to $-1/2$ (approaching it from above and below, respectively). Lemmas 1′, 2′, and 3′ generalize Lemmas 1, 2, and 3 for the case of $(pq)^{1/2}$ from Δp, $(\Delta p)'$, and $(\Delta p)''$ being substituted by $(pq)^\gamma$, $\gamma \in \boldsymbol{R}$.

Lemmas 1, 2, and 3 taken together answer the second part of the question by stating that N_+/N finds its values on the interval $[p - \eta\Delta p, p - (\Delta p)'' \cup (p + (\Delta p)'', p + \eta\Delta p]$, where $0 < \eta < \infty$ is the confidence coefficient and where n from $(\Delta p)''$ is arbitrarily close to $-1/2$, with the probability given by the right side of eq. (10). The frequency N_+/N finds its values on the interval $[p - (\Delta p)', p - \eta\Delta p) \cup (p + \eta\Delta p, p + (\Delta p)']$ with the probability equal to the previous one subtracted from unity.

Lemmas 1′, 2′, and 3′ taken together give a corresponding answer for $\gamma \neq 1/2$, stating in addition that for p approaching either 0 or 1 the frequency N_+/N finds values on $[p - \eta(pq)^\gamma N^{-1/2}, p + \eta(pq)^\gamma N^{-1/2}]$ with probability 1 and 0 as N approaches infinity for $\gamma < 1/2$ and $\gamma > 1/2$, respectively, no matter which confidence coefficient η we choose.

The lemmas read as follows:

Lemma 1. Let p, N_+, and N be defined as above. If $0 < p < 1$, $-1/2 < k < -1/3$, and $(\Delta p)' = (pq)^{1/2} N^k$, then

$$\lim_{N\to\infty} P[p - (\Delta p)' \leqslant \frac{N_+}{N} \leqslant p + (\Delta p)'] = 1 . \tag{5}$$

Proof. The DeMoivre-Laplace central-limit theorem [4] reads as follows: If $\{m_N\}$ and $\{M_N\}$ are sequences of nonnegative integers satisfying $m_N \leqslant M_N$, where $N \geqslant 1$, for which

$$(m_N - Np)^3/N^2 \to 0 \quad \text{and} \quad (M_N - Np)^3/N^2 \to 0 \tag{6}$$

as $N \to \infty$, where $0 < p < 1$, and if $\mu = (m_N - Np - 1/2)(pqN)^{-1/2}$ and $\nu = (M_N - Np + 1/2)(pqN)^{-1/2}$, then

$$P(m_N \leqslant N_+ \leqslant M_N) = [1 + o(1)](2\pi)^{-1/2} \int_\mu^\nu \exp\left[-\frac{x^2}{2}\right] dx , \tag{7}$$

where $1 + o(1) \to 1$ uniformly as $N \to \infty$. ($o(1)$ is Hardy's «little o» [4].)

Let us choose $m_N = Np - N^{1+k}(pq)^{1/2} + c(N)$ and $M_N = Np + N^{1+k}(pq)^{1/2} - C(N)$, where $-1/2 < k < -1/3$, and $0 \leqslant c(N)$, $C(N) < 1$ are chosen so as to make m_N and M_N integers. We can easily check that m_N and M_N satisfy conditions (6). To ensure $m_N \leqslant M_N$ and $N \geqslant 1$, we choose $N > (qp/4)^{-1/(2+2k)}$. By introducing our m_N and M_N in μ and ν we get

$$\mu = -N^{k+1/2} + [c(N) - 1/2](pqN)^{-1/2}, \qquad \nu = N^{k+1/2} + [1/2 - C(N)](pqN)^{-1/2} . \tag{8}$$

Hence, $\mu \to -\infty$ and $\nu \to +\infty$ when $N \to \infty$. Thus, the right-hand side of (7) approaches 1 as N approaches infinity. Dividing the inequalities, to which the probability on the left-hand side of eq. (7) refers, by N—an operation that does not clash with the way in which eq. (7) is derived in ref. [4]—and taking into account that $Np - N(\Delta p)' \leqslant m_N$ and $M_N \leqslant Np + N(\Delta p)'$, we obtain eq. (5). □

Lemma 2. Let p, N_+, and N be defined as above. If $0 < p < 1$ and $n < -1/2$, then

$$\lim_{N\to\infty} P[p - (pq)^{1/2} N^n \leqslant \frac{N_+}{N} \leqslant p + (pq)^{1/2} N^n] = 0 . \tag{9}$$

Proof. Proceeding as in the proof of Lemma 1, we obtain expressions for μ and ν which differ from (8) only in k being substituted by n. Hence, $\mu \to 0$ and $\nu \to 0$ when $N \to \infty$. Thus the right-hand side of eq. (7) approaches 0 as N approaches infinity. Proceeding further, as in the proof of Lemma 1, we obtain eq. (9). □

Lemma 3. Let p, Δp, N_+, and N be defined as above, in particular Δp by eq. (3). If $0 < p < 1$ and $0 < \eta < \infty$, where η is the confidence coefficient, then

$$\lim_{N\to\infty} P\left(p - \eta\Delta p \leqslant \frac{N_+}{N} \leqslant p + \eta\Delta p\right) = (2\pi)^{-1/2} \int_{-\eta}^{\eta} \exp\left(-\frac{x^2}{2}\right) dx . \tag{10}$$

Proof. Proceeding as in the proof of Lemma 1, we obtain the expressions for μ and

ν which differ from (8) in k being substituted by $-1/2$. Hence, $\mu \to -\eta$ and $\nu \to +\eta$ when $N \to \infty$. Thus the right-hand side of eq. (7) boils down to eq. (9) as N approaches infinity. □

The lemmas put together state that the convergence of the frequency N_+/N to probability p takes place on a set defined as

$$[p-(pq)^{1/2}N^k, p-(pq)^{1/2}N^n) \cup (p+pq)^{1/2}N^n, p+(pq)^{1/2}N^k],$$

$$-1/2 < k < -1/3, \qquad n < -1/2,$$

where k and n are arbitrary close to $-1/2$. This means that a frequency N_+/N which converges to an appropriate probability p is «improperly» Gaussian-like distributed, so far as only arbitrary large N's, instead of the ones approaching infinity, are considered; *i.e.* an arbitrary narrow strip is «cut out» from the middle of the Gaussian. (The lemmas remain «almost unchanged» when N is only «sufficiently large» instead of approaching infinity.)

When N tends to infinity we can hardly speak about a distribution, because of $(pq)^{1/2}N^k \to 0$, but we can express the result by saying the N_+/N acquires a value which is strictly equal to p by probability zero:

$$\lim_{N\to\infty} P\left(\frac{N_+}{N} = p\right) = 0. \tag{11}$$

The equation shows that the values of the frequency never cluster strictly at p but only around p, and puts forward an interesting characterization of the stochasticity of frequencies of the Bernoulli trials. In this case, the above statement that N_+/N is not «properly» Gaussian-like distributed means that a line centred at p is «cut off» from an infinitely narrow strip to which the Gaussian «shrinks».

Taken together, it is clear that the lemmas and all their consequences heavily depend on the fact that N approaches infinity; therefore we are tempted to conclude that p itself does not play any role in the measure of deviation of N_+/N from p, *i.e.* from its mean. However, so far as Lemma 3 is concerned, this is not the case, as we are now going to show.

Retaining the previous notation and conditions it is obvious that the lemmas hold in the following form as well ($\gamma \in \boldsymbol{R}; 0 < \eta < \infty$):

Lemma 1′.

$$\lim_{N\to\infty} P\left[p-\eta(pq)^\gamma N^k \leqslant \frac{N_+}{N} \leqslant p+\eta(pq)^\gamma N^k\right] = 1, \quad -1/2 < k < -1/3.$$

Lemma 2′.

$$\lim_{N\to\infty} P\left[p-\eta(pq)^\gamma N^n \leqslant \frac{N_+}{N} \leqslant p+\eta(pq)^\gamma N^n\right] = 0, \qquad n < -1/2.$$

Lemma 3′.

$$\lim_{N\to\infty} P\left[p-\eta(pq)^{\gamma}N^{-1/2}\leqslant \frac{N_+}{N}\leqslant p+\eta(pq)^{\gamma}N^{-1/2}\right]=\frac{1}{2\pi}\int\limits_{-\eta(pq)^{\gamma-1/2}}^{+\eta(pq)^{\gamma-1/2}}\exp\left[-\frac{x^2}{2}\right]\mathrm{d}x\,.$$

Only Lemma 3′ is changed with respect to Lemma 3 by the introduction of γ and it will serve us in sect. 3 to discuss a possible estimation of N_+/N by means of $p+ +\chi(N)(pq)^{\gamma}N^{-1/2}$, where $-\infty<\chi(N)<0$, $0<\chi(N)<+\infty$.

First, we have to restrict γ to the positive values. Otherwise we are in the position of estimating frequency by something which is greater than one for all N's which are not large enough, and this is absurd.

For arbitrary large $\gamma>1/2$, the frequency N_+/N finds its value on the interval $[p-\eta(pq)^{\gamma}N^{-1/2}, p+\eta(pq)^{\gamma}N^{-1/2}]$ with a probability which is arbitrarily close to zero, no matter which confidence coefficient η is chosen. On the other hand, for $\gamma<1/2$, the smaller γ is, the closer the expression on the right side is to one, no matter which η we choose. In a word, the values we should allow γ depend on the nature of the problem we treat.

Let us now consider the particular cases of probabilities equal to zero and one. Theorems concerning the Bernoulli trials, our lemmas included, do not say anything about the strict values $p=0$ and $p=1$. It follows from Lemma 1 that $N_+(p,N)/N\to 0$ when $p\to 0$ and $N\to\infty$, and that $N_+(p,N)/N\to 1$ when $p\to 1$ and $N\to\infty$. Thus we are tempted to expect to have strictly $N_+(0,N)=0$ and strictly $N_+(1,N)=N$. However, as we show in the next section, so far as the quantum YES-NO measurements are concerned, we should be tempted to expect just the opposite.

3. – Quantum YES-NO measurements.

Let us consider polarization and spin preparation-detection measurements for photons, spin-(1/2) particles, and spin-1 particles. The appropriate probabilities are then either of the Malus form $-p=\cos^2(C\alpha)$—with $C=1/2$ for spin-(1/2) particles, and $C=1$ for photons and for the zero projections of spin-1 particles; or of the form $p= =\cos^4(\alpha/2)$ for ± 1 projections of spin-1 particles.

Quantum systems are prepared, one by one, by a preparation device (a polarizer or a Stern-Gerlach device) and detected, one by one, by a detection device (an analyser or another Stern-Gerlach device) deflected at an angle relative to the preparation device. In a word, we carry out quantum YES-NO measurements on individual systems. Quantum mechanics then predicts that the relative frequency N_+/N of the number N_+ of detections of the prepared property («prepared» in the statistical sense of the word—see sect. 1) on the systems among the total number N of the prepared systems, approaches $\cos^2(C\alpha)$ and $\cos^4(\alpha/2)$ as N approaches infinity. And the other way round, whenever we register the frequency N_+/N, we find the detection device deflected at angles $(1/C)\cos^{-1}(N_+/N)^{1/2}$ and $2\cos^{-1}(N_+/N)^{1/4}$ as N approaches infinity.

The main point in our reasoning is that the angle α is not considered as a macro-observable measured directly by classical means, but as a function of the obtained frequency, in symbols $\alpha=\alpha(N_+/N)$, and as a function of the corresponding probability (being the mean of the frequency), in symbols $\alpha = \alpha(p)$. Of course, the value of the latter coincides with its macro-value.

The function $\alpha = \alpha(p)$, no matter which of the two afore-mentioned forms it acquires, is a continuous monotonic increasing function defined on $[0, 1]$ and differentiable on $(0, 1)$. Hence, it is invertible and its inversion function $p = p(\alpha)$ is also differentiable and $\mathrm{d}\alpha/\mathrm{d}p = (\mathrm{d}p/\mathrm{d}\alpha)^{-1}$ holds [5]. Therefore, in the definition of the derivative

$$\left|\frac{\mathrm{d}\alpha}{dp}\right| = \lim_{h\to 0}\left|\frac{\alpha(p+h)-\alpha(p)}{h}\right| = \left|\frac{\mathrm{d}p}{\mathrm{d}\alpha}\right|^{-1} \tag{12}$$

we are allowed to replace h by a particular net or series converging to zero and we shall do so using a particular expression, which will eventually serve ascribing values of the frequency N_+/N to $p + h$. As a candidate of such an expression, the estimation analysis for N_+/N carried out in sect. 2 singles out $\chi(N)(pq)^\gamma N^\beta$, where $0 < |\chi(N)| < \infty$, $0 < \gamma < \infty$, and $-\infty < \beta < 0$. Let us proceed along this line.

The points $p \pm |\chi(N)|(pq)^\gamma N^\beta$ are the end points of the interval $[p - |\chi(N)|(pq)^\gamma N^\beta, p + |\chi(N)|(pq)^\gamma N^\beta]$ and, according to Lemmas 1′ and 2′, we can accomplish the ascription only when $\beta = -1/2$. For, according to Lemma 2′, the frequency N_+/N almost never acquires a value on the interval when the latter is defined by $\beta < -1/2$; thus, according to Lemma 1′, the frequency N_+/N has almost all values on the interval, but perhaps none at its end points when the interval is defined by β, which satisfies: $-1/2 < \beta < -1/3$.

On the other hand, according to Lemma 3′ and the discussion following it, for $\gamma > 1/2$, when p approaches zero and one—these are the cases in which we are primarily interested—the frequency N_+/N almost never acquires values on the interval. When $0 < \gamma < 1/2$ and p approaches zero or one the frequency again has almost all values on the interval, but perhaps none at its end points.

Thus, we are left with $\beta = -1/2$ and $\gamma = 1/2$, our points boiling down to $p + \chi(N)\,\Delta p$, where Δp is given by eq. (3). In other words, only for these values of β and γ is the convergence of N_+/N—on the minimal set from sect. 2—to p assured.

According to Lemma 3, for every N, for $\eta \geqslant |\chi(N)|$ large enough, almost all (*i.e.* with a probability approaching one) values of N_+/N, as well as of $p + \chi(N)\,\Delta p$, lie within the interval $[p - \eta\Delta p, p + \eta\Delta p]$. On the other hand, for $\eta' \leqslant |\chi(N)|$ small enough almost all values of N_+/N as well as of $p + \chi(N)\,\Delta p$, lie outside the interval $[p - \eta'\,\Delta p, p + \eta'\,\Delta p]$. Besides, according to Lemma 2 and eq. (11) the probability of N_+/N being strictly equal to p is zero.

Hence, for every N there is almost certainly such $\chi(N)$, so as to make $p + \chi(N)\,\Delta p$ equal to N_+/N and $\chi(N)\,\Delta p$ converging to zero (without making it equal to zero) as N approaches infinity.

In eq. (12) we can, therefore, choose $h = \chi(N)\,\Delta p$ and substitute N_+/N for $p + h$, thus obtaining

$$\left|\frac{\mathrm{d}\alpha}{\mathrm{d}p}\right| = \lim_{N\to\infty}\frac{|\alpha(N_+/N)-\alpha(p)|\,N^{1/2}}{|\chi(N)|(pq)^{1/2}} = \left|\frac{\mathrm{d}p}{\mathrm{d}\alpha}\right|^{-1}.$$

After a rearrangement it gives (for $0 < p < 1$)

$$G(p) := L^{-1}\lim_{N\to\infty}[|\alpha(N_+/N)-\alpha(p)|\,N^{1/2}] = (pq)^{1/2}\left|\frac{\mathrm{d}p}{\mathrm{d}\alpha}\right|^{-1} =: H(p)\,, \tag{13}$$

After a rearrangement it gives (for $0 < p < 1$)

$$G(p) := L^{-1} \lim_{N\to\infty} [|\alpha(N_+/N) - \alpha(p)|\, N^{1/2}] = (pq)^{1/2} \left| \frac{\mathrm{d}p}{\mathrm{d}\alpha} \right|^{-1} =: H(p)\,, \tag{13}$$

where $L = \lim_{N\to\infty} |\chi(N)|$ is a bounded ($0 < L < \infty$) random (stochastic) variable which does not have a definite value, in the same way in which, *e.g.*, $\lim_{x\to\infty} \sin x$ does not have a definite value. However, the randomness of L is matched by the randomness of the other limit in eq. (13), so that $G(p)$ turns out to be a continuous nonvanishing function on $(0, 1)$. Namely, for our probabilities $p = \cos^2(C\alpha)$ and $p = \cos^4(\alpha/2)$ we get

$$G_1(p) = H_1[p(\alpha)] = \frac{\sin(2C\alpha)}{2C \sin(2C\alpha)}\,, \qquad G_2(p) = H_2[p(\alpha)] = \frac{\sin(\alpha/2)[1 + \cos^2(\alpha/2)]^{1/2}}{2 \sin(\alpha/2)|\cos(\alpha/2)|}\,,$$

respectively. Therefore, for $p \in (0, 1)$, *i.e.*$\alpha \in (0, \pi/(2C))$ and $\alpha \in (0, \pi)$, the functions H_1 and H_2 are well defined:

$$G_1(p) = H_1(p) = (2C)^{-1} =: \widetilde{H}_1(p)\,, \qquad G_2(p) = H_2(p) = (1/2)[1 + \cos^{-2}(\alpha/2)]^{1/2} =: \widetilde{H}_2(p)\,.$$

Turning our attention to the probabilities equal to zero and one, we see that H's are not defined for these values: $H_1(0) = H_1(1) = H_2(1) = 0/0$; $H_2(0) = \infty$. However, their limits exist:

$$\lim_{p\to 0} H_1(p) = \lim_{p\to 0} G_1(p) = \lim_{p\to 1} H_1(p) = \lim_{p\to 1} G_1(p) = (2C)^{-1} = \widetilde{H}_1(0) = \widetilde{H}_1(1)\,,$$

$$\lim_{p\to 1} H_2(p) = \lim_{p\to 1} G_2(p) = 2^{-1/2} = \widetilde{H}_2(1)\,,$$

$$\lim_{p\to 0} H_2(p) = \lim_{p\to 0} G_2(p) = \infty = \widetilde{H}_2(0)\,.$$

Thus continuous extensions of H_1 and H_2 to $[0, 1]$ exist and these are given by $\widetilde{H}_1$ and $\widetilde{H}_2$, respectively.

The functions G_1 and G_2, on the other hand, cannot be approached in the same way because we do not know whether $G_1(0)$, $G_1(1)$, $G_2(0)$, and $G_2(1)$ are well defined.

There are three physically acceptable possibiities [6]:

1) $G(p)$ is continuous at 0 and 1. A necessary and sufficient condition for that is $G(0) = \lim_{p\to 0} G(p)$ and $G(1) = \lim_{p\to 1} G(p)$, respectively. In this case we cannot strictly have $N_+(0, N) = 0$ and $N_+(1, N) = N$, since then $G(0) = 0 \neq \lim_{p\to 0} G(p)$ and $G(1) = 0 \neq \lim_{p\to 1} G(p)$ obtains a contradiction.

2) $G(0)$ and $G(1)$ are undefined. In this case we also cannot have $N_+(0, N) = 0$ and $N_+(1, N) = N$ since the latter equations make $G(0)$ and $G(1)$ defined, *i.e.* equal to zero. A continuous extension of $G(P)$ to $[0, 1]$ exists and given by $\widetilde{G}(p) := \widetilde{H}(p)$.

3) $G(0) = G(1) = 0$. In this case we must have $N_+(0, N) = 0$ and $N_+(1, N) = N$. And *vice versa*: if the latter equations hold we get $G(0) = G(1) = 0$.

Thus we reach the conclusion that quantum YES-NO measurements of the dislimit as p^s_{mm} approaches 1, *i.e.* as α approaches 0, and mostly an infinite limit as p^s_{mm} approaches 0, for any s and m.

Hence, a YES-NO measurement of a discrete spin observable s can be considered repeatable with respect to individual measured systems, if and only if $G(p^s_{mm})$, defined by eq. (13), exhibits a jump discontinuity for $p^s_{mm} = 0$ and $p^s_{mm} = 1$ in the sense of point 3 above.

4. – Discussion.

We considered polarization and spin preparation-detection measurements, so as to treat the angle α at which a detection device is deflected relative to the preparation device, as a function of the frequency N_+/N measured on individual systems and as a function of the corresponding probability $p = \langle N_+/N \rangle$. Thus, $\alpha = \alpha(N_+/N)$ can be expressed by means of the frequency N_+/N as $\alpha(N_+/N) = (1/C)\cos^{-1}(N_+/N)^{1/2}$ and $\alpha(N_+/N) = 2\cos^{-1}(N_+/N)^{1/4}$ for the considered cases, respectively. For, according to eq. (4), $N_+/N \xrightarrow{\text{a.c.}} p$ as N approaches infinity. On the other hand, $\alpha = \alpha(p)$ is expressed by the corresponding «objective» probability ascribed to the systems, and therefore it represents the actual «macroscopic» angle of deflection.

The obtained result, expressed by eq. (15), then states that the difference $|\alpha(N_+/N) - \alpha(p)|$ between the «microscopically» measured angle $\alpha(N_+/N)$ and the «macroscopically» measured angle $\alpha(p) = \alpha$ multiplied by $N^{1/2}$, *i.e.* the expression $|\alpha(N_+/N) - \alpha(p)|\,N^{1/2}$, never vanishes as N approaches infinity. For strict values $p = 0$ and $p = 1$ we, however, face a dilemma. Shall we assume that the difference in question suddenly drops to zero for these values of p, thus adopting individual interpretation of quantum mechanics and the repeatability of YES-NO measurements? Or shall we assume the continuity of undefiniteness of $G(p)$ and the validity of eq. (11) for these values of p, thus adopting the statistical interpretation and banishing repeatable measurements on individual systems from quantum mechanics altogether?

By adopting the former interpretation we cannot but assume that Nature differentiates open intervals from closed ones, *i.e.* distinguishes between two infinitely close points. One can try to object that the consequence of the obtained result is overemphasized and try to argue that for α's, which are *infinitely close* to zero, *i.e.* for p's *infinitely close* to one, the expression $|\alpha(N_+/N) - \alpha(p)|\,N^{1/2}$ approaches zero for an *arbitrary large* N. However, such an argument does not hold water, since $p = \langle N_+/N \rangle$ and whenever p is not *strictly* equal to one, N_+ (being an integer) cannot be greater than $N - 1$. Namely, given the data (N_+, N) one can easily, by means of beta distribution, find the inverse probability p which satisfies

$$N_+/N - [N_+(N - N_+)]^{1/2} N^k \leqslant p \leqslant N_+/N + [N_+(N - N_+)]^{1/2} N^k, \quad -7/4 < k < -5/3,$$

with a probability which is closer to one the larger N is. In this way, p simply cannot be let *infinitely* close to 1 without *first* N being *infinitely large*. And we have already shown in sect. 3 that $\lim\limits_{p\to 1}\lim\limits_{N\to\infty}[|\alpha(N_+/N) - \alpha(p)|\,N^{1/2}] \neq 0$. We point out here that N and p cannot simultaneously tend to infinity and one, respectively, since the existence of such a simultaneous limit implies the existence and equality of the appropriate successive limits. As one of the successive limits turned out not to exist, a simultaneous limit cannot exist either. The above reasoning applies to the case $p \to 0$ in an analogous manner.

By adopting the latter interpretation, we allow but a bare statistical meaning to quantum probabilities, including those ones which are equal to unity and zero, and deny them a unique experimental meaning with which all individual measured systems would comply.

Since the consideration of an experiment which would differentiate between the two interpretations is rather beyond the scope of the present paper, we shall close the discussion by comparing the obtained result with a possible classical one.

Had electrons obeyed a classical «linear» probability law, the deflection angle under consideration would have been expressible [7] as $\alpha = \pi(1-p)$. In this case, eq. (13) gives $G(p) = \pi(pq)^{1/2}$, which is, as opposed to the quantum case, well defined for $p \in [0, 1]$ and we meet no problems for $p = 0$ and $p = 1$. Thus, for a classical probability, the assumptions $N_+(0, N) = 0$ and $N_+(1, N) = N$ invoke no dilemma.

* * *

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Note added in proofs.

The generalization we referred to in the last but one paragraph of sect. 3 is carried out in ref. [8].

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[6] Mathematically, there are also possibilities that one of $G(0)$ and $G(1)$ is undefined and the other defined, as well as both being defined but unequal either to zero or to $\lim_{p\to 0} G(p)$ and $\lim_{p\to 1} G(p)$, respectively. In neither of these cases $N_+(0, N) = 0$ and $N_+(1, N) = N$ can hold. However, such possibilities seem to be utterly «unphysical».
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A Relative Frequency Criterion for the Repeatability of Quantum Measurements.

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PACS 99.10 – Errata.

On p. 1104, 1st line from below, the expression

$$P(X_i = 1,\ X_j = 1) = p$$

should read

$$P(X_i = 1,\ X_i = 1) = p\,.$$

On p. 1105, 11th and 12th line from below, the expression

$$[p - \eta\,\Delta p,\ p - (\Delta p)'' \cup (p + (\Delta p)'',\ p + (\Delta p)'']$$

should read

$$[p - \eta\,\Delta p,\ p - (\Delta p)'') \cup (p + (\Delta p)'',\ p + (\Delta p)'']\,.$$

On p. 1107, 6th line from above, the expression

$$[p - (pq)^{1/2} N^k,\ p - (pq)^{1/2} N^n) \cup (p + pq)^{1/2} N^n,\ p + (pq)^{1/2} N^k]\,,$$

should read

$$[p - (pq)^{1/2} N^k,\ p - (pq)^{1/2} N^n) \cup (p + (pq)^{1/2} N^n,\ p + (pq)^{1/2} N^k]\,.$$

On p. 1110, 11th line from below, «possibiities» should read «possibilities».

On p. 1110, 4th line from below, «and given» should read «and is given».

Due to a technical incovenience, on p. 1109 the last two lines are doubled; and on p. 1110, 1st line from below:

Thus we reach the conclusion that quantum YES-NO measurements of the dis-

should read:

Thus we reach the conclusion that quantum YES-NO measurements of the discrete spin observables considered is repeatable with respect to individual measured systems, if and only if $G(p)$ is jump discontinuous in the sense of point 3 above.

In general, for spin s and its projection m we have $p = p^s_{mm} = (\delta^s_{mm})^2$, where δ^s_{mm} is an element of the rotation matrix. It is not difficult to show that $G(p^s_{mm})$ has a finite

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