entropy

MDPI

Article

Non-Kochen-Specker Contextuality

Mladen Pavičić

Center of Excellence for Advanced Materials and Sensors, Research Unit Photonics and Quantum Optics, Institute Ruder Bošković and Institute of Physics, Zagreb, Croatia.
* Correspondence: mpavicic@irb.hr

Received: date; Accepted: date; Published: date

Abstract: Quantum contextuality supports quantum computation and communication. One of its main vehicles are hypergraphs. Most elaborated are the Kochen-Specker ones but there is also another class of contextual sets which are not of this kind. Their representation has been mostly operator-based and limited to special constructs in 3- to 6-dim spaces, a notable example of which being the Yu-Oh set. Previously, we have shown that hypergraphs underlie all of them and in this paper we give general methods—whose complexity does not scale up with dimension—for generating such non-Kochen-Specker hypergraphs in any dimension and give examples in up to 16-dim spaces. Our automated generation is probabilitic and random but the statistics of accumulated data enable one to filter out sets with required size and structure.

Keywords: quantum contextuality; hypergraph contextuality; MMP hypergraphs; operator contextuality; qutrits; Yu-Oh contextuality; random generation

1. Introduction

Quantum contextuality, which precludes assignments of predetermined values to dense sets of states, has found applications in quantum communication [1–3], quantum computation [4,5], quantum nonlocality [6], quantum steering [7], and lattice theory [8,9]. Small contextual set experiments were carried out with photons [10–21], classical light [22–25], neutrons [26–28], trapped ions [29], solid state molecular nuclear spins [30], and superconducting quantum systems [31].

There are three classes contextual sets elaborated on in the literature which are not of the more common kind of the Kochen-Specker (KS) sets [32–34] and for which we shall provide a hypergraph generalization in this paper.

The first class consists of those operator-based state-independent contextual (SIC) sets put forward by Klyachko et al. [35], Yu and Oh [36], Bengtsson, Blanchfield, and Cabello [37], Xu, Chen, and Su [38], Ramanathan and Horodecki [39], and Cabello, Kleinmann, and Budroni [40] which are not Kochen-Specker sets.

The second class consists of hypergraphs built by multiples of mutually orthogonal vectors where at least one of the multiples contains less than n vectors, where n is the dimension of space in which a hypergraph resides [4,34,41].

The third class consists of the so-called true-implies-false and true-implies-true sets [42,43].

All sets from these three classes as well as their hypergraph generalization we shall elaborate on are contextual and therefore we call them non-KS contextual sets.

We provide a general method of generating arbitrarily many non-KS hypergraphs in up to 16-dim spaces. In order to achieve these goals we make use of non-binary non-KS McKay-Megill-Pavičić hypergraphs (MMPHs) and their language. By means of our algorithms and programs we obtain arbitrarily many of MMPHs which can serve for various applications as, e.g., to generate new entropic tests of contextuality or new operator-based contextual sets.

The paper is organized as follows.

In Sec. 2.1 we present the hypergraph language and formalism and define non-binary MMPHs (NBMMPH) and binary MMPHs (BMMPH). We explain how vertices and hyperedges in an MMPH in and n-dim space correspond to vectors and their orthogonalities, i.e., m-tuples ($2 \leq m \leq n$) of mutually orthogonal vectors, respectively.

In Sec. 2.2 we present three methods of generating non-KS MMPHs.

In Sec. 2.3 we give examples of aforementioned non-KS sets.

In Secs. 2.3 to 2.4 we generate 4- to 8-dim critical non-KS NBMMPHs from master sets themselves generated from simple vector components.

In Secs. 2.5 to 2.6 we obtain 9- to 16-dim critical non-KS NBMMPHs via dimensional upscaling method which does not scale up with dimension.

In Sec. 3 we discuss and review the steps and details of our methods.

In Sec. 4 we give technical methods used in the paper.

In Sec. 5 we summarize the results achieved in the paper.

2. Results

We consider a set of quantum states represented by vectors in n-dim Hilbert space $\mathcal{H}^n$ grouped in m-tuples ($m \leq n$) of mutually orthogonal vectors with $m < n$ holding for at least one m. We describe such a set by means of MMPHs. In it, vectors themselves are represented by vertices and mutually orthogonal m-tuples of them by hyperedges. However, an MMPH itself has a definition which is independent of a possible representation of vertices by means of vectors. For instance, there are MMPHs without a coordinatization, i.e., MMPHs for whose vertices vectors do not exist. When a coordinatization exist, that does not mean that $n - m$ vertices in considered hyperedges do not or cannot exist, but only that we do not take the remaining $n - m$ vertices/vectors into account while elaborating on properties of vertices and hyperedges.

2.1. Formalism

Let us define the MMPH formalism/language [34].

Definition 1. *An* MMPH *is an n-dim hypergraph k-l with k vertices and l hyperedges in which*

1. *Every vertex belongs to at least one hyperedge;*
2. *Every hyperedge contains at least 2 and at most n vertices;*
3. *No hyperedge shares only one vertex with another hyperedge;*
4. *Hyperedges my intersect each other in at most $n - 2$ vertices*
5. *Graphically, vertices are represented as dots and hyperedges as (curved) lines passing through them.*

Definition 2. *An n-dim* non-binary MMPH (NBMMPH)*, $n \geq 3$, [44] is an* MMPH *whose each hyperedge contains m vertices, $2 \leq m \leq n$ and to which it is impossible to assign* 1*s and* 0*s in such a way that*

1. *No two vertices within any of its edges are both assigned the value* 1*;*
2. *In any of its edges, not all of the vertices are assigned the value* 0.

Definition 3. *An* NBMMPH *in which $m = n$ holds for all hyperedges is a* KS MMPH.

For $m = n$ an NBMMPH reduces to a KS contextuality set, i.e., to a set satisfying the Kochen-Specker theorem [32,34,45,46].

Definition 4. *An* NBMMPH *in which $m < n$ holds for at least one hyperedge is a* non-KS MMPH.

In this paper we shall consider only those non-KS MMPHs for which $m = n$ for at least one hyperedge.

Definition 5. *An n-dim* binary MMPH (BMMPH)*, $n \geq 3$, is an* MMPH *whose each hyperedge contains m vertices, $2 \leq m \leq n$ and to which it is possible to assign 1s and 0s in such a way that*

1. *No two vertices within any of its edges are both assigned the value 1;*
2. *In any of its edges, not all of the vertices are assigned the value 0.*

Definition 6. *A* non-KS NBMMPH *to which vertices are added so as to make the number of vertices equal to n in every hyperedge is called a* filled MMPH*.*

Filled MMPHs are mostly BMMPHs.

Definition 7. *A* critical NBMMPH *is a* NBMMPH *which is minimal in the sense that removing any of its hyperedges turns it into a* BMMPH*.*

Definition 8. *Vertex multiplicity is the number of hyperedges vertex 'i' belongs to; we denote it by $m(i)$.*

Definition 9. *A* master *is a non-critical* MMPH *which contains smaller critical and non-critical sub-*MMPH*s. A collection of sub-*MMPH*s of an* MMPH *master forms its* class*.*

A *parity proof* of contextuality of a *k-l* NBMMPH with odd l and each vertex sharing an even number of edges stems from its inherent contradiction: because each vertex shares an even number of hyperedges, there should be an even number of hyperedges with 1's. At the same time, each edge can contain only one 1 by definition, and since there are an odd number of hyperedges in the MMPH, there should also be an odd number of edges with 1's

Definition 10. *A* coordinatization *of a non-*KS NBMMPH *is a set of vectors assigned to its vertices which is a subset of n-dim vectors vectors in $\mathcal{H}^n$, $n \geq 3$, assigned to vertices of its filled* MMPH *or its smallest master (they need not coincide) or any of its masters..*

In other words, a "coordinatization" of each hyperedge of a filled MMPH or a smallest master MMPH is represented by an n-tuple of orthogonal vectors, while a "coordinatization" of each hyperedge of the original non-KS NBMMPH is represented by a vector m-tuple ($m \leq n$) which is a subset of that n-tuple. That means that the former MMPH inherits its coordinatization from the coordinatization of its master or its filled set (they may but usually do not coincide) or any its masters. In our present approach, a coordinatization is automatically assigned to each hypergraph by the very procedure of its generation from master MMPHs as we shall see below.

An MMPH is encoded with the help of printable ASCII characters, with the exception of 'space', '0', '+', ',' and '.', organized in single strings; its hyperedges are separated by commas; each string ends with a period. When all ASCII characters are exhausted one reuses them prefixed by '+', then again by '++', and so forth. An MMPH with k vertices and l edges is denoted as a *k-l* MMPH. ASCII string representation is used for computer processing. MMPH strings are handled by means of algorithms embedded in the programs SHORTD, MMPSTRIP, MMPSUBGRAPH, VECFIND, STATES01, and others [8,47–51].

2.2. Generation of non-KS MMPHs

To generate non-KS NBMMPHs we make use of the following methods.

- **M1** consists in dropping vertices contained in single hyperedges (multiplicity $m = 1$) [34] of either NBMMPHs or BMMPHs and a possible subsequent stripping of their hyperedges. The obtained smaller MMPHs are often non-KS although never KS.
- **M2** consists in a random addition of hyperedges to MMPHs so as to obtain bigger ones which then serve us to generate smaller non-KS NBMMPHs by stripping hyperedges randomly again;

- **M3** consists in random deleting of vertices in either an NBMMPHs or a BMMPHs until a non-KS NBMMPH is reached.

We combine all three methods to obtain an arbitrary number of non-KS NBMMPHs in an arbitrary dimension. The methods rely on the property of MMPHs that by stripping an MMPH, or NBMMPH (critical or not), or BMMPH of its hyperedges we can arrive at smaller non-KS NBMMPHs in contrast to a critical KS NBMMPH whose stripping of hyperedges can never yield another (smaller) NBMMPH.

2.3. Dimensions 3 to 5 and the three classes of non-KS contextual sets from the literature

In Fig. 1 we give examples from each of the three classes of non-KS sets we referred to in the Introduction. Here we remind the reader that *k*-*l* MMPHs refer to hypergraphs with *k* vertices and *l* hyperedges (Def. 1) while the corresponding graphs have more than *l* edges. E.g., in Fig. 1(a) the hypergraph hyperedge `ALK` corresponds to a graph clique with 3 edges `AL`, `LK`, and `KA`.

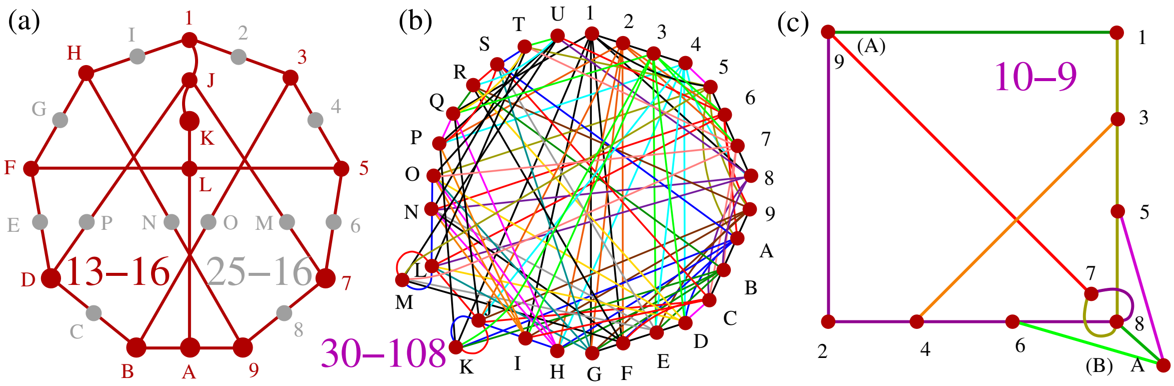

Figure 1. (a) Yu-Oh's 3-dim non-KS 13-16 non-KS NBMMPH [36, Fig. 2]; gray vertices that enlarge 13-16 to 25-16 are necessary for a coordinatization and implementation; (b) Howard, Wallman, Veitech, and Emerson's 4-dim 30-108 non-KS NBMMPH [4, Fig. 2]; (c) Cabello, Portillo, Solís, and Svozil's 5-dim 10-9 non-KS NBMMPH [42, Fig. 5(a)]; original symbols are in brackets (A,B).

Yu-Oh's 3-dim non-KS NBMMPH shown in Fig. 1(a) is presumably the earliest of the kind. It was operator-based but the operators were defined via states/vectors/vertices of 13-16 MMPH as reviewed in [41]. Since orthogonal vectors in a 3-dim space form triples full representation requires 25-16 as indicated by gray vertices in the figure. The 25-16 can be obtained from Peres' 33-40 [41] by stripping hyperedges and the 13-16 from it by removing $m = 1$ vertices, i.e., via **M1**. The 13-16 is not critical and it contains 4 critical sub-MMPHs the smallest of which is 10-9 [41].

Howard, Wallman, Veitech, and Emerson's 4-dim 30-108 non-KS NBMMPH shown in Fig. 1(b) obtained from the set of stabilizer states served them to prove that the underlying contextuality is essential for quantum computation. We discuss its filled 232-108 MMPH and its critical 24-71 MMPH in [34].

Cabello, Portillo, Solís, and Svozil's 5-dim 10-9 non-KS NBMMPH shown in Fig. 1(b) is one of the minimal 5-dim true-implies-false sets (TIFS) [42, Fig. 5(a)]. It is not critical and the only critical it contains is a 10-7 but it is not a TIFS any more. The coordinatization of the filled 10-9 (31-9, which includes the coordinatization of 10-9 itself) can be built from the $\{0, \pm 1, 2\}$ components and is given in the appendices.

Our methods can generate NBMMPHs that are critical as well as those that are not. Therefore, although none of the aforementioned examples are critical we shall focus on critical ones because they offer the simplest implementation and presentation. The rationale for adopting such an approach is that only minimal contextual sets, i.e., critical NBMMPHs, are relevant for experimental implementations since their supersets just contain additional orthogonalities that do not change the contextuality property of their smallest critical set. Hence, while designing MMPHs for particular implementations we should attempt to find the ones that are critical and provided via automated generations of MMPHs.

In [41] we give ample distributions of 3-dim non-KS NBMMPHs obtained via **M1** and **M2**. Therefore, below we give distributions and samples of just 4- and 5-dim critical non-KS NBMMPHs presented in Figs. 2(a,f). Here, we only point out that the KS "bug," 8-7 non-KS NBMMPH shown in [41, Fig. 3(a)] is the smallest 3-dim non-KS NBMMPH which satisfies our requirement that at least one of the hyperedges contains n vertices (n being the dimension of the considered MMPH), none of which has the multiplicity $m = 1$. Its string, the string of its filled MMPH, and their coordinatizations are given in the appendices, as are the strings and coordinatizations of any other MMPH considered in the paper.

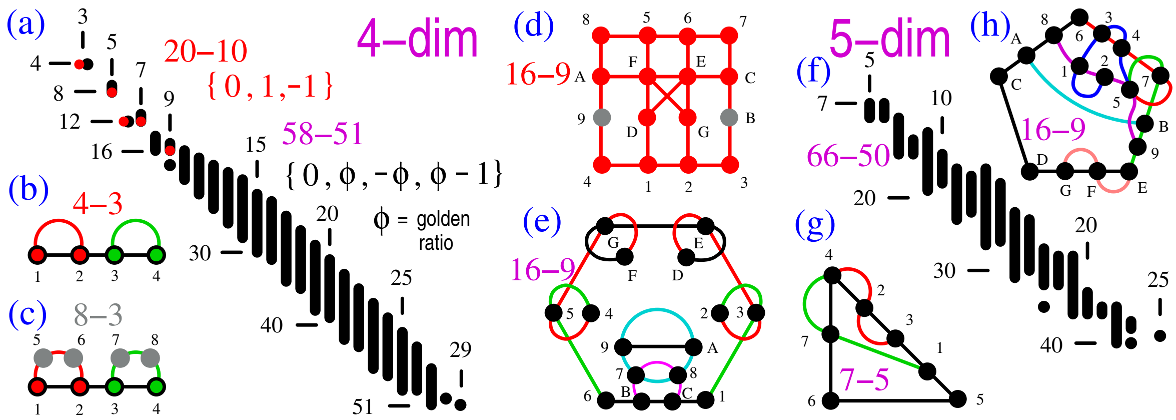

Figure 2. (a) Distributions of critical 4-dim non-KS NBMMPHs obtained from a submaster 20-10 which is obtained from (Peres') 24-24 super-master (generated by vector components $\{0, \pm 1\}$) by **M1** (dots in red) and from a submaster 58-51 itself obtained from the 60-72 super-master (generated by vector components $\{0, \pm\phi, \phi - 1\}$, ϕ is the golden ratio: $\frac{1+\sqrt{5}}{2}$) by **M1** (in black); abscissa is l (number of hyperedges); ordinate is k (number of vertices). Dots represent (k, l). Consecutive dots (same l) are shown as strips; (b) the smallest non-KS in the distributions: 4-3; (c) BMMPH 8-3—filled 4-3—one needs for getting the coordinatization and implementation of 4-3; (d) the 16-9 critical obtained from the 20-10 master; (e) the 16-9 critical obtained from the 58-51 master; (f) distributions of critical 5-dim non-KS NBMMPHs obtained from a submaster 66-50 which is obtained from the 105-136 super-master (generated by vector components $\{0, \pm 1\}$); (g) the smallest critical; (h) a 16-9 critical for the sake of comparison with 4-dim 16-9's; strings and coordinatizations are given in the appendices.

To obtain non-KS NBMMPHs via **M1** we first generate the super-masters from the vector components. In the 4-dim space, we obtain the 24-24 super-master from the $\{0, \pm 1\}$ components and the 60-72 supermaster from the $\{0, \pm\phi, \phi - 1\}$ components, where $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio). Their strings and coordinatizations are given in the appendices. Then we randomly strip hyperedges from them, e.g., 14 from the 24-24 and 21 from the 60-72 supermaster so as to obtain the 20-10 and 58-51 masters, respectively. From the latter masters we remove $m = 1$ vertices and from any of them we generate the classes of critical MMPHs by stripping them further till we obtain critical MMPHs that form the 20-10 and 58-51 non-KS classes. In the 5-dim space, we obtain the 105-136 super-master from the $\{0, \pm 1\}$ components. Its string and coordinatization are given in the appendices. Further, we randomly strip 86 hyperedges to obtain a 66-50 master and eventually its class of critical non-KS NBMMPHs.

We generate n-dim critical non-KS MMPHs under a requirement that at least one of their hyperedges contains n vertices no one of which has the multiplicity 1 ($m = 1$). (All examples from Fig. 1 satisfy these conditions.) For instance, the smallest critical we obtain in the 4-dim distribution Fig. 2(a) is the 4-3 shown Fig. 2(b) whose hyperedge 1234 of such a kind. Its filled MMPH shown in Fig. 2(c) provides a coordinatization necessary for an implementation of the 4-3. The 16-9 critical of the

20-10 master shown in Fig. 2(d) contains two $m = 1$ vertices (9,B) because $m = 1$ vertices are stripped only once (from the master) when we started the generation of the 20-10 class. We can remove one or both of these vertices and still have a critical non-KS MMPH (15-9 or 14-9, respectively) if we wanted to for some reason. The 16-9 critical shown in Fig. 2(e) has a parity proof since in it each vertex shares exactly two hyperedges while there is an odd number of them (9). Strings and coordinatizations are given in the appendices.

2.4. Dimensions 6 to 8

Cabello, Portillo, Solís, and Svozil also give a number of minimal 6-dim TIFS non-KS NBMMPHs in [42, Fig. 7] along the same line as for their 5-dim one shown in Fig. 1(c). To our knowledge there are no explicit examples of non-KS NBMMPH in dimensions 7 and 8 in the literature. Therefore, we straightforwardly go to a generation of 6- to 8-dim non-KS NBMMPHs.

An NBMMPH in the 6-dim Hilbert space corresponds to a qubit entangled to a qutrit ($\mathcal{H}^6 = \mathcal{H}^2 \otimes \mathcal{H}^3$) or to a $\frac{5}{2}$-spin system. So far, to obtain KS NBMMPH masters the following vector components were used: $\{0, \pm\omega\}$, [44,52,53] (ω is a cube root of 1, $\omega = e^{2\pi i/3} = (i\sqrt{3} - 1)/2$), $\{0, \pm\omega, \omega^2\}$ [53,54] and $\{0, \pm 1\}$ [52]. Since the first set of components yields a master with only three MMPHs, we shall make use of the other two to generate 6-dim non-KS NBMMPHs.

The $\{0, 1, \omega, \omega^2\}$ set generates two unconnected masters: 591-1123 and 81-162 [53]. To obtain non-KS NBMMPHs we apply **M1** to the 81-162 class. Their distribution is shown in Fig. 3(a) in black. The $\{0, \pm 1\}$ set generates a 236-1216 master. Its non-KS NBMMPHs are also obtained via **M1** and shown in Fig. 3(a) in green.

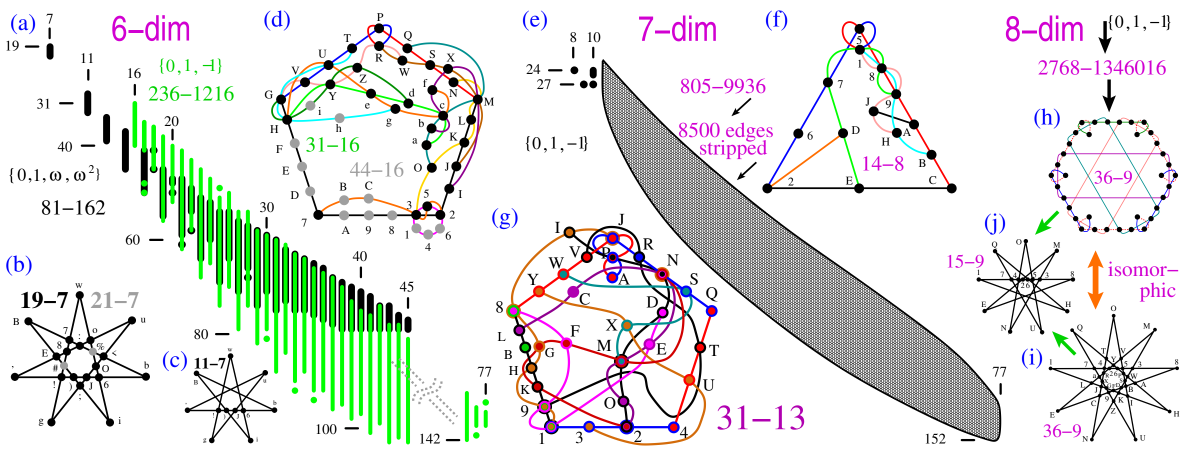

Figure 3. (a) Distributions of 6-dim critical non-KS NBMMPHs obtained from two different submasters—see text; (b) the smallest critical non-KS NBMMPH obtained from the former class by **M3**; it has a parity proof; (c) an even smaller critical non-KS NBMMPH obtained from it by hand; has a parity proof; (d) the smallest critical non-KS NBMMPH obtained from the latter class by **M1**; (e) distributions of 7-dim critical non-KS NBMMPHs—see text; (f) 14-8 non-KS NBMMPH, one of the smallest non-KS NBMMPHs obtained via **M3** from the smallest KS NBMMPH 34-14; (g) 31-13 also obtained from the 34-14 (no $m = 1$ vertices essential for criticality); (h,i) isomorphic 8-dim KS MMPHs with the smallest number of hyperedges (9) serve us to generate the 15-9 non-KS NBMMPH in (j); (h-j) MMPHs have parity proofs; Strings and coordinatizations are given in the appendices.

In the 7-dim space masters obtained from simple vector components, like $\{0, \pm 1\}$, is too big to be used for an exhaustive generation of a complete non-KS NBMMPH class. Instead, as in the previous 6-dim case, we strip a significant portion of hyperedges from a master obtained from $\{0, \pm 1\}$ components and make use of the remaining MMPHs to obtain a non-KS class as shown in Fig. 3(e); $\{0, \pm 1\}$ yield the 805-9936 master; stripping of 8,500 hyperedges leave us with NBMMPHs with 436 hyperedge NBMMPHs which generate a 436-hyperedge class. Since this class is still big we have to repeat **M1** several times to obtain small non-KS critical NBMMPHs. As a result, hyperedges of all small NBMMPHs may contain some $m = 1$ vertices essential for criticality as shown in Fig. 3(f) (removal of vertex `6` would terminate the criticality of the MMPH). In dimensions higher than 9 such vertices do not appear although even here we can avoid their generation by applying **M3** to KS NBMMPHs as shown Fig. 3(g).

The 8-dim MMPH master is big (2768-1346016) but the stripping technique can still provide us with non-KS NBMMPHs via **M1**. However, the MMPHs with $m = 1$ vertices are also big and obtaining small criticals with up to 40 hyperedges would require roughly a week on a supercomputer with 200 2.5 GHz CPUs working in parallel. We might go around this problem by exploiting previously generated small KS criticals [52] so as to use them as masters for non-KS MMPHs while applying **M3** as shown in Fig. 3(h-j). (Cf. the 6-dim star in Fig. 3(b)). Notice the graphical similarity of 4-dim 18-9 [55, Fig. 3(a)] and 8-dim 36-9 (shown in Fig. 3(h)): each vertex from the 18-9 vs. a pair of vertices in the 36-9. Since the distribution of 8-dim KS MMPHs in Ref. [52] is abundant we can generate arbitrarily many non-KS NBMMPHs in this manner via **M3**.

2.5. Dimensions 9 to 11

The 9-dim NBMMPH master obtained from $\{0, \pm 1\}$ has 9,586 vertices and 12,068,705 hyperedges and that proves to be too big for a direct generation of critical MMPHs (via stripping and filtering). (Higher dimensions—even more so.) However, billions of BMMPHs can be generated from the master and as we already stressed stripping them of $m = 1$ often provide us with NBMMPHs. This renders **M1** applicable. Thus, after random stripping of 12,068,200 hyperedges we obtained submasters with 505 hyperedges. Requiring that at least one of the hyperedges contains n vertices and that some of them can have the multiplicity $m = 1$, our program STATES01 yields a series of critical NBMMPHs, the smallest of which is 13-6 shown in Fig. 4(a); hyperedge `4ac7efhK2` contains 9 vertices. (Notice also that the 13-6 NBMMPH remains critical non-KS NBMMPH with any or some or all of `a,c,e,f,h,K` removed.) The filled 13-16, i.e., 44-6, also shown in Fig. 4(a), obtains the coordinatization directly from the supermaster since the programs preserve the names of the vertices in the process of stripping and of yielding sub-MMPHs. Obtaining a coordinatization via VECFIND takes too many CPU hours. The latter feature also makes **M2** inapplicable.

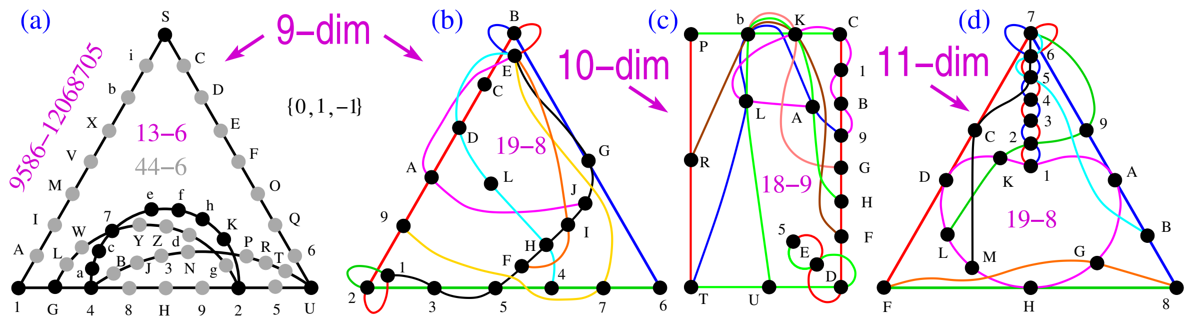

Figure 4. (a) 44-6 BMMPH and its critical subgraph 13-6 non-KS NBMMPH directly obtained from the super-master via **M1**; (b) critical 9-dim 19-8 obtained via **M3** from the master 47-16; (c) critical 10-dim 18-9 non-KS NBMMPH obtained via **M3** from the 50-15 master; (d) critical 11-dim 19-8 non-KS NBMMPH obtained via **M3** from the 50-14 master. Strings and coordinatizations are given in the appendices

If we wanted to keep our n-vertex requirement in full ("no $m = 1$ vertices"), in order to obtain critical non-KS NBMMPHs, we should employ **M3** so as to apply it on KS NBMMPHs obtained via dimensional upscaling [56,57] as follows. We remove several vertices from the smallest critical 47-16 we obtained in [57] till it is not critical any more. Then STATES01 yields the 19-8 critical shown in Fig. 4(b). (Removal of vertex `L` would terminate the criticality of the MMPH as with the 7-dim one shown in Fig. 3(f), but that does not affect the full n-vertex requirement.)

A 10-dim or any higher dimensional masters are too big to be generated from vector components. Therefore for obtaining non-KS MMPH in those dimensions we rely on minimal KS NBMMPHs obtained via dimensional upscaling [57] while applying **M3**. The procedure consists in removing vertices and/or hyperedges in such a way that an NBMMPHs stops being critical what enables us to generate smaller critical non-KS NBMMPHs from it via STATES01.

In Fig. 4(c) we show an 18-9 10-dim critical obtained via this approach from the 50-15 KS MMPH master [57].

In Fig. 4(d) we show a 19-8 11-dim critical obtained via the same approach from the 50-14 KS MMPH master [57].

In the following sections we shall keep to this approach while applying **M3**.

2.6. Dimensions 12 to 16

It has been proven that the minimal complexity (minimal number of hyperedges or vertices) of dimensional upscaling of KS MMPHs does not scale up with dimension [56]. In [57] we give constructive proofs that the minimal number of hyperedges of KS MMPHs repeatedly fluctuate between 9 and 16 in confirmation to that result. In the previous sections and in this one we provide constructive generations of critical non-KS NBMMPHs in dimensions 7 to 16 whose minimal number of hyperedges fluctuates between 8 (odd dimensions) and 9 (even dimensions) under the requirement that at least one the hyperedges contains n vertices none of which has the multiplicity $m = 1$. In lower dimensions (3-6) the minimal number of hyperedges is even smaller.

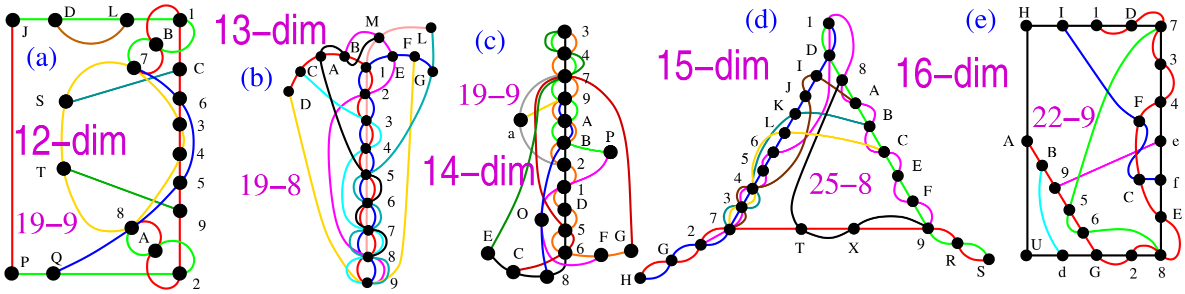

Figure 5. (a) 12-dim 19-9 critical non-KS NBMMPH directly obtained from the master 52-9 via **M3**; (b) 13-dim 19-8 critical non-KS NBMMPH obtained from the master 63-16; hyperedges do not form any loop of order 3 or higher; (c) 14-dim critical obtained from 66-15; maximal loop is also of order 2; (d) 15-dim 25-8 critical from the 66-14 master; (e) 16-dim 22-9 critical from the 70-9 master; all criticals are obtained via **M3**; all criticals and masters are given in the appendices.

3. Discussion

In this paper, we generate non-KS contextual NBMMPHs (non-binary MMP hypergraphs) first with the help of master sets generated from simple vector components whose complexity exponentially scales with dimension—dimensions 4 to 8—and then by means of methods whose complexity does not scale with dimension. The need for developing such methods and obtaining MMPHs in higher dimensions emerges from recent elaborations of classes of contextual sets that are not of the KS kind, all of which have an MMP hypergraph representation. Examples of such elaborations in the literature and their correspondence with MMPHs are given in Sec. 2.3. In subsequent sections we presented generations of non-KS NBMMPHs in up to 16-dim spaces.

In Sec. 2.1 the formalism and language of MMPH and in Sec. 2.2 the methods of generating them. In Sec. 2.3 we review the most prominent examples of non-KS sets from the literature in dimensions 3 to 5, represent them via MMPH formalism, and generate several new non-KS MMPHs in dimensions 4 and 5 with several coordinatizations. In Sec. 2.3 we then go up to the 8-dim spaces and show that arbitrarily exhaustive generation of MMPHs gets more and more computationally demanding from

3-dim to 8-dim spaces due to the exponentially increasing size of the MMPH masters obtained from vector components and the exponential complexity of extracting of NBMMPH classes from them. This is exacerbated by the ratio of NBMMPHs and BMMPHs which starts with under 0.1% in 4-dim spaces and grows exponentially with the dimension. So, in the 9-dim space in Sec. 2.5 with a master containing 9,586 vertices and 12,068,705 hyperedges we can strip any number of hyperedges from the master but a probability of finding any NBMMPH among the obtained MMPHs decreases with their sizes (e.g., search for them in MMPHs with more than a few thousand hyperedges would take "for ever" for any practical purpose). In 10 and higher dimensional spaces no method for obtaining MMPH masters from vector components is available any more.

Therefore to ensure arbitrarily exhaustive generation of MMPHs in ever higher dimensions we need a method whose complexity does not grow with dimension. For comparatively small KS MMPHs such a method—dimensional upscaling—was recently developed in [57] based on previous results in [56]. In this paper, we put forward a method of generating non-KS NBMMPHs whose complexity also does not scale up with dimension and which makes use of KS MMPHs obtained by the former KS method (in Secs. 2.5 and 2.6). Although the method applies to a generation of comparatively small MMPHs which are still suitable for any practical implementation since we can always obtain bigger MMPHs at the cost of the time a generation would take and since really big MMPHs cannot be generated at all and even if they could, they would be un-implementable. Minimal complexity (minimal number of hyperedges or vertices) of KS MMPHs repeatedly fluctuates between 9 and 16 while for non-KS NBMMPHs it fluctuates between 8 (odd dimensions) and 9 (even dimensions) in 7- to 16-dim spaces. In 3- to 6-dim it even goes down to 3. We provide a list of them in Table 1.

Table 1. The smallest critical non-KS MMPHs obtained via small vector-component method and by the dimensional upscaling method via **M1** and **M3**. Notice the steady fluctuation of the number of hyperedges over dimensions which is consistent with our previous result showing that the minimum complexity of NBMMPHs does not grow with dimension. The MMPH strings and coordinatizations of both the criticals and their masters are given in the appendices. ϕ is the Golden ratio and ω is the cube root of 1.

dim	Smallest critical MMPHs	Master	Vector components
3-dim	8-7 (Kochen-Specker's "bug")	49-36 (Bub's KS MMPH)	$\{0, \pm 1, \pm 2, 5\}$
4-dim	4-3	8-3	$\{0, \pm 1\}$
4-dim	16-9	58-51	$\{0, \pm\phi, \phi - 1\}$
5-dim	7-5	16-5	$\{0, \pm 1\}$
6-dim	11-7	19-7	$\{0, 1, \omega, \omega^2\}$
7-dim	14-8	34-14	$\{0, \pm 1\}$
8-dim	15-9	2768-1346016	$\{0, \pm 1\}$
9-dim	13-6	9586-12068705	$\{0, \pm 1\}$
9-dim	19-8	47-16	$\{0, \pm 1\}$
10-dim	18-9	50-15	$\{0, \pm 1\}$
11-dim	19-8	50-14	$\{0, \pm 1\}$
12-dim	19-9	52-9	$\{0, \pm 1\}$
13-dim	19-8	63-16	$\{0, \pm 1\}$
14-dim	19-9	66-15	$\{0, \pm 1\}$
15-dim	25-8	66-14	$\{0, \pm 1\}$
16-dim	22-9	70-9	$\{0, \pm 1\}$

4. Methods

The methods we use to handle quantum contextual sets rely on algorithms and programs within the MMP language: VECFIND, STATES01, MMPSTRIP, MMPSHUFFLE, SUBGRAPH, LOOP, and SHORTD developed in [8,47–49,51,58–60]. They are freely available at http://puh.srce.hr/s/Qegixzz2BdjYwFL. MMPHs can be visualized via hypergraph figures consisting of dots and lines and represented as a string of ASCII characters. The latter representation enables processing billions of MMPHs simultaneously via supercomputers and clusters. For the latter elaboration, we developed other dynamical programs specific program to handle and parallelize jobs with arbitrary number of MMP hypergraph vertices and edges.

5. Conclusion

To summarize, based on elaborations of non-KS sets which recently appeared in the literature and of which we give several examples in Sec. 2.3, we develop methods of generating comparatively small non-KS contextual sets in high dimensional spaces whose complexity does not grow with dimension. We give examples in all dimensions up to 16. A more detailed summary of the achieved results is given in Sec. 3

Author Contributions: Conceptualization; Data curation; Formal analysis; Funding acquisition; Investigation; Methodology; Project administration; Resources; Supervision; Validation; Visualization; Writing – original draft; Writing – review & editing.

Funding: Supported by the Ministry of Science and Education of Croatia through the Center of Excellence for Advanced Materials and Sensing Devices (CEMS) funding, and by MSE grants Nos. KK.01.1.1.01.0001 and 533-19-15-0022. Computational support was provided by the cluster Isabella of the Zagreb University Computing Centre.

Acknowledgments: Technical supports of Emir Imamagić from the University of Zagreb Computing Centre is gratefully acknowledged.

Conflicts of Interest: The author declares no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MMPH	McKay-Megill-Pavičić hypergraph (Definition 1)
NBMMPH	Non-binary McKay-Megill-Pavičić hypergraph (Definition 2)
BMMPH	Binary McKay-Megill-Pavičić hypergraph (Definition 5)
KS	Kochen-Specker (Definition 3
non-KS	Non-Kochen-Specker (Definition 4)
M1,M2,M3	Methods 1,2,3 (Section 2.2)

Appendix A. ASCII strings of non-KS MMPH classes and their masters and supermasters

Below we give strings and coordinatizations of all MMPHs referred to in the main body of the paper. The first hyperedges in a line of a critical NBMMPH often correspond to the biggest loops in the figures.

Appendix A.1. 3-dim MMPHs

8-7 (KS "bug") 123,34,45,567,78,81,26.

13-7 (filled **8-7**) `123,394,4A5,567,7B8,8C1,2D6.` 1=(0,0,1), 2=(0,1,0), 3=(1,0,0), 4=(0,1,1), 5=(1,1,-1), 6=(1,0,1), 7=(-1,2,1), 8=(2,1,0), 9=(0,1,-1), A=(2,-1,1), B=(1,-2,5), C=(1,-2,0), D=(1,0,-1)

Appendix A.2. 4-dim MMPHs

4-3 `12,34,1234.`

8-3 (filled **4-3**) `1562,3784,1234.` 1=(0,0,0,1), 2=(0,0,1,0), 3=(0,1,0,0), 4=(1,0,0,0), 5=(1,1,0,0), 6=(1,-1,0,0), 7=(0,0,1,1), 8=(0,0,1,-1)

16-9 `3124,49A8,8567,7BC3,DE,FG,GE62,FD51,FECA.`

20-9 (filled **16-9**) 1=(1,0,0,-1), 2=(0,1,1,0), 3=(1,1,-1,1), 4=(1,-1,1,1), 5=(1,0,0,1), 6=(0,1,-1,0), 7=(1,1,1,-1), 8=(-1,1,1,1), 9=(0,0,1,-1), A=(1,1,0,0), B=(0,0,1,1), C=(1,-1,0,0), D=(0,1,0,0), E=(0,0,0,1), F=(0,0,1,0), G=(1,0,0,0), H=(1,0,1,0), I=(1,0,-1,0), J=(0,1,0,1), K=(0,1,0,-1)

24-24 (Peres' super-master) LMNO,HIJK,DEFG,BCFG,9ADE,78EG,56DF,5678,9ABC,68JK,57HI, ACIK,9BHJ,1234,4DGO,3EFN,258M,167L,19CM,2ABL,3HKO,4IJN,34NO,12LM. 1=(0,0,0,1), 2=(0,0,1,0), 3=(1,1,0,0), 4=(1,-1,0,0), 5=(0,1,0,-1), 6=(1,0,-1,0), 7=(1,0,1,0), 8=(0,1,0,1), 9=(0,1,-1,0), A=(1,0,0,-1), B=(1,0,0,1), C=(0,1,1,0), D=(1,1,1,1), E=(1,-1,-1,1), F=(1,-1,1,-1), G=(1,1,-1,-1), H=(-1,1,1,1), I=(1,1,-1,1), J=(1,1,1,-1), K=(1,-1,1,1), L=(0,1,0,0), M=(1,0,0,0), N=(0,0,1,1), O=(0,0,1,-1)

16-9 231,1BC6,654,45GF,FGED,DE32,789A,7BC8,9A.

20-9 (filled **16-9**) 2H31,1BC6,6I54,45GF,FGED,DE32,789A,7BC8,9JKA. 1=$(0,0,0,\phi)$, 2=$(\phi\text{-}1,0,\text{-}\phi,0)$, 3=$(\phi,0,\phi\text{-}1,0)$, 4=$(0,\phi,0,\phi)$, 5=$(0,\phi,0,\text{-}\phi)$, 6=$(0,0,\phi,0)$, 7=$(0,0,\phi,\phi)$, 8=$(0,0,\phi,\text{-}\phi)$, 9=$(\phi,\phi\text{-}1,0,0)$, A=$(\phi\text{-}1,\text{-}\phi,0,0)$, B=$(\phi,\phi,0,0)$, C=$(\phi,\text{-}\phi,0,0)$, D=$(0,\phi\text{-}1,0,\text{-}\phi)$, E=$(0,\phi,0,\phi\text{-}1)$, F=$(\phi,0,\phi,0)$, G=$(\phi,0,\text{-}\phi,0)$, H=$(0,\phi,0,0)$, I=$(\phi,0,0,0)$, J=$(0,0,\phi,\phi\text{-}1)$, K=$(0,0,\phi\text{-}1,\text{-}\phi)$

60-72 (supper-master) 1234,1256,1278,129A,13BC,13DE,13FG,1HI4,1JK4,1LM4,23NO,23PQ, 23RS,2TU4,2VW4,2XY4,Za34,Za56,Za78,Za9A,Z5bc,Zde6,fg34,fg56,fg78,fg9A,hi34,hi56,hi78, hi9A,ajk6,a5lm,ano6,apq6,TUBC,TUDE,TUFG,TBbo,TCmd,VWBC,VWDE,VWFG,XYBC,XYDE,XYFG,HINO, HIPQ,HIRS,HNco,HOme,JKNO,JKPQ,JKRS,LMNO,LMPQ,LMRS,UBle,UnCc,UrsC,UtCu,jkbc,INld,InOb, IvwO,IxOy,nbco,rsbo,pbcq,vwco,tbuo,xyco,lmde. 1=$(0,0,0,\phi)$, 2=$(0,0,\phi,0)$, 3=$(0,\phi,0,0)$, 4=$(\phi,0,0,0)$, 5=$(\phi,\phi,0,0)$, 6=$(\phi,\text{-}\phi,0,0)$, 7=$(\phi,\phi\text{-}1,0,0)$, 8=$(\phi\text{-}1,\text{-}\phi,0,0)$, 9=$(\text{-}\phi,\phi\text{-}1,0,0)$, A=$(\phi\text{-}1,\phi,0,0)$, B=$(\phi,0,\phi,0)$, C=$(\phi,0,\text{-}\phi,0)$, D=$(\text{-}\phi,0,\phi\text{-}1,0)$, E=$(\phi\text{-}1,0,\phi,0)$, F=$(\phi,0,\phi\text{-}1,0)$, G=$(\phi\text{-}1,0,\text{-}\phi,0)$, H=$(0,\phi,\phi,0)$, I=$(0,\phi,\text{-}\phi,0)$, J=$(0,\phi\text{-}1,\phi,0)$, K=$(0,\text{-}\phi,\phi\text{-}1,0)$, L=$(0,\phi\text{-}1,\text{-}\phi,0)$, M=$(0,\phi,\phi\text{-}1,0)$, N=$(\phi,0,0,\phi)$, O=$(\phi,0,0,\text{-}\phi)$, P=$(\phi,0,0,\phi\text{-}1)$, Q=$(\phi\text{-}1,0,0,\text{-}\phi)$, R=$(\text{-}\phi,0,0,\phi\text{-}1)$, S=$(\phi\text{-}1,0,0,\phi)$, T=$(0,\phi,0,\phi)$, U=$(0,\phi,0,\text{-}\phi)$, V=$(0,\phi,0,\phi\text{-}1)$, W=$(0,\phi\text{-}1,0,\text{-}\phi)$, X=$(0,\text{-}\phi,0,\phi\text{-}1)$, Y=$(0,\phi\text{-}1,0,\phi)$, Z=$(0,0,\phi,\phi)$, a=$(0,0,\phi,\text{-}\phi)$, b=$(\phi,\text{-}\phi,\text{-}\phi,\phi)$, c=$(\phi,\text{-}\phi,\phi,\text{-}\phi)$, d=$(\phi,\phi,\phi,\text{-}\phi)$, e=$(\phi,\phi,\text{-}\phi,\phi)$, f=$(0,0,\phi,\phi\text{-}1)$, g=$(0,0,\phi\text{-}1,\text{-}\phi)$, h=$(0,0,\text{-}\phi,\phi\text{-}1)$, i=$(0,0,\phi\text{-}1,\phi)$, j=$(\phi,\phi,\phi\text{-}1,\phi\text{-}1)$, k=$(\phi\text{-}1,\phi\text{-}1,\text{-}\phi,\text{-}\phi)$, l=$(\text{-}\phi,\phi,\phi,\phi)$, m=$(\phi,\text{-}\phi,\phi,\phi)$, n=$(\phi,\phi,\phi,\phi)$, o=$(\phi,\phi,\text{-}\phi,\text{-}\phi)$, p=$(\text{-}\phi,\text{-}\phi,\phi\text{-}1,\phi\text{-}1)$, q=$(\phi\text{-}1,\phi\text{-}1,\phi,\phi)$, r=$(\phi,\phi\text{-}1,\phi,\phi\text{-}1)$, s=$(\phi\text{-}1,\text{-}\phi,\phi\text{-}1,\text{-}\phi)$, t=$(\text{-}\phi,\phi\text{-}1,\text{-}\phi,\phi\text{-}1)$, u=$(\phi\text{-}1,\phi,\phi\text{-}1,\phi)$, v=$(\phi,\phi\text{-}1,\phi\text{-}1,\phi)$, w=$(\phi\text{-}1,\text{-}\phi,\text{-}\phi,\phi\text{-}1)$, x=$(\text{-}\phi,\phi\text{-}1,\phi\text{-}1,\text{-}\phi)$, y=$(\phi\text{-}1,\phi,\phi,\phi\text{-}1)$

Appendix A.3. 5-dim MMPHs

7-5 41235,56,674,234,714.

16-5 (**7-5** filled) 41235,589A6,6BC74,2DE34,7FG14. 1=(0,0,1,0,0), 2=(1,-1,0,0,0), 3=(1,1,0,0,0), 4=(0,0,0,0,1), 5=(0,0,0,1,0), 6=(0,1,1,0,0), 7=(1,0,0,1,0), 8=(1,0,0,0,-1), 9=(0,1,-1,0,0), A=(1,0,0,0,1), B=(1,-1,1,-1,0), C=(1,1,-1,-1,0), D=(0,0,1,1,0), E=(0,0,1,-1,0), F=(0,1,0,0,0), G=(1,0,0,-1,0)

16-9 63457,75B9E,EFGD,DC,CA86,12345,89125,AB,FGE.

26-9 (**16-9** filled) 63457,75B9E,EFHGD,DIJKC,CAL86,12345,89125,AMNOB,FPQGE. 1=(1,1,1,-1,0), 2=(1,1,-1,1,0), 3=(1,-1,1,1,0), 4=(-1,1,1,1,0), 5=(0,0,0,0,1), 6=(0,0,1,-1,0), 7=(1,1,0,0,0), 8=(0,0,1,1,0), 9=(1,-1,0,0,0), A=(0,1,0,0,1), B=(0,0,1,0,0), C=(1,0,0,0,0), D=(0,1,1,0,0), E=(0,0,0,1,0), F=(1,0,0,0,1), G=(0,1,-1,0,0), H=(1,0,0,0,-1), I=(0,0,0,1,1), J=(0,1,-1,1,-1), K=(0,1,-1,-1,1), L=(0,1,0,0,-1), M=(1,1,0,-1,-1), N=(1,-1,0,-1,1), O=(1,0,0,1,0), P=(1,1,1,0,-1), Q=(-1,1,1,0,1)

105-136 (super-master) 12345,12367,12489,12AB5,134CD,13EF5,1GH45,1GH67,1G6IJ,1GKL7, 1H6MN,1HOP7,1EF89,1E8IP,1E9KN,1F8ML,1F9OJ,1ABCD,1ACJP,1ADLN,1QRST,1QUVW,1XYSZ,1XabW, 1BCMK,1BDOI,1cYVd,1caeT,1fRbd,1fUeZ,1OIJP,1MKLN,234gh,23ij5,2kl45,2kl67,2k6mn,2kop7, 2l6qr,2lst7,2ij89,2i8mt,2i9or,2j8qp,2j9sn,2ABgh,2Agtn,2Ahpr,2uvSw,2uxyW,2z!S",2z#$W, 2Bgoq,2Bhsm,2%!yd,2%#ew,2&v$d,2&xe",2smtn,2oqpr,'(345,'(367,'(489,'(AB5,'36)*,'3-/7, '48:;,'4<=9,'A>?5,'@[B5,(36\],(3^_7,(48‘{,(4|}9,(A~+15,(+2+3B5,3ijCD,3iC_*,3iD-\, 3jC/],3jD^),3EFgh,3Eg_),3Eh/\,3+4+5Vw,3+4+6yT,3+7+8V",3+7+9$T,3Fg-],3Fh^*,3+A+8yZ, 3+A+9bw,3+B+5$Z,3+B+6b",3^_)*,3-/\],kl4CD,klEF5,k4C};,k4D<‘,kE+3?5,kF@~5,l4C={,l4D|:, lE[+15,lF+2>5,GH4gh,GHij5,G4g}:,G4h=‘,Gi+3>5,Gj[~5,+C4+5Rx,+C4+6vU,+C+7X%5,+C+Azc5, +D4+8R#,+D4+9!U,+D+4X&5,+D+Buc5,H4g<{,H4h|;,Hi@+15,Hj+2?5,+E4+8va,+E4+9Yx,+E+4zf5, +E+BQ%5,+F4+5!a,+F4+6Y#,+F+7uf5,+F+AQ&5,4|}:;,4<=‘{,+2+3>?5,@[~+15. 1=(0,0,0,0,1), 2=(0,0,0,1,0), ’=(0,0,0,1,1), (=(0,0,0,1,-1), 3=(0,0,1,0,0), k=(0,0,1,0,1), l=(0,0,1,0,-1), G=(0,0,1,1,0), +C=(0,0,1,1,1), +D=(0,0,1,1,-1), H=(0,0,1,-1,0), +E=(0,0,1,-1,1), +F=(0,0,-1,1,1), 4=(0,1,0,0,0), i=(0,1,0,0,1), j=(0,1,0,0,-1), E=(0,1,0,1,0), +4=(0,1,0,1,1), +7=(0,1,0,1,-1), F=(0,1,0,-1,0), +A=(0,1,0,-1,1), +B=(0,-1,0,1,1), A=(0,1,1,0,0), u=(0,1,1,0,1), z=(0,1,1,0,-1), Q=(0,1,1,1,0), +2=(0,1,1,1,1), @=(0,1,1,1,-1), X=(0,1,1,-1,0), [=(0,1,1,-1,1), +3=(0,1,1,-1,-1), B=(0,1,-1,0,0), %=(0,1,-1,0,1), &=(0,-1,1,0,1), c=(0,1,-1,1,0), ∼=(0,1,-1,1,1), >=(0,1,-1,1,-1), f=(0,-1,1,1,0), ?=(0,1,-1,-1,1), +1=(0,-1,1,1,1), 5=(1,0,0,0,0), g=(1,0,0,0,1), h=(1,0,0,0,-1), C=(1,0,0,1,0), +8=(1,0,0,1,1), +5=(1,0,0,1,-1), D=(1,0,0,-1,0), +6=(1,0,0,-1,1), +9=(-1,0,0,1,1), 8=(1,0,1,0,0), !=(1,0,1,0,1), v=(1,0,1,0,-1), Y=(1,0,1,1,0), |=(1,0,1,1,1), <=(1,0,1,1,-1), R=(1,0,1,-1,0), ==(1,0,1,-1,1), }=(1,0,1,-1,-1), 9=(1,0,-1,0,0), x=(1,0,-1,0,1), #=(-1,0,1,0,1), U=(1,0,-1,1,0), ‘=(1,0,-1,1,1), :=(1,0,-1,1,-1), a=(-1,0,1,1,0), ;=(1,0,-1,-1,1), {=(-1,0,1,1,1), 6=(1,1,0,0,0), $=(1,1,0,0,1), y=(1,1,0,0,-1), b=(1,1,0,1,0), ^=(1,1,0,1,1), -=(1,1,0,1,-1), V=(1,1,0,-1,0), /=(1,1,0,-1,1), _=(1,1,0,-1,-1), e=(1,1,1,0,0), s=(1,1,1,0,1), o=(1,1,1,0,-1), O=(1,1,1,1,0), M=(-1,1,1,1,0), I=(1,-1,-1,1,0), K=(1,1,1,-1,0), q=(-1,1,1,0,1), m=(1,-1,-1,0,1), S=(1,1,-1,0,0), p=(1,1,-1,0,1), t=(1,1,-1,0,-1), L=(1,1,-1,1,0), d=(-1,1,1,0,0), J=(1,-1,1,-1,0), P=(1,1,-1,-1,0), N=(1,-1,1,1,0), n=(1,-1,1,0,-1), 7=(1,-1,0,0,0), w=(1,-1,0,0,1), "=(-1,1,0,0,1), T=(1,-1,0,1,0), \=(1,-1,0,1,1),)=(1,-1,0,1,-1), Z=(-1,1,0,1,0), *=(1,-1,0,-1,1),]=(-1,1,0,1,1), W=(1,-1,1,0,0), r=(1,-1,1,0,1)

10-9 [42, Fig. 5(a)] 17835,27846,91,97,92,8A,34,6A,5A.

31-9 (**10-9** filled) 17835,27846,9BCD1,9EFG7,9HIJ2,8KLMA,3NOP4,6QRSA,5TUVA. 1=(0,0,1,0,-1), 2=(0,0,0,1,-1), 3=(0,0,0,1,0), 4=(0,0,1,0,0), 5=(0,0,1,0,1), 6=(0,0,0,1,1), 7=(0,1,0,0,0), 8=(1,0,0,0,0), 9=(0,0,1,1,1), A=(0,1,1,1,-1), B=(0,0,-1,2,-1), C=(1,2,0,0,0), D=(2,-1,0,0,0), E=(0,0,-1,-1,2), F=(1,0,1,-1,0), G=(2,0,-1,1,0), H=(0,0,2,-1,-1), I=(1,1,0,0,0), J=(1,-1,0,0,0), K=(0,0,1,-1,0), L=(0,1,0,0,1), M=(0,-1,1,1,1), N=(0,0,0,0,1), O=(-1,2,0,0,0), P=(2,1,0,0,0), Q=(0,1,1,-1,1), R=(1,1,-1,0,0), S=(2,-1,1,0,0), T=(0,1,0,-1,0), U=(2,1,-1,1,1), V=(2,-1,1,-1,-1)

Appendix A.4. 6-dim MMPHs

19-7 7!8gw,woO6i,i;)EB,B<:b7,’!)Jb6,’o:8Eu,;<OJgu.

11-7 w!g,wi6,’!)Jb6,’u,B)i,Jgu,Bb.

21-7 (**11-7, 19-7** filled) w!78#g,woOi6%,’!)Jb6,’o:8Eu,;B)i#E,;<OJgu,B<7:b%. w=$(\omega,1,1,\omega^2,1,\omega)$, ’=$(\omega^2,\omega,\omega,1,1,1)$, ;=$(\omega,1,1,\omega^2,\omega,1)$, !=$(\omega,1,\omega^2,1,\omega,1)$, B=$(1,\omega^2,\omega,1,\omega,1)$, <=$(\omega,1,\omega,1,1,\omega^2)$, o=$(1,\omega^2,1,1,1,\omega^2)$,)=$(\omega,\omega^2,1,1,1,\omega)$, 7=$(1,\omega,\omega^2,1,1,\omega)$, O=$(1,\omega,\omega,\omega^2,1,1)$, :=$(\omega^2,1,1,1,\omega,\omega)$, 8=$(1,\omega,1,\omega^2,\omega,1)$, i=$(\omega^2,1,\omega^2,1,1,1)$, J=$(1,\omega,1,1,\omega,\omega^2)$, #=$(1,1,1,1,\omega^2,\omega^2)$, b=$(1,1,1,\omega,1,1)$, 6=$(1,1,\omega,1,\omega^2,\omega)$, g=$(\omega^2,\omega^2,1,1,1,1)$, E=$(1,1,\omega,\omega^2,1,\omega)$, %=$(\omega,\omega,1,1,\omega^2,1)$, u=$(1,1,\omega^2,1,\omega^2,1)$

79-162 (81-162 stripped; %,#) 123456,12789A,1BCD5E,1B7FGH,1ICJ9K,1I3LGM,1NODAP,1NQ4HR, 1SOJ6T,1SU8MR,1VQLET,1VUFKP,WXY45Z,WX7abA,Wcde5E,Wc7Ffg,WIdJbh,WIYifM,WjkeAP,WjQ4gl, WSkJZm,WSnaMl,WoQiEm,WonFhP,pqY89Z,pq3ab6,pcre9K,pc3Lsg,pBrDbh,pBYisH,pjte6T,pjU8gu, pNtDZm,pNnaHu,pvnLhT,pvUiKm,wxyD5Z,wx7azH,w!"e56,w!78g,wI"Lzh,wIyiK,w$keHR,w$ODgl, wVkLZ,wV&aKl,woOi6,wo&8hR,’(yDbA,’(Y4zH,’!)Jb6,’!Y8*M,’c)LzE,’cyF*K,’-kJHu,’-tDMl, ’vkLA/,’v:4Kl,’otF6/,’o:8Eu,;("e9A,;(34g,;x)J9Z,;x3a*M,;B)iE,;B"F*h,;<teMR,;<OJgu, ;vOiA=,;v>4hR,;VtFZ=,;V>aEu,?!reGM,?!CJsg,?qyFGZ,?qCazE,?2yisA,?2r4zh,?$:eET,?$UFg/, ?-&JhT,?-UiM,?N:4Z,?N&aA/,@(deGH,@(CDfg,@X)LGZ,@XCa*K,@2)if6,@2d8*h,@<:eKP,@<QLg/, @->DhP,@-QiH=,@S:8Z=,@S>a6/,[xdJsH,[xrDfM,[X"LsA,[Xr4K,[q"Ff6,[qd8E,[<&JKm,[<nLM, [$>DEm,[$nFH=,[j>46,[j&8A=,(S"Uzm,(SynT,(VdUb,(VY&fT,(oCn9,(o3&Gm,xj)UzP,xjyQ*T, xvdU5/,xv7:fT,xorQ9/,xo3:sP,!N)nP,!N"Q*m,!vCn5=,!v7>Gm,!VrQb=,!VY>sP,X$)UbR,X$YO*T, X-"U5u,X-7tT,XorOGu,XoCtsR,q<yQbR,q<YOzP,q-"Q9l,q-3kP,qvdOGl,qvCkfR,2<yn5u,2<7tzm, 2$)n9l,2$3k*m,2Vdtsl,2Vrkfu,c-3&5=,c-7>9,cN)&fR,cNdO*,cSy>sR,cSrOz=,B<Y&5/,B<7:b, Bj)&Gl,BjCk*,BS":sl,BSrk/,I$Y>9/,I$3:b=,Ijy>Gu,IjCtz=,IN":fu,INdt/.

81-162 123456,12789A,1BCD5E,1B7FGH,1ICJ9K,1I3LGM,1NODAP,1NQ4HR,1SOJ6T,1SU8MR, 1VQLET,1VUFKP,WXY45Z,WX7abA,Wcde5E,Wc7Ffg,WIdJbh,WIYifM,WjkeAP,WjQ4gl,WSkJZm,WSnaMl,

WoQiEm,WonFhP,pqY89Z,pq3ab6,pcre9K,pc3Lsg,pBrDbh,pBYisH,pjte6T,pjU8gu,pNtDZm,pNnaHu,
pvnLhT,pvUiKm,wxyD5Z,wx7azH,w!"e56,w!78#g,wI"Lzh,wIyi#K,w$keHR,w$ODgl,wVkLZ%,wV&aKl,
wo0i6%,wo&8hR,'(yDbA,'(Y4zH,'!)Jb6,'!Y8*M,'c)LzE,'cyF*K,'-kJHu,'-tDMl,'vkLA/,'v:4Kl,
'otF6/,'o:8Eu,;("e9A,;(34#g,;x)J9Z,;x3a*M,;B)i#E,;B"F*h,;<teMR,;<OJgu,;vOiA=,;v>4hR,
;VtFZ=,;V>aEu,?!reGM,?!CJsg,?qyFGZ,?qCazE,?2yisA,?2r4zh,?$:eET,?$UFg/,?-&JhT,?-UiM%,
?N:4Z%,?N&aA/,@(deGH,@(CDfg,@X)LGZ,@XCa*K,@2)if6,@2d8*h,@<:eKP,@<QLg/,@->DhP,
@-QiH=,@S:8Z=,@S>a6/,[xdJsH,[xrDfM,[X"LsA,[Xr4#K,[q"Ff6,[qd8#E,[<&JKm,[<nLM%,
[$>DEm,[$nFH=,[j>46%,[j&8A=,(S"Uzm,(Syn#T,(VdUb%,(VY&fT,(oCn9%,(o3&Gm,xj)UzP,xjyQ*T,
xvdU5/,xv7:fT,xorQ9/,xo3:sP,!N)n#P,!N"Q*m,!vCn5=,!v7>Gm,!VrQb=,!VY>sP,X$)UbR,X$YO*T,
X-"U5u,X-7t#T,XorOGu,XoCtsR,q<yQbR,q<YOzP,q-"Q9l,q-3k#P,qvdOGl,qvCkfR,2<yn5u,
2<7tzm,2$)n9l,2$3k*m,2Vdtsl,2Vrkfu,c-3&5=,c-7>9%,cN)&fR,cNdO*%,cSy>sR,cSrOz=,
B<Y&5/,B<7:b%,Bj)&Gl,BjCk*%,BS":sl,BSrk#/,I$Y>9/,I$3:b=,Ijy>Gu,IjCtz=,IN":fu,
INdt#/. 1=$(\omega,1,1,1,1,1)$, W=$(\omega,1,1,1,\omega^2,\omega)$, p=$(\omega,1,1,1,\omega,\omega^2)$, w=$(\omega,1,1,\omega^2,1,\omega)$, '=$(\omega^2,\omega,\omega,1,1,1)$,
;=$(\omega,1,1,\omega^2,\omega,1)$, ?=$(\omega,1,1,\omega,1,\omega^2)$, @=$(\omega,1,1,\omega,\omega^2,1)$, [=$(1,\omega^2,\omega^2,1,1,1)$, (=$(\omega,1,\omega^2,1,1,\omega)$,
x=$(\omega^2,\omega,1,\omega,1,1)$, !=$(\omega,1,\omega^2,1,\omega,1)$, X=$(\omega^2,\omega,1,1,\omega,1)$, q=$(\omega^2,\omega,1,1,1,\omega)$, 2=$(1,\omega^2,\omega,\omega,1,1)$,
c=$(\omega,1,\omega^2,\omega,1,1)$, B=$(1,\omega^2,\omega,1,\omega,1)$, I=$(1,\omega^2,\omega,1,1,\omega)$, <=$(\omega,1,\omega,1,1,\omega^2)$, $\$=(\omega,1,\omega,1,\omega^2,1)$,
-=$(1,\omega^2,1,\omega^2,1,1)$, j=$(\omega,1,\omega,\omega^2,1,1)$, N=$(1,\omega^2,1,\omega,\omega,1)$, S=$(1,\omega^2,1,\omega,1,\omega)$, v=$(1,\omega^2,1,1,\omega^2,1)$,
V=$(1,\omega^2,1,1,\omega,\omega)$, o=$(1,\omega^2,1,1,1,\omega^2)$,)=$(\omega,\omega^2,1,1,1,\omega)$, "=$(\omega^2,1,\omega,\omega,1,1)$, y=$(\omega,\omega^2,1,1,\omega,1)$,
d=$(\omega^2,1,\omega,1,\omega,1)$, r=$(\omega^2,1,\omega,1,1,\omega)$, C=$(1,\omega,\omega^2,\omega,1,1)$, Y=$(\omega,\omega^2,1,\omega,1,1)$, 3=$(1,\omega,\omega^2,1,\omega,1)$,
7=$(1,\omega,\omega^2,1,1,\omega)$, k=$(\omega^2,1,1,\omega,\omega,1)$, t=$(\omega^2,1,1,\omega,1,\omega)$, O=$(1,\omega,\omega,\omega^2,1,1)$, :=$(\omega^2,1,1,1,\omega,\omega)$,
>=$(\omega^2,1,1,1,1,\omega^2)$, &=$(\omega^2,1,1,1,\omega^2,1)$, Q=$(1,\omega,\omega,1,\omega^2,1)$, n=$(\omega^2,1,1,\omega^2,1,1)$, U=$(1,\omega,\omega,1,1,\omega^2)$,
a=$(\omega,\omega^2,\omega,1,1,1)$, 8=$(1,\omega,1,\omega^2,\omega,1)$, 4=$(1,\omega,1,\omega^2,1,\omega)$, F=$(1,\omega,1,\omega,\omega^2,1)$, i=$(\omega^2,1,\omega^2,1,1,1)$,
L=$(1,\omega,1,\omega,1,\omega^2)$, D=$(1,\omega,1,1,\omega^2,\omega)$, J=$(1,\omega,1,1,\omega,\omega^2)$, *=$(1,1,1,1,1,\omega)$, z=$(1,1,1,1,\omega,1)$, #=$(1,1,1,1,\omega^2,\omega^2)$,
e=$(1,\omega,1,1,1,1)$, b=$(1,1,1,\omega,1,1)$, 5=$(1,1,1,\omega,\omega,\omega^2)$, 9=$(1,1,1,\omega,\omega^2,\omega)$, f=$(1,1,1,\omega^2,1,\omega^2)$, G=$(1,1,1,\omega^2,\omega,\omega)$,
s=$(1,1,1,\omega^2,\omega^2,1)$, Z=$(1,1,\omega,1,1,1)$, A=$(1,1,\omega,1,\omega,\omega^2)$, 6=$(1,1,\omega,1,\omega^2,\omega)$, H=$(1,1,\omega,\omega,1,\omega^2)$,
g=$(\omega^2,\omega^2,1,1,1,1)$, M=$(1,1,\omega,\omega,\omega^2,1)$, E=$(1,1,\omega,\omega^2,1,\omega)$, K=$(1,1,\omega,\omega^2,\omega,1)$, h=$(\omega,\omega,\omega^2,1,1,1)$,
==$(\omega,\omega,1,1,1,\omega^2)$, %=$(\omega,\omega,1,1,\omega^2,1)$, l=$(1,1,\omega^2,1,1,\omega^2)$, R=$(1,1,\omega^2,1,\omega,\omega)$, u=$(1,1,\omega^2,1,\omega^2,1)$,
/=$(1,1,\omega^2,\omega^2,1,1)$, m=$(\omega,\omega,1,\omega^2,1,1)$, P=$(1,1,\omega^2,\omega,1,\omega)$, T=$(1,1,\omega^2,\omega,\omega,1)$

31-16 237,7HG,GTUVRP,PRNQSM,MIJKL2,235,7235,3NOKLM,WXJRSM,WYZGRV,aOQbcM,aYdHce,
XIfbcM,fgUHce,gTUGH,YZdGH.

44-16 (master for **31-16**) 123456,72389A,723B5C,7DEFGH,2IJKLM,3NOKLM,PNQRSM,PTUGRV,
WXJRSM,WYZGRV,aOQbcM,aYdHce,XIfbcM,fgUHce,gTUhGH,iYZdGH. 1=(0,0,1,-1,1,0), 7=(0,0,-1,1,1,0),
2=(0,1,0,0,0,1), 3=(0,1,0,0,0,-1), P=(0,1,0,0,1,0), W=(0,1,0,0,-1,0), a=(0,1,0,1,0,0), X=(0,1,0,1,1,1),
D=(0,1,0,1,-1,0), N=(0,1,0,1,-1,1), I=(0,1,0,1,-1,-1), f=(0,1,0,-1,0,0), O=(0,1,0,-1,1,1), J=(0,1,0,-1,1,-1),
Q=(0,-1,0,1,1,1), E=(0,1,1,0,1,0), g=(0,1,1,1,1,0), i=(0,1,1,1,-1,0), Y=(0,1,1,-1,1,0), T=(0,1,1,-1,-1,0),
Z=(0,1,-1,1,1,0), U=(0,1,-1,1,-1,0), F=(0,-1,1,1,0,0), h=(0,1,-1,-1,1,0), d=(0,-1,1,1,1,0), G=(1,0,0,0,0,1),
H=(1,0,0,0,0,-1), 8=(1,0,0,1,-1,0), B=(1,0,0,-1,1,0), 4=(-1,0,0,1,1,0), 9=(1,0,1,0,1,0), b=(1,0,1,0,1,-1),
c=(1,0,1,0,-1,1), 5=(1,0,1,1,0,0), R=(1,0,1,1,0,-1), K=(1,0,1,1,1,0), S=(1,0,1,-1,0,1), L=(1,0,1,-1,-1,0),
M=(1,0,-1,0,0,0), 6=(1,0,-1,0,1,0), e=(1,0,-1,0,1,1), C=(-1,0,1,0,1,0), A=(-1,0,1,1,0,0), V=(-1,0,1,1,0,1)

117-116 (strip-master of 236-1216 super-master) 123456,123789,1AB4CD,1AB7EF,1GHI5F,
1GHJC9,1KLI8D,1KLJE6,MN8COP,MNF6QR,STE5OP,ST9DQR,UVWXYZ,UVabcd,UefgYZ,Uefabh,Uijgcd,
UijWXh,UBk4XZ,UBk7ac,U3l4bd,U3l7WY,ULmIXY,ULmJbc,UHnIad,UHnJWZ,opVgYZ,opVabh,opijgh,
opabqR,opYZPr,oistuv,oiwxyz,oj!"#$,oj%&'(,)*Vgcd,)*VWXh,)*efgh,)*WXqR,)*cdPr,)e-/uv,
)e:;yz,)f!"<=,)f%&>?,*e@[#$,*e\]'(,*fst^_,*fwx‘{,pi@[<=,pi\]>?,pj-/^_,pj:;‘{,
2|V4XZ,2|V7ac,2|3l47,2|ac}∼,2|XZ+1+2,2389u{,2356^z,2l\"+3+4,2l%[+5+6,A+7V4bd,A+7V7WY,
A+7Bk47,A+7WY}∼,A+7bd+1+2,ABEFu{,ABCD^z,Ak\"+8+9,Ak%[+A+B,G+CVIXY,G+CVJbc,G+CHnIJ,
G+Cbc+D+E,G+CXY+F+G,GHC9#?,GH5F<(,Gn:t+3+B,Gnw/+8+6,K+HVIad,K+HVJWZ,K+HLmIJ,
K+HWZ+D+E,K+Had+F+G,KLE6#?,KL8D<(,Km:t+5+9,Kmw/+A+4,+7B@&+3+4,+7B!]+5+6,+7k89y_,
+7k56‘v,|3@&+8+9,|3!]+A+B,|lEFy_,|lCD‘v,+HL-x+3+B,+HLs;+8+6,+HmC9'=,+Hm5F>$,

+CH-x+5+9,+CHs;+A+4,+CnE6’=,+Cn8D>$,efab+IO,efYZQ+J,ijWX+IO,ijcdQ+J,Bkac+K+L, BkXZ+M+N,3lWY+K+L,3lbd+M+N,Lmbc+O+P,LmXY+Q+R,HnWZ+O+P,Hnad+Q+R. 1=(0,1,0,-1,0,0), M=(0,1,0,-1,1,1), S=(0,1,0,-1,1,-1), T=(0,1,0,-1,-1,1), N=(0,-1,0,1,1,1), U=(0,1,1,0,0,0), o=(0,1,1,0,1,1),)=(0,1,1,0,1,-1), *=(0,1,1,0,-1,1), p=(0,1,1,0,-1,-1), 2=(0,1,1,1,0,1), A=(0,1,1,1,0,-1), G=(0,1,1,1,1,0), K=(0,1,1,1,-1,0), +7=(0,1,1,-1,0,1), |=(0,1,1,-1,0,-1), +H=(0,1,1,-1,1,0), +C=(0,1,1,-1,-1,0), V=(0,1,-1,0,0,0), e=(0,1,-1,0,1,1), i=(0,1,-1,0,1,-1), j=(0,1,-1,0,-1,1), f=(0,-1,1,0,1,1), B=(0,1,-1,1,0,1), 3=(0,1,-1,1,0,-1), L=(0,1,-1,1,1,0), H=(0,1,-1,1,-1,0), l=(0,1,-1,-1,0,1), k=(0,-1,1,1,0,1), n=(0,1,-1,-1,1,0), m=(0,-1,1,1,1,0), I=(1,0,0,0,0,1), J=(1,0,0,0,0,-1), 4=(1,0,0,0,1,0), 7=(1,0,0,0,-1,0), g=(1,0,0,1,0,0), W=(1,0,0,1,1,1), a=(1,0,0,1,1,-1), b=(1,0,0,1,-1,1), X=(1,0,0,1,-1,-1), h=(1,0,0,-1,0,0), c=(1,0,0,-1,1,1), Y=(1,0,0,-1,1,-1), Z=(1,0,0,-1,-1,1), d=(-1,0,0,1,1,1), E=(1,0,1,0,1,1), 8=(1,0,1,0,1,-1), C=(1,0,1,0,-1,1), 5=(1,0,1,0,-1,-1), -=(1,0,1,1,0,1), s=(1,0,1,1,0,-1), @=(1,0,1,1,1,0), !=(1,0,1,1,-1,0), :=(1,0,1,-1,0,1), w=(1,0,1,-1,0,-1), \=(1,0,1,-1,1,0), %=(1,0,1,-1,-1,0), 9=(1,0,-1,0,1,1), F=(1,0,-1,0,1,-1), 6=(1,0,-1,0,-1,1), D=(-1,0,1,0,1,1), t=(1,0,-1,1,0,1), /=(1,0,-1,1,0,-1), "=(1,0,-1,1,1,0), [=(1,0,-1,1,-1,0), x=(1,0,-1,-1,0,1), ;=(-1,0,1,1,0,1), &=(1,0,-1,-1,1,0),]=(-1,0,1,1,1,0), +A=(1,1,0,0,1,1), +5=(1,1,0,0,1,-1), +8=(1,1,0,0,-1,1), +3=(1,1,0,0,-1,-1), >=(1,1,0,1,0,1), ’=(1,1,0,1,0,-1), ‘=(1,1,0,1,1,0), y=(1,1,0,1,-1,0), <=(1,1,0,-1,0,1), # =(1,1,0,-1,0,-1), ^=(1,1,0,-1,1,0), u=(1,1,0,-1,-1,0), +Q=(1,1,1,0,0,1), +O=(1,1,1,0,0,-1), +M=(1,1,1,0,1,0), +K=(1,1,1,0,-1,0), Q=(1,1,1,1,0,0), +I=(1,1,1,-1,0,0), O=(-1,1,1,1,0,0), +F=(1,1,-1,0,0,1), +D=(1,1,-1,0,0,-1), +1=(1,1,-1,0,1,0), }=(1,1,-1,0,-1,0), P=(1,1,-1,1,0,0), +J=(1,-1,-1,1,0,0), q=(1,1,-1,-1,0,0), +L=(-1,1,1,0,1,0), +N=(1,-1,-1,0,1,0), +6=(1,-1,0,0,1,1), +B=(1,-1,0,0,1,-1), +4=(1,-1,0,0,-1,1), +9=(-1,1,0,0,1,1), (=(1,-1,0,1,0,1), ?=(1,-1,0,1,0,-1), z=(1,-1,0,1,1,0), {=(1,-1,0,1,-1,0), $=(1,-1,0,-1,0,1), ==(-1,1,0,1,0,1), v=(1,-1,0,-1,1,0), _=(-1,1,0,1,1,0), +G=(1,-1,1,0,0,1), +E=(1,-1,1,0,0,-1), +2=(1,-1,1,0,1,0), ∼=(1,-1,1,0,-1,0), r=(1,-1,1,1,0,0), +P=(-1,1,1,0,0,1), +R=(1,-1,-1,0,0,1), R=(1,-1,1,-1,0,0)

Appendix A.5. 7-dim MMPHs

14-8 12567,189A5BC,189DE7,189HJ,189HB,2D,2EC,AJ.

31-13 1234,189DEF,189GHIJ,189KHBL,2MNDOIP,2MNEOCL,2MNGKF,QRNSAJP,QT4U,RTV9,WXMS, WYV8AJP,XY3U.

34-14 (master for **14-8** and **31-13**) 1234567,189A5BC,189DE7F,189GHIJ,189KHBL,2MNDOIP, 2MNEOCL,2MNGK6F,QRNSAJP,QT4U567,RTV9567,WXMS567,WYV8AJP,XY3U567. 1=(0,0,0,1,0,0,0); 2=(0,0,1,0,0,0,0); 3=(1,-1,0,0,0,0,0); 4=(1,1,0,0,0,0,0); 5=(0,0,0,0,0,0,1); 6=(0,0,0,0,1,1,0); 7=(0,0,0,0,1,-1,0); 8=(0,1,-1,0,0,0,0); 9=(0,1,1,0,0,0,0); A=(0,0,0,0,1,0,0); B=(1,0,0,0,0,-1,0); C=(1,0,0,0,0,1,0); D=(1,0,0,0,1,1,-1); E=(-1,0,0,0,1,1,1); F=(1,0,0,0,0,0,1); G=(1,0,0,0,1,-1,-1); H=(1,0,0,0,1,1,1); I=(1,0,0,0,-1,0,0); J=(0,0,0,0,0,1,-1); K=(1,0,0,0,-1,1,-1); L=(0,0,0,0,1,0,-1); M=(0,1,0,1,0,0,0); N=(0,1,0,-1,0,0,0); O=(1,0,0,0,1,-1,1); P=(0,0,0,0,0,1,1); Q=(-1,1,1,1,0,0,0); R=(1,1,-1,1,0,0,0); S=(1,0,1,0,0,0,0); T=(1,-1,1,1,0,0,0); U=(0,0,1,-1,0,0,0); V=(1,0,0,-1,0,0,0); W=(1,-1,-1,1,0,0,0); X=(1,1,-1,-1,0,0,0); Y=(1,1,1,1,0,0,0).

Appendix A.6. 8-dim MMPHs

15-9 17426538,8E,E2M,M3N,N4O,O5U,U7Q,Q6H,H1.

36-9 (master for **15-9**) 17426538,8ABDF9CE,ELaR2YVM,M3WSDKZN,NCJXR4TO,OV5PSBIU,UZ9GXa7Q, QTY6PWAH,HIKFGJL1. 1=(0,0,0,0,0,0,0,1), 2=(0,0,0,0,0,0,1,0), 3=(0,0,0,0,0,1,0,0), 4=(0,0,0,0,1,0,0,0), 5=(0,0,1,1,0,0,0,0), 6=(0,0,-1,1,0,0,0,0), 7=(1,1,0,0,0,0,0,0), 8=(-1,1,0,0,0,0,0,0), 9=(0,0,0,0,0,0,1,1), A=(0,0,1,1,1,-1,0,0), B=(1,1,0,0,0,0,-1,1), C=(1,1,0,0,0,0,1,-1), D=(0,0,-1,0,1,0,0,0), E=(0,0,0,1,0,1,0,0), F=(0,0,1,-1,1,1,0,0), G=(0,0,0,1,1,0,0,0), H=(0,0,1,1,-1,1,0,0), I=(1,0,0,0,0,0,1,0), J=(0,0,-1,0,0,1,0,0), K=(-1,0,0,0,0,0,1,0), L=(0,1,0,0,0,0,0,0), M=(0,0,1,0,1,0,0,0), N=(0,0,0,1,0,0,0,0), O=(-1,1,0,0,0,0,1,1), P=(0,0,0,0,1,1,0,0), Q=(-1,1,0,0,0,0,1,-1), R=(1,0,0,0,0,0,0,1), S=(0,-1,0,0,0,0,0,1), T=(0,-1,0,0,0,0,1,0), U=(0,0,-1,1,-1,1,0,0), V=(0,0,-1,1,1,-1,0,0), W=(1,1,0,0,0,0,1,1), X=(0,0,1,0,0,1,0,0), Y=(-1,0,0,0,0,0,0,1), Z=(-1,1,0,0,0,0,-1,1), a=(0,0,-1,-1,1,1,0,0)

Appendix A.7. 9-dim MMPHs

13-6 SU,1G42U,1S,472acefhK,G72,4U.

44-6 (**13-6** filled) SUCDEFOQ6,1G42U8H95,1SAIMSVXbi,472acefhK,G72LWYZdg,4UBJ3NPRT. 1=(0,0,0,0,0,0,0,1,0); 2=(0,0,0,0,0,0,1,0,0); 3=(0,0,0,0,1,1,0,0,0); 4=(0,0,0,1,0,0,0,0,1); 5=(0,0,0,1,0,0,0,0,-1); 6=(0,0,0,1,0,0,-1,1,0); 7=(0,0,1,0,0,0,0,1,0); 8=(0,0,1,0,0,1,0,0,0); 9=(0,0,1,0,0,-1,0,0,0); A=(0,0,1,0,-1,0,1,0,0); B=(0,0,1,0,-1,1,1,0,0); C=(0,0,-1,-1,1,1,0,1,1); D=(0,0,1,-1,-1,-1,0,1,1); H=(0,1,0,0,1,0,0,0,0); E=(0,1,0,0,1,-1,1,1,-1); F=(0,1,0,0,1,-1,-1,-1,1); G=(0,1,0,0,-1,0,0,0,0); I=(0,1,0,1,1,1,1,0,1); J=(0,1,0,-1,0,0,0,-1,1); K=(1,-1,1,1,-1,1,0,-1,-1); L=(0,1,0,-1,1,1,0,0,0); M=(0,1,0,-1,1,-1,1,0,-1); N=(0,1,-1,0,0,0,1,1,0); O=(0,1,1,1,0,1,1,0,1); P=(0,1,1,1,1,-1,1,-1,-1); Q=(0,1,1,-1,0,1,-1,0,-1); R=(0,1,-1,0,0,0,1,1,0); S=(0,-1,1,0,1,0,0,0,0); U=(1,0,0,0,0,0,0,0,0); V=(1,0,0,0,0,-1,0,0,1); W=(1,0,0,1,0,1,0,0,1); X=(1,0,0,-1,0,1,0,0,0); Y=(1,0,0,-1,0,-1,0,0,1); Z=(1,0,1,0,0,0,0,-1,-1); a=(1,1,0,0,-1,-1,0,0,0); b=(1,1,1,1,0,0,-1,0,-1); c=(1,1,1,-1,1,1,0,-1,1); d=(-1,1,1,1,1,-1,0,-1,1); e=(1,-1,-1,-1,-1,1,0,1,1); f=(1,1,-1,1,1,1,0,1,-1); g=(1,1,-1,1,1,-1,0,1,-1); h=(1,-1,0,0,1,-1,0,0,0); i=(1,-1,-1,1,0,0,1,0,-1).

19-8 1234567,129ABCDE,13FGH5IJE,GB6E,AJE,97E,LH4DE,FIE.

47-16 (master for **19-8**) 123456789,12ABCDEF,13HIJ5KL,1AMCLNOPQ,1BRSETUVQ,1H4567VWX, 1ICDEFYOX,1ZJ4FTUW9,1ZCDEFYP8,23abRSET,cdIeMD6f,cgaZMCLN,dghA4567,ijhBRk7f,ilbZJ4FT, jlHeSkKN. 1=(1,0,0,0,0,0,0,0,0), 2=(0,1,0,0,0,0,0,0,0), 3=(0,0,1,0,0,0,0,0,0), c=(1,1,1,1,0,0,0,0,0), d=(1,-1,1,-1,0,0,0,0,0), g=(1,-1,-1,1,0,0,0,0,0), i=(1,-1,-1,-1,0,0,0,0,0), j=(1,-1,1,1,0,0,0,0,0), l=(1,1,1,-1,0,0,0,0,0), h=(1,1,0,0,0,0,0,0,0), A=(0,0,1,1,0,0,0,0,0), B=(0,0,1,-1,0,0,0,0,0), H=(0,1,0,1,0,0,0,0,0), I=(0,1,0,-1,0,0,0,0,0), e=(1,0,-1,0,0,0,0,0,0), a=(1,0,0,-1,0,0,0,0,0), b=(1,0,0,1,0,0,0,0,0), Z=(0,1,-1,0,0,0,0,0,0), J=(0,0,0,0,1,0,0,0,0), 4=(0,0,0,0,0,1,0,0,0), 5=(0,0,0,0,0,0,1,0,0), M=(0,0,0,0,1,1,1,1,0), C=(0,0,0,0,1,-1,1,-1,0), D=(0,0,0,0,1,-1,-1,1,0), R=(0,0,0,0,1,-1,-1,-1,0), S=(0,0,0,0,1,-1,1,1,0), k=(0,0,0,0,1,1,1,-1,0), E=(0,0,0,0,1,1,0,0,0), F=(0,0,0,0,0,0,1,1,0), T=(0,0,0,0,0,0,1,-1,0), K=(0,0,0,0,0,1,0,1,0), L=(0,0,0,0,0,1,0,-1,0), N=(0,0,0,0,1,0,-1,0,0), 6=(0,0,0,0,1,0,0,-1,0), 7=(0,0,0,0,1,0,0,1,0), f=(0,0,0,0,0,1,-1,0,0), Y=(0,1,1,1,0,0,0,0,1), O=(0,1,-1,1,0,0,0,0,-1), P=(0,1,1,-1,0,0,0,0,-1), U=(0,1,1,1,0,0,0,0,-1), V=(0,1,-1,-1,0,0,0,0,-1), W=(0,1,1,-1,0,0,0,0,1), Q=(0,1,0,0,0,0,0,0,1), X=(0,0,1,0,0,0,0,0,-1), 8=(0,0,0,1,0,0,0,0,-1), :9=(0,0,0,1,0,0,0,0,1)

Appendix A.8. 10-dim MMPHs

18-9 1BC5DEFGH9,1BCKL9A,T5DEU,TPR,TbL9A,CbKP,bKG,bKFR,bKUHA.

50-15 (master for **18-9**) 12BCUVfgik,1DEJXYceoq,1DELMVWajk,1DEMNRSblm,1DEOPRTajk, 1DEOQYZajk,2GHLNXZajk,2GHPQUWblm,45EFUVcdmn,46GIJSTajk,46GIUVceoq,56ABUVdeij, 78ACUVfhpq,79HIUVabop,89DFUVghln. 1=(1,0,0,0,0,0,0,0,0,0), 2=(0,1,0,0,0,0,0,0,0,0), 3=(0,0,1,0,0,0,0,0,0,0), 4=(1,1,1,1,0,0,0,0,0,0), 5=(1,-1,1,-1,0,0,0,0,0,0), 6=(1,-1,-1,1,0,0,0,0,0,0), 7=(1,-1,-1,-1,0,0,0,0,0,0), 8=(1,-1,1,1,0,0,0,0,0,0), 9=(1,1,1,-1,0,0,0,0,0,0), A=(1,1,0,0,0,0,0,0,0,0), B=(0,0,1,1,0,0,0,0,0,0), C=(0,0,1,-1,0,0,0,0,0,0), D=(0,1,0,1,0,0,0,0,0,0), E=(0,1,0,-1,0,0,0,0,0,0), F=(1,0,-1,0,0,0,0,0,0,0), G=(1,0,0,-1,0,0,0,0,0,0), H=(1,0,0,1,0,0,0,0,0,0), I=(0,1,-1,0,0,0,0,0,0,0), J=(0,0,0,0,1,0,0,0,0,0), K=(0,0,0,0,0,1,0,0,0,0), L=(0,0,1,0,1,1,1,0,0,0), M=(0,0,1,0,-1,1,-1,0,0,0), N=(0,0,1,0,-1,-1,1,0,0,0), O=(0,0,1,0,-1,-1,-1,0,0,0), P=(0,0,1,0,-1,1,1,0,0,0), Q=(0,0,1,0,1,1,-1,0,0,0), R=(0,0,1,0,1,0,0,0,0,0), S=(0,0,0,0,0,1,1,0,0,0), T=(0,0,0,0,0,1,-1,0,0,0), U=(0,0,0,0,1,0,1,0,0,0), V=(0,0,0,0,1,0,-1,0,0,0), W=(0,0,1,0,0,-1,0,0,0,0), X=(0,0,1,0,0,0,-1,0,0,0), Y=(0,0,1,0,0,0,1,0,0,0), Z=(0,0,0,0,1,-1,0,0,0,0), a=(0,0,0,0,0,0,0,1,0,0), b=(0,0,0,0,0,0,0,0,1,0), c=(0,0,0,0,0,1,0,1,1,1), d=(0,0,0,0,0,1,0,-1,1,-1), e=(0,0,0,0,0,1,0,-1,-1,1), f=(0,0,0,0,0,1,0,-1,-1,-1), g=(0,0,0,0,0,1,0,-1,1,1), h=(0,0,0,0,0,1,0,1,1,-1), i=(0,0,0,0,0,1,0,1,0,0), j=(0,0,0,0,0,0,0,0,1,1), k=(0,0,0,0,0,0,0,0,1,-1), l=(0,0,0,0,0,0,0,1,0,1), m=(0,0,0,0,0,0,0,1,0,-1), n=(0,0,0,0,0,1,0,0,-1,0), o=(0,0,0,0,0,1,0,0,0,-1), p=(0,0,0,0,0,1,0,0,0,1), q=(0,0,0,0,0,0,0,1,-1,0)

Appendix A.9. 11-dim MMPHs

19-8 123456789AB,1234567CDF,1GHKLMDA,27KL9,567MC,567B,H8F,G8F.

50-14 (master for **19-8**) 123456789AB,1234567CDEF,1GHIJKLMDAN,1GHIJKLOPQR,27STUVKL8FW, 27STUVKL9QX,347YZabMDAN,567cdefMCXR,567cdefgOEW,567cdefgPBN,cdhijkJV8FW,eYlmnIUk9QX, fZoHTjmn8FW,abGShilo8FW. 1=(0,0,1,1,1,1,0,0,0,0,0), 2=(0,0,1,-1,1,-1,0,0,0,0,0), 3=(0,0,0,1,0,-1,0,0,0,0,0), 4=(0,0,1,0,-1,0,0,0,0,0,0), 5=(0,1,0,0,0,0,0,0,0,0,0), 6=(1,0,0,0,0,0,0,0,0,0,0), 7=(0,0,0,0,0,0,1,0,0,0,0), 8=(0,0,0,1,0,0,0,0,0,0,0), 9=(0,0,1,0,0,0,0,0,0,0,0), A=(0,0,0,0,0,1,0,0,0,0,0), B=(0,0,0,0,1,0,0,0,0,0,0), C=(1,-1,1,0,1,0,0,0,0,0,0), D=(1,1,0,1,0,1,0,0,0,0,0), E=(1,1,0,-1,0,-1,0,0,0,0,0), F=(-1,1,1,0,1,0,0,0,0,0,0), G=(0,1,-1,1,0,0,1,0,0,0,0), H=(1,0,1,1,0,0,0,-1,0,0,0), I=(1,0,0,0,1,1,0,1,0,0,0), J=(0,1,0,0,-1,1,-1,0,0,0,0), K=(0,0,1,0,-1,0,1,1,0,0,0), L=(0,0,0,1,0,-1,-1,1,0,0,0), M=(1,0,1,0,0,-1,1,0,0,0,0), N=(0,-1,1,0,0,1,0,1,0,0,0), O=(-1,1,0,0,0,0,1,1,0,0,0), P=(1,0,-1,-1,0,0,0,1,0,0,0), Q=(0,1,1,-1,0,0,-1,0,0,0,0), R=(1,0,0,1,-1,0,-1,0,0,0,0), S=(0,1,0,1,1,0,0,1,0,0,0), T=(1,1,0,0,0,0,1,-1,0,0,0), U=(0,1,0,0,1,-1,-1,0,0,0,0), V=(1,0,0,0,-1,-1,0,1,0,0,0), W=(1,1,0,-1,0,1,0,0,0,0,0), X=(1,-1,-1,0,1,0,0,0,0,0,0), Y=(0,0,0,0,0,0,0,0,1,0,0), Z=(0,0,0,0,0,0,0,0,0,1,0), a=(0,0,0,0,0,0,0,1,1,1,1), b=(0,0,0,0,0,0,0,1,-1,1,-1), c=(0,0,0,0,0,0,0,1,-1,-1,1), d=(0,0,0,0,0,0,0,1,-1,-1,-1), e=(0,0,0,0,0,0,0,1,-1,1,1), f=(0,0,0,0,0,0,0,1,1,1,-1), g=(0,0,0,0,0,0,0,1,1,0,0), h=(0,0,0,0,0,0,0,0,0,1,1), i=(0,0,0,0,0,0,0,0,0,1,-1), j=(0,0,0,0,0,0,0,0,1,0,1), k=(0,0,0,0,0,0,0,0,1,0,-1), l=(0,0,0,0,0,0,0,1,0,-1,0), m=(0,0,0,0,0,0,0,1,0,0,-1), n=(0,0,0,0,0,0,0,1,0,0,1), o=(0,0,0,0,0,0,0,0,1,-1,0)

Appendix A.10. 12-dim MMPHs

19-9 123456789ABC,17DJBL,28PQA,PJ,3478ST,5678Q,SC,T9,DL

52-9 (master for **19-9**) 123456789ABC,17DEFGHIJBKL,28MNOPHIQAKR,3478STUVWXYR, 5678ZabcdWQe,ZafghiGPjJke,bSlmnFOijdoC,cTpENhmn9qYo,UVDMfglpkXqL. 1=(0,0,1,1,1,1,0,0,0,0,0,0), 2=(0,0,1,-1,1,-1,0,0,0,0,0,0), 3=(0,0,0,1,0,-1,0,0,0,0,0,0), 4=(0,0,1,0,-1,0,0,0,0,0,0,0), 5=(0,1,0,0,0,0,0,0,0,0,0,0), 6=(1,0,0,0,0,0,0,0,0,0,0,0), 7=(0,0,0,0,0,0,0,1,0,0,0,0), 8=(0,0,0,0,0,0,1,0,0,0,0,0), Z=(0,0,0,1,0,0,0,0,0,0,0,0), a=(0,0,1,0,0,0,0,0,0,0,0,0), b=(0,0,0,0,0,1,0,0,0,0,0,0), c=(0,0,0,0,1,0,0,0,0,0,0,0), S=(1,-1,1,0,1,0,0,0,0,0,0,0), T=(1,1,0,1,0,1,0,0,0,0,0,0), U=(1,1,0,-1,0,-1,0,0,0,0,0,0), V=(-1,1,1,0,1,0,0,0,0,0,0,0), D=(0,1,-1,1,0,0,1,0,0,0,0,0), M=(1,0,1,1,0,0,0,-1,0,0,0,0), f=(1,0,0,0,1,1,0,1,0,0,0,0), g=(0,1,0,0,-1,1,-1,0,0,0,0,0), l=(0,0,1,0,-1,0,1,1,0,0,0,0), p=(0,0,0,1,0,-1,-1,1,0,0,0,0), E=(1,0,1,0,0,-1,1,0,0,0,0,0), N=(0,-1,1,0,0,1,0,1,0,0,0,0), h=(-1,1,0,0,0,0,1,1,0,0,0,0), m=(1,0,-1,-1,0,0,0,1,0,0,0,0), n=(0,1,1,-1,0,0,-1,0,0,0,0,0), F=(1,0,0,1,-1,0,-1,0,0,0,0,0), O=(0,1,0,1,1,0,0,1,0,0,0,0), i=(1,1,0,0,0,0,1,-1,0,0,0,0), G=(0,1,0,0,1,-1,-1,0,0,0,0,0), P=(1,0,0,0,-1,-1,0,1,0,0,0,0), H=(1,1,0,-1,0,1,0,0,0,0,0,0), I=(1,-1,-1,0,1,0,0,0,0,0,0,0), j=(0,0,0,0,0,0,0,0,1,0,0,0), d=(0,0,0,0,0,0,0,0,0,1,0,0), J=(0,0,0,0,0,0,0,0,0,1,1,0), k=(0,0,0,0,0,0,0,0,0,1,-1,0), W=(0,0,0,0,0,0,0,0,1,0,1,0), Q=(0,0,0,0,0,0,0,0,1,0,-1,0), 9=(0,0,0,0,0,0,0,0,1,-1,0,0), e=(0,0,0,0,0,0,0,0,0,0,0,1), A=(0,0,0,0,0,0,0,0,1,1,1,1), B=(0,0,0,0,0,0,0,0,1,1,-1,-1), K=(0,0,0,0,0,0,0,0,1,-1,1,-1), X=(0,0,0,0,0,0,0,0,1,-1,-1,-1), q=(0,0,0,0,0,0,0,0,1,1,1,-1), Y=(0,0,0,0,0,0,0,0,1,1,-1,1), L=(0,0,0,0,0,0,0,0,1,0,0,1), o=(0,0,0,0,0,0,0,0,0,0,1,1), C=(0,0,0,0,0,0,0,0,0,0,1,-1), R=(0,0,0,0,0,0,0,0,0,1,0,-1)

Appendix A.11. 13-dim MMPHs

19-8 123456789ABCD,123456789EFG,12345678ILM,289EBM,34789C,56789LG,5678ABM,9FD.

63-16 (master for **19-8**) 123456789ABCD,123456789EFGH,12345678IJKLM,17NOPQRSTUVWM, 28XYZaRS9EbBM,3478cdef9bghH,3478cdef9WhiC,3478cdefIjVkM,5678lmno9LpGq,5678lmnoATrBM, lmstuvQa9WpFD,lmstuvQaEKwxM,ncyz!PZv9AkLM,od"OYuz!9kgiq,od"OYuz!bUjxM,efNXsty"rJwWM. 1=(0,0,1,1,1,1,0,0,0,0,0,0,0), 2=(0,0,1,-1,1,-1,0,0,0,0,0,0,0), 3=(0,0,0,1,0,-1,0,0,0,0,0,0,0), 4=(0,0,1,0,-1,0,0,0,0,0,0,0,0), 5=(0,1,0,0,0,0,0,0,0,0,0,0,0), 6=(1,0,0,0,0,0,0,0,0,0,0,0,0), 7=(0,0,0,0,0,0,0,1,0,0,0,0,0), 8=(0,0,0,0,0,0,1,0,0,0,0,0,0), l=(0,0,0,1,0,0,0,0,0,0,0,0,0), m=(0,0,1,0,0,0,0,0,0,0,0,0,0), n=(0,0,0,0,0,1,0,0,0,0,0,0,0), o=(0,0,0,0,1,0,0,0,0,0,0,0,0), c=(1,-1,1,0,1,0,0,0,0,0,0,0,0), d=(1,1,0,1,0,1,0,0,0,0,0,0,0), e=(1,1,0,-1,0,-1,0,0,0,0,0,0,0), f=(-1,1,1,0,1,0,0,0,0,0,0,0,0), N=(0,1,-1,1,0,0,1,0,0,0,0,0,0), X=(1,0,1,1,0,0,0,-1,0,0,0,0,0), s=(1,0,0,0,1,1,0,1,0,0,0,0,0), t=(0,1,0,0,-1,1,-1,0,0,0,0,0,0), y=(0,0,1,0,-1,0,1,1,0,0,0,0,0), "=(0,0,0,1,0,-1,-1,1,0,0,0,0,0), O=(1,0,1,0,0,-1,1,0,0,0,0,0,0), Y=(0,-1,1,0,0,1,0,1,0,0,0,0,0),

u=(-1,1,0,0,0,0,1,1,0,0,0,0,0), z=(1,0,-1,-1,0,0,0,1,0,0,0,0,0), !=(0,1,1,-1,0,0,-1,0,0,0,0,0,0),
P=(1,0,0,1,-1,0,-1,0,0,0,0,0,0), Z=(0,1,0,1,1,0,0,1,0,0,0,0,0), v=(1,1,0,0,0,0,1,-1,0,0,0,0,0),
Q=(0,1,0,0,1,-1,-1,0,0,0,0,0,0), a=(1,0,0,0,-1,-1,0,1,0,0,0,0,0), R=(1,1,0,-1,0,1,0,0,0,0,0,0,0),
S=(1,-1,-1,0,1,0,0,0,0,0,0,0,0), 9=(0,0,0,0,0,0,0,0,1,0,0,0,0), A=(0,0,0,0,0,0,0,0,0,1,0,0,0),
E=(0,0,0,0,0,0,0,0,0,1,1,0,0), b=(0,0,0,0,0,0,0,0,0,1,-1,0,0), T=(0,0,0,0,0,0,0,0,1,0,1,0,0),
r=(0,0,0,0,0,0,0,0,1,0,-1,0,0), I=(0,0,0,0,0,0,0,0,1,-1,0,0,0), B=(0,0,0,0,0,0,0,0,0,0,0,1,0),
J=(0,0,0,0,0,0,0,0,1,1,1,1,0), K=(0,0,0,0,0,0,0,0,1,1,-1,-1,0), w=(0,0,0,0,0,0,0,0,1,-1,1,-1,0),
U=(0,0,0,0,0,0,0,0,1,-1,-1,-1,0), j=(0,0,0,0,0,0,0,0,1,1,1,-1,0), V=(0,0,0,0,0,0,0,0,1,1,-1,1,0),
x=(0,0,0,0,0,0,0,0,1,0,0,1,0), k=(0,0,0,0,0,0,0,0,0,0,1,1,0), L=(0,0,0,0,0,0,0,0,0,0,1,-1,0),
W=(0,0,0,0,0,0,0,0,0,1,0,-1,0), M=(0,0,0,0,0,0,0,0,0,0,0,0,1), p=(0,0,0,0,0,0,0,0,0,1,1,1,1),
F=(0,0,0,0,0,0,0,0,0,1,-1,1,-1), G=(0,0,0,0,0,0,0,0,0,1,-1,-1,1), g=(0,0,0,0,0,0,0,0,0,1,1,-1,1),
h=(0,0,0,0,0,0,0,0,0,1,1,1,-1), i=(0,0,0,0,0,0,0,0,0,1,-1,1,1), H=(0,0,0,0,0,0,0,0,0,0,0,1,1),
C=(0,0,0,0,0,0,0,0,0,0,1,0,1), D=(0,0,0,0,0,0,0,0,0,0,1,0,-1), q=(0,0,0,0,0,0,0,0,0,1,0,0,-1)

Appendix A.12. 14-dim MMPHs

19-9 123456789ABCDE,12345679ABFGD,1OPF,27a,347E,3479ABP,567CG,9a,O89AB.

66-15 (master for **19-9**) 123456789ABCDE,12345679ABFGDH,1IJKLMNOPFQRST,27UVWXMNYZTabc, 347defghijkElm,347defg9ABPHnm,347defgFijkGDH,567opqrOBPFsZt,567opqr9ABaunl, 567opqrijbkCGu,opvwxyLX9iQYab,qdz!"KWyAt#Sab,re$JVx!"s%abck,fgIUvwz$Oh89AB, fgIUvwz$#R%jab. 1=(0,0,1,1,1,1,0,0,0,0,0,0,0,0), 2=(0,0,1,-1,1,-1,0,0,0,0,0,0,0,0),
3=(0,0,0,1,0,-1,0,0,0,0,0,0,0,0), 4=(0,0,1,0,-1,0,0,0,0,0,0,0,0,0), 5=(0,1,0,0,0,0,0,0,0,0,0,0,0,0),
6=(1,0,0,0,0,0,0,0,0,0,0,0,0,0), 7=(0,0,0,0,0,0,1,0,0,0,0,0,0,0), o=(0,0,0,1,0,0,0,0,0,0,0,0,0,0),
p=(0,0,1,0,0,0,0,0,0,0,0,0,0,0), q=(0,0,0,0,0,1,0,0,0,0,0,0,0,0), r=(0,0,0,0,1,0,0,0,0,0,0,0,0,0),
d=(1,-1,1,0,1,0,0,0,0,0,0,0,0,0), e=(1,1,0,1,0,1,0,0,0,0,0,0,0,0), f=(1,1,0,-1,0,-1,0,0,0,0,0,0,0,0),
g=(-1,1,1,0,1,0,0,0,0,0,0,0,0,0), I=(0,1,-1,1,0,0,1,0,0,0,0,0,0,0), U=(1,0,1,1,0,0,0,-1,0,0,0,0,0,0),
v=(1,0,0,0,1,1,0,1,0,0,0,0,0,0), w=(0,1,0,0,-1,1,-1,0,0,0,0,0,0,0), z=(0,0,1,0,-1,0,1,1,0,0,0,0,0,0),
$=(0,0,0,1,0,-1,-1,1,0,0,0,0,0,0), J=(1,0,1,0,0,-1,1,0,0,0,0,0,0,0), V=(0,-1,1,0,0,1,0,1,0,0,0,0,0,0),
x=(-1,1,0,0,0,0,1,1,0,0,0,0,0,0), !=(1,0,-1,-1,0,0,0,1,0,0,0,0,0,0), "=(0,1,1,-1,0,0,-1,0,0,0,0,0,0,0),
K=(1,0,0,1,-1,0,-1,0,0,0,0,0,0,0), W=(0,1,0,1,1,0,0,1,0,0,0,0,0,0), y=(1,1,0,0,0,0,1,-1,0,0,0,0,0,0),
L=(0,1,0,0,1,-1,-1,0,0,0,0,0,0,0), X=(1,0,0,0,-1,-1,0,1,0,0,0,0,0,0), M=(1,1,0,-1,0,1,0,0,0,0,0,0,0,0),
N=(1,-1,-1,0,1,0,0,0,0,0,0,0,0,0), O=(0,0,0,0,0,0,0,0,0,0,1,0,0,0), h=(0,0,0,0,0,0,0,0,0,1,-1,0,0,0,0),
8=(0,0,0,0,0,0,0,0,0,1,1,0,0,0,0), 9=(0,0,0,0,0,0,0,0,0,0,0,0,0,1), A=(0,0,0,0,0,0,0,0,0,0,0,0,1,1,0),
B=(0,0,0,0,0,0,0,0,0,0,0,0,1,-1,0), P=(0,0,0,0,0,0,0,0,1,0,-1,0,0,0,0), F=(0,0,0,0,0,0,0,0,1,0,1,0,0,0,0),
i=(0,0,0,0,0,0,0,0,0,0,0,0,1,0,0), Q=(0,0,0,0,0,0,0,0,1,0,0,0,-1,0), Y=(0,0,0,0,0,0,0,0,1,0,0,0,1,0),
s=(0,0,0,0,0,0,0,0,1,0,0,1,1,-1), Z=(0,0,0,0,0,0,0,0,1,0,0,-1,-1,-1), t=(0,0,0,0,0,0,0,0,1,0,0,0,0,1),
#=(0,0,0,0,0,0,0,0,1,0,0,1,-1,-1), R=(0,0,0,0,0,0,0,0,1,0,0,1,1,1), %=(0,0,0,0,0,0,0,0,1,0,0,-1,0,0),
j=(0,0,0,0,0,0,0,0,0,0,0,0,0,1,-1), S=(0,0,0,0,0,0,0,0,1,0,0,-1,1,-1), T=(0,0,0,0,0,0,0,0,0,0,0,1,0,-1),
a=(0,0,0,0,0,0,0,0,0,1,1,0,0,0,0), b=(0,0,0,0,0,0,0,0,0,1,-1,0,0,0,0), c=(0,0,0,0,0,0,0,0,1,0,0,1,-1,1),
k=(0,0,0,0,0,0,0,0,0,0,0,0,0,1,1), C=(0,0,0,0,0,0,0,0,1,-1,1,1,0,0,0), G=(0,0,0,0,0,0,0,0,1,-1,-1,-1,0,0,0),
u=(0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0), D=(0,0,0,0,0,0,0,0,1,1,-1,1,0,0,0), E=(0,0,0,0,0,0,0,0,1,0,0,-1,0,0,0),
H=(0,0,0,0,0,0,0,0,0,1,0,-1,0,0,0), n=(0,0,0,0,0,0,0,0,1,-1,1,-1,0,0,0), l=(0,0,0,0,0,0,0,0,1,-1,-1,1,0,0,0),
m=(0,0,0,0,0,0,0,0,1,1,1,1,0,0,0)

Appendix A.13. 15-dim MMPHs

25-8 123456789ABCDEF,1234567GHIJKLD,1RS89ABCDEF,27TRSX9GH,347CL,347BK,347AIJ,T8X9.

66-14 (master for **25-8**) 123456789ABCDEF,1234567GHIJKLDM,1NOPQRS89ABCDEF, 27TUVWRSX9YGHZa,347bcdeZfghiCjL,347bcdeaklBmKhi,347bcdenoApIJgl,567qrstuYpmjMEF, 567qrstvw9Yfkno,qrxyz!QW8uvwX9Y,qrxyz!QWX9YGHZa,sb"#$PV!8uvwX9Y,tc%OUz#$8uvwX9Y, deNTxy"%8uvwX9Y. 1=(0,0,1,1,1,1.0,0,0,0,0,0,0,0,0), 2=(0,0,1,-1,1,-1.0,0,0,0,0,0,0,0,0),

3=(0,0,0,1,0,-1.0,0,0,0,0,0,0,0,0), 4=(0,0,1,0,-1,0.0,0,0,0,0,0,0,0,0), 5=(0,1,0,0,0,0.0,0,0,0,0,0,0,0,0),
6=(1,0,0,0,0,0.0,0,0,0,0,0,0,0,0), 7=(0,0,0,0,0,0.1,0,0,0,0,0,0,0,0), q=(0,0,0,1,0,0.0,0,0,0,0,0,0,0,0),
r=(0,0,1,0,0,0.0,0,0,0,0,0,0,0,0), s=(0,0,0,0,0,1.0,0,0,0,0,0,0,0,0), t=(0,0,0,0,1,0.0,0,0,0,0,0,0,0,0),
b=(1,-1,1,0,1,0.0,0,0,0,0,0,0,0,0), c=(1,1,0,1,0,1.0,0,0,0,0,0,0,0,0), d=(1,1,0,-1,0,-1.0,0,0,0,0,0,0,0,0),
e=(-1,1,1,0,1,0.0,0,0,0,0,0,0,0,0), N=(0,1,-1,1,0,0.1,0,0,0,0,0,0,0,0), T=(1,0,1,1,0,0.0,-1,0,0,0,0,0,0,0),
x=(1,0,0,0,1,1.0,1,0,0,0,0,0,0,0), y=(0,1,0,0,-1,1.-1,0,0,0,0,0,0,0,0), "=(0,0,1,0,-1,0.1,1,0,0,0,0,0,0,0),
%=(0,0,0,1,0,-1.-1,1,0,0,0,0,0,0,0), O=(1,0,1,0,0,-1.1,0,0,0,0,0,0,0,0), U=(0,-1,1,0,0,1.0,1,0,0,0,0,0,0,0),
z=(-1,1,0,0,0,0.1,1,0,0,0,0,0,0,0), #=(1,0,-1,-1,0,0.0,1,0,0,0,0,0,0,0), $=(0,1,1,-1,0,0.-1,0,0,0,0,0,0,0,0),
P=(1,0,0,1,-1,0.-1,0,0,0,0,0,0,0,0), V=(0,1,0,1,1,0.0,1,0,0,0,0,0,0,0), !=(1,1,0,0,0,0.1,-1,0,0,0,0,0,0,0),
Q=(0,1,0,0,1,-1.-1,0,0,0,0,0,0,0,0), W=(1,0,0,0,-1,-1.0,1,0,0,0,0,0,0,0), R=(1,1,0,-1,0,1.0,0,0,0,0,0,0,0,0),
S=(1,-1,-1,0,1,0.0,0,0,0,0,0,0,0,0), 8=(0,0,0,0,0,0.0,0,0,1,1,1,1,0,0), u=(0,0,0,0,0,0.0,0,0,1,-1,1,-1,0,0),
v=(0,0,0,0,0,0.0,0,0,0,1,0,-1,0,0), w=(0,0,0,0,0,0.0,0,0,1,0,-1,0,0,0), X=(0,0,0,0,0,0.0,0,1,0,0,0,0,0,0),
9=(0,0,0,0,0,0.0,0,0,0,0,0,0,0,1), Y=(0,0,0,0,0,0.0,0,0,0,0,0,0,1,0), G=(0,0,0,0,0,0.0,0,0,0,1,0,0,0,0),
H=(0,0,0,0,0,0.0,0,0,1,0,0,0,0,0), Z=(0,0,0,0,0,0.0,0,0,0,0,0,1,0,0), a=(0,0,0,0,0,0.0,0,0,0,0,1,0,0,0),
f=(0,0,0,0,0,0.0,1,-1,1,0,1,0,0,0), k=(0,0,0,0,0,0.0,1,1,0,1,0,1,0,0), n=(0,0,0,0,0,0.0,1,1,0,-1,0,-1,0,0),
o=(0,0,0,0,0,0.0,1,-1,-1,0,-1,0,0,0), A=(0,0,0,0,0,0.0,0,1,-1,1,0,0,1,0), p=(0,0,0,0,0,0.0,1,0,1,1,0,0,0,-1),
I=(0,0,0,0,0,0.0,1,0,0,0,1,1,0,1), J=(0,0,0,0,0,0.0,0,1,0,0,-1,1,-1,0), g=(0,0,0,0,0,0.0,0,0,1,0,-1,0,1,1),
l=(0,0,0,0,0,0.0,0,0,0,1,0,-1,-1,1), B=(0,0,0,0,0,0.0,1,0,1,0,0,-1,1,0), m=(0,0,0,0,0,0.0,0,1,-1,0,0,-1,0,-1),
K=(0,0,0,0,0,0.0,1,-1,0,0,0,0,-1,-1), h=(0,0,0,0,0,0.0,1,0,-1,-1,0,0,0,1), i=(0,0,0,0,0,0.0,0,1,1,-1,0,0,-1,0),
C=(0,0,0,0,0,0.0,1,0,0,1,-1,0,-1,0), j=(0,0,0,0,0,0.0,0,1,0,1,1,0,0,1), L=(0,0,0,0,0,0.0,1,1,0,0,0,0,1,-1),
D=(0,0,0,0,0,0.0,0,1,0,0,1,-1,-1,0), M=(0,0,0,0,0,0.0,1,0,0,0,-1,-1,0,1), E=(0,0,0,0,0,0.0,1,1,0,-1,0,1,0,0),
F=(0,0,0,0,0,0.0,1,-1,-1,0,1,0,0,0)

Appendix A.14. 16-dim MMPHs

22-9 123456789ABCDEFG,17HID,28UdG,3478efE,5678,Bd,e9,fICF,HUA.

70-9 (master for **22-9**) 123456789ABCDEFG,17HIJKLMNOPQRDST,28UVWXLMYZabScdG, 3478efghijPbkElm,5678nopqrZsQtukT,novwxyKXzasBu!dm,pe"#$JWy9iY%tR&’,qf(IVx#$zOjC)lF&, ghHUvw"(ArN%)c!’. 1=(0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0), 2=(0,0,1,-1,1,-1,0,0,0,0,0,0,0,0,0,0), 3=(0,0,0,1,0,-1,0,0,0,0,0,0,0,0,0,0), 4=(0,0,1,0,-1,0,0,0,0,0,0,0,0,0,0,0), 5=(0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0), 6=(1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0), 7=(0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0), 8=(0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0), n=(0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0), o=(0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0), p=(0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0), q=(0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0), e=(1,-1,1,0,1,0,0,0,0,0,0,0,0,0,0,0), f=(1,1,0,1,0,1,0,0,0,0,0,0,0,0,0,0), g=(1,1,0,-1,0,-1,0,0,0,0,0,0,0,0,0,0), h=(-1,1,1,0,1,0,0,0,0,0,0,0,0,0,0,0), H=(0,1,-1,1,0,0,1,0,0,0,0,0,0,0,0,0), U=(1,0,1,1,0,0,0,-1,0,0,0,0,0,0,0,0), v=(1,0,0,0,1,1,0,1,0,0,0,0,0,0,0,0), w=(0,1,0,0,-1,1,-1,0,0,0,0,0,0,0,0,0), "=(0,0,1,0,-1,0,1,1,0,0,0,0,0,0,0,0), (=(0,0,0,1,0,-1,-1,1,0,0,0,0,0,0,0,0), I=(1,0,1,0,0,-1,1,0,0,0,0,0,0,0,0,0), V=(0,-1,1,0,0,1,0,1,0,0,0,0,0,0,0,0), x=(-1,1,0,0,0,0,1,1,0,0,0,0,0,0,0,0), #=(1,0,-1,-1,0,0,0,1,0,0,0,0,0,0,0,0), $=(0,1,1,-1,0,0,-1,0,0,0,0,0,0,0,0,0), J=(1,0,0,1,-1,0,-1,0,0,0,0,0,0,0,0,0), W=(0,1,0,1,1,0,0,1,0,0,0,0,0,0,0,0), y=(1,1,0,0,0,0,1,-1,0,0,0,0,0,0,0,0), K=(0,1,0,0,1,-1,-1,0,0,0,0,0,0,0,0,0), X=(1,0,0,0,-1,-1,0,1,0,0,0,0,0,0,0,0), L=(1,1,0,-1,0,1,0,0,0,0,0,0,0,0,0,0), M=(1,-1,-1,0,1,0,0,0,0,0,0,0,0,0,0,0), 9=(0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0), A=(0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0), i=(0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0), Y=(0,0,0,0,0,0,0,0,0,1,-1,0,0,0,0,0), r=(0,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0), N=(0,0,0,0,0,0,0,0,1,0,-1,0,0,0,0,0), z=(0,0,0,0,0,0,0,0,1,-1,0,0,0,0,0,0), %=(0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0), O=(0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0), j=(0,0,0,0,0,0,0,0,1,1,-1,-1,0,0,0,0), P=(0,0,0,0,0,0,0,0,1,-1,1,-1,0,0,0,0), Z=(0,0,0,0,0,0,0,0,1,-1,-1,-1,0,0,0,0), a=(0,0,0,0,0,0,0,0,1,1,1,-1,0,0,0,0), s=(0,0,0,0,0,0,0,0,1,1,-1,1,0,0,0,0), b=(0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0), B=(0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0), C=(0,0,0,0,0,0,0,0,0,0,1,-1,0,0,0,0), Q=(0,0,0,0,0,0,0,0,0,1,0,-1,0,0,0,0), t=(0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0), R=(0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0), u=(0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0), k=(0,0,0,0,0,0,0,0,0,0,0,0,0,1,-1,0), D=(0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,0), S=(0,0,0,0,0,0,0,0,0,0,0,0,1,0,-1,0),)=(0,0,0,0,0,0,0,0,0,0,0,0,1,-1,0,0), T=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1), c=(0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1), !=(0,0,0,0,0,0,0,0,0,0,0,0,1,1,-1,-1), d=(0,0,0,0,0,0,0,0,0,0,0,0,1,-1,1,-1), E=(0,0,0,0,0,0,0,0,0,0,0,0,1,-1,-1,-1), l=(0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,-1), F=(0,0,0,0,0,0,0,0,0,0,0,0,1,1,-1,1), m=(0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1), &=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1), ’=(0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,-1), G=(0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,-1)

1. Cabello, A.; D’Ambrosio, V.; Nagali, E.; Sciarrino, F. Hybrid Ququart-Encoded Quantum Cryptography Protected by Kochen-Specker Contextuality. *Phys. Rev. A* **2011**, **84**, 030302(R)–1–4.
2. Nagata, K. Kochen-Specker Theorem as a Precondition for Secure Quantum Key Distribution. *Phys. Rev. A* **2005**, **72**, 012325–1–4.
3. Saha, D.; Horodecki, P.; Pawłowski, M. State Independent Contextuality Advances One-Way Communication. *New J. Phys.* **2019**, **21**, 093057–1–32. doi:10.1088/1367-2630/ab4149.
4. Howard, M.; Wallman, J.; Veitech, V.; Emerson, J. Contextuality Supplies the ‘Magic’ for Quantum Computation. *Nature* **2014**, **510**, 351–355. doi:10.1038/nature13460.
5. Bartlett, S.D. Powered by Magic. *Nature* **2014**, **510**, 345–346. doi:10.1038/nature13504.
6. Kurzyński, P.; Cabello, A.; Kaszlikowski, D. Fundamental Monogamy Relation between Contextuality and Nonlocality. *Phys. Rev. Lett.* **2014**, **112**, 100401–1–5. doi:10.1103/PhysRevLett.112.100401.
7. Tavakoli, A.; Uola, R. Measurement Incompatibility and Steering Are Necessary and Sufficient for Operational Contextuality. *Phys. Rev. Research* **2020**, **2**, 013011–1–7. doi:10.1103/PhysRevResearch.2.013011.
8. Pavičić, M.; McKay, B.D.; Megill, N.D.; Fresl, K. Graph Approach to Quantum Systems. *J. Math. Phys.* **2010**, **51**, 102103–1–31. doi:10.1063/1.3491766.
9. Megill, N.D.; Pavičić, M. Kochen-Specker Sets and Generalized Orthoarguesian Equations. *Ann. Henri Poinc.* **2011**, **12**, 1417–1429. doi:10.1007/s00023-011-0109-0.
10. Simon, C.; Żukowski, M.; Weinfurter, H.; Zeilinger, A. Feasible Kochen-Specker Experiment with Single Particles. *Phys. Rev. Lett.* **2000**, **85**, 1783–1786.
11. Michler, M.; Weinfurter, H.; Żukowski, M. Experiments towards Falsification of Noncontextual Hidden Variables. *Phys. Rev. Lett.* **2000**, **84**, 5457–5461.
12. Amselem, E.; Rådmark, M.; Bourennane, M.; Cabello, A. State-Independent Quantum Contextuality with Single Photons. *Phys. Rev. Lett.* **2009**, **103**, 160405–1–4. doi:10.1103/PhysRevLett.103.160405.
13. Liu, B.H.; Huang, Y.F.; Gong, Y.X.; Sun, F.W.; Zhang, Y.S.; Li, C.F.; Guo, G.C. Experimental Demonstration of Quantum Contextuality with Nonentangled Photons. *Phys. Rev. A* **2009**, **80**, 044101–1–4. doi:10.1103/PhysRevA.80.044101.
14. D’Ambrosio, V.; Herbauts, I.; Amselem, E.; Nagali, E.; Bourennane, M.; Sciarrino, F.; Cabello, A. Experimental Implementation of a Kochen-Specker Set of Quantum Tests. *Phys. Rev. X* **2013**, **3**, 011012–1–10. doi:10.1103/PhysRevX.3.011012.
15. Huang, Y.F.; Li, C.F.; Yong-Sheng Zhang, J.W.P.; Guo, G.C. Realization of All-or-nothing-type Kochen-Specker Experiment with Single Photons. *Phys. Rev. Lett.* **2002**, **88**, 240402–1–4.
16. Huang, Y.F.; Li, C.F.; Zhang, Y.S.; Pan, J.W.; Guo, G.C. Experimental Test of the Kochen-Specker Theorem with Single Photons. *Phys. Rev. Lett.* **2003**, **90**, 250401–1–4. doi:10.1103/PhysRevLett.90.250401.
17. Lapkiewicz, R.; Li, P.; Schaeff, C.; Langford, N.K.; Ramelow, S.; Wieśniak, M.; Zeilinger, A. Experimental Non-Classicality of an Indivisible Quantum System. *Nature* **2011**, **474**, 490–493.
18. Zu, C.; Wang, Y.X.; Deng, D.L.; Chang, X.Y.; Liu, K.; Hou, P.Y.; Yang, H.X.; Duan, L.M. State-Independent Experimental Test of Quantum Contextuality in an Indivisible System. *Phys. Rev. Lett.* **2012**, **109**, 150401–1–5. doi:10.1103/PhysRevLett.109.150401.
19. Cañas, G.; Etcheverry, S.; Gómez, E.S.; Saavedra, C.; Xavier, G.B.; Lima, G.; Cabello, A. Experimental Implementation of an Eight-Dimensional Kochen-Specker Set and Observation of Its Connection with the Greenberger-Horne-Zeilinger Theorem. *Phys. Rev. A* **2014**, **90**, 012119–1–8. doi:10.1103/PhysRevA.90.012119.
20. Cañas, G.; Arias, M.; Etcheverry, S.; Gómez, E.S.; Cabello, A.; Saavedra, C.; Xavier, G.B.; Lima, G. Applying the Simplest Kochen-Specker Set for Quantum Information Processing. *Phys. Rev. Lett.* **2014**, **113**, 090404–1–5. doi:10.1103/PhysRevLett.113.090404.
21. Zhan, X.; Zhang, X.; Li, J.; Zhang, Y.; Sanders, B.C.; Xue, P. Realization of the Contextuality-Nonlocality Tradeoff with a Qubit-Qutrit Photon Pair. *Phys. Rev. Lett.* **2016**, **116**, 090401.

22. Li, T.; Zeng1, Q.; Song, X.; Zhang, X. Experimental Contextuality in Classical Light. *Scientific Reports* **2017**, **7**, 44467–1–8.
23. Li, T.; Zeng, Q.; Zhang, X.; Chen, T.; Zhang, X. State-Independent Contextuality in Classical Light. *Scientific Reports* **2019**, **9**, 17038–1–12.
24. Frustaglia, D.; Baltanás, J.P.; Velázquez-Ahumada, M.C.; Fernández-Prieto, A.; Lujambio, A.; Losada, V.; Freire, M.J.; Cabello, A. Classical Physics and the Bounds of Quantum Correlations. *Phys. Rev. Lett.* **2016**, **116**, 250404–1–5.
25. Zhang, A.; Xu, H.; Xie, J.; Zhang, H.; Smith, B.J.; Kim, M.S.; Zhang, L. Experimental Test of Contextuality in Quantum and Classical Systems. *Phys. Rev. Lett.* **2004**, **122**, 080401–1–6.
26. Hasegawa, Y.; Loidl, R.; Badurek, G.; Baron, M.; Rauch, H. Quantum Contextuality in a Single-Neutron Optical Experiment. *Phys. Rev. Lett.* **2006**, **97**, 230401–1–4. doi:10.1103/PhysRevLett.97.230401.
27. Cabello, A.; Filipp, S.; Rauch, H.; Hasegawa, Y. Proposed Experiment for Testing Quantum Contextuality with Neutrons. *Phys. Rev. Lett.* **2008**, **100**, 130404–1–4.
28. Bartosik, H.; Klepp, J.; Schmitzer, C.; Sponar, S.; Cabello, A.; Rauch, H.; Hasegawa, Y. Experimental Test of Quantum Contextuality in Neutron Interferometry. *Phys. Rev. Lett.* **2009**, **103**, 040403–1–4. doi:0.1103/PhysRevLett.103.040403.
29. Kirchmair, G.; Zähringer, F.; Gerritsma, R.; Kleinmann, M.; Gühne, O.; Cabello, A.; Blatt, R.; Roos, C.F. State-Independent Experimental Test of Quantum Contextuality. *Nature* **2009**, **460**, 494–497. doi:10.1038/nature08172.
30. Moussa, O.; Ryan, C.A.; Cory, D.G.; Laflamme, R. Testing Contextuality on Quantum Ensembles with One Clean Qubit. *Phys. Rev. Lett.* **2010**, **104**, 160501–1–4. doi:10.1103/PhysRevLett.104.160501.
31. Jerger, M.; Reshitnyk, Y.; Oppliger, M.; Potočnik, A.; Mondal, M.; Wallraff, A.; Goodenough, K.; Wehner, S.; Juliusson, K.; Langford, N.K.; Fedorov, A. Contextuality without Nonlocality in a Superconducting Quantum System. *Nature Commun.* **2016**, **7**, 12930–1–6.
32. Kochen, S.; Specker, E.P. The problem of hidden variables in quantum mechanics. *J. Math. Mech.* **1967**, **17**, 59–87.
33. Pavičić, M.; Megill, N.D. Automated Generation of Arbitrarily Many Kochen-Specker and Other Contextual Sets in Odd Dimensional Hilbert Spaces. *Phys. Rev. A* **2022**, **106**, L060203–1–5. doi:10.1103/PhysRevA.106.L060203.
34. Pavičić, M. Quantum Contextuality. *Quantum* **2023**, **7**, 953–1–68. doi:10.22331/q-2023-03-17-953.
35. Klyachko, A.A.; Can, M.A.; Binicioğlu, S.; Shumovsky, A.S. Simple Test for Hidden Variables in Spin-1 Systems. *Phys. Rev. Lett.* **2008**, **101**, 020403–1–4. doi:10.1103/PhysRevLett.101.020403.
36. Yu, S.; Oh, C.H. State-Independent Proof of Kochen-Specker Theorem with 13 Rays. *Phys. Rev. Lett.* **2012**, **108**, 030402–1–5. doi:10.1103/PhysRevLett.108.030402.
37. Bengtsson, I.; Blanchfield, K.; Cabello, A. A Kochen–Specker Inequality from a SIC. *Phys. Lett. A* **2012**, **376**, 374–376. doi:10.1016/j.physleta.2011.12.011.
38. Xu, Z.P.; Chen, J.L.; Su, H.Y. State-independent contextuality sets for a qutrit. *Phys. Lett. A* **2015**, **379**, 1868–1870. doi:10.1016/j.physleta.2015.04.024.
39. Ramanathan, R.; Horodecki, P. Necessary and Sufficient Condition for State-Independent Contextual Measurement Scenarios. *Phys. Rev. Lett.* **2014**, **112**, 040404–1–5. doi:10.1103/PhysRevLett.112.040404.
40. Cabello, A.; Kleinmann, M.; Budroni, C. Necessary and Sufficient Condition for Quantum State-Independent Contextuality. *Phys. Rev. Lett.* **2014**, **114**, 250402–1–5. doi:10.1103/PhysRevLett.114.250402.
41. Pavičić, M. Hypergraph Contextuality. *Entropy* **2019**, **21***(11)*, 1107–1–20. doi:10.3390/e21111107.
42. Cabello, A.; Portillo, J.R.; Solís, A.; Svozil, K. Minimal True-Implies-False and True-Implies-True Sets of Propositions in Noncontextual Hidden-Variable Theories. *Phys. Rev. A* **2018**, **98**, 012106–1–8. doi:10.1103/PhysRevA.98.012106.
43. Svozil, K. Extensions of Hardy-Type True-Implies-False Gadgets to Classically Obtain Indistinguishability. *Phys. Rev. A* **2021**, **103**, 022204–1–13. doi:10.1103/PhysRevA.103.022204.
44. Pavičić, M.; Megill, N.D. Vector Generation of Quantum Contextual Sets in Even Dimensional Hilbert Spaces. *Entropy* **2018**, **20**, 928–1–12. doi:10.3390/e20120928.
45. Budroni, C.; Cabello, A.; Gühne, O.; Kleinmann, M.; Larsson, J.Å. Kochen-Specker Contextuality. *Rev. Mod. Phys.* **2022**, **94**, 0450007–1–62, [arXiv:2102.13036]. doi:10.1103/RevModPhys.94.045007.

46. Zimba, J.; Penrose, R. On Bell Non-Locality without Probabilities: More Curious Geometry. *Stud. Hist. Phil. Sci.* **1993**, **24**, 697–720. doi:10.1016/0039-3681(93)90061-N Get.
47. McKay, B.D.; Megill, N.D.; Pavičić, M. Algorithms for Greechie Diagrams. *Int. J. Theor. Phys.* **2000**, **39**, 2381–2406.
48. Pavičić, M.; Merlet, J.P.; McKay, B.D.; Megill, N.D. CORRIGENDUM Kochen-Specker Vectors. *J. Phys. A* **2005**, **38**, 3709. doi:10.1088/0305-4470/38/16/C01.
49. Pavičić, M.; Megill, N.D.; Merlet, J.P. New Kochen-Specker Sets in Four Dimensions. *Phys. Lett. A* **2010**, **374**, 2122–2128. doi:10.1016/j.physleta.2010.03.019.
50. Megill, N.D.; Fresl, K.; Waegell, M.; Aravind, P.K.; Pavičić, M. Probabilistic Generation of Quantum Contextual Sets; Supplementary Material. *Phys. Lett. A* **2011**, **375**, 3419–3424. *Supplementary Material*.
51. Pavičić, M.; Megill, N.D.; Aravind, P.K.; Waegell, M. New Class of 4-Dim Kochen-Specker Sets. *J. Math. Phys.* **2011**, **52**, 022104–1–9. doi:10.1063/1.3549586.
52. Pavičić, M. Arbitrarily Exhaustive Hypergraph Generation of 4-, 6-, 8-, 16-, and 32-Dimensional Quantum Contextual Sets. *Phys. Rev. A* **2017**, **95**, 062121–1–25. doi:10.1103/PhysRevA.95.062121.
53. Pavičić, M.; Waegel, M.; Megill, N.D.; Aravind, P. Automated Generation of Kochen-Specker Sets. *Scientific Reports* **2019**, **9**, 6765–1–11. doi:10.1038/s41598-019-43009-9.
54. Lisoněk, P.; Badziąg, P.; Portillo, J.R.; Cabello, A. Kochen-Specker Set with Seven Contexts. *Phys. Rev. A* **2014**, **89**, 042101–1–7. doi:10.1103/PhysRevA.89.042101.
55. Pavičić, M.; Merlet, J.P.; McKay, B.D.; Megill, N.D. Kochen-Specker Vectors. *J. Phys. A* **2005**, **38**, 1577–1592. doi:10.1088/0305-4470/38/7/013.
56. Waegell, M.; Aravind, P.K. Minimal Complexity of Kochen-Specker Sets Does not Scale with Dimension. *Phys. Rev. A* **2017**, **95**, 050101. Supplemental Material https://journals.aps.org/pra/supplemental/10.1103/PhysRevA.95.050101, doi:10.1103/PhysRevA.95.050101.
57. Pavičić, M.; Waegel, M. Kochen-Specker Contextuality. *ArXiv* **2023**, **2305.08267 [quant-ph]**, 1–12. doi:10.48550/arXiv.2305.08267.
58. Pavičić, M.; Megill, N.D. Quantum Logic and Quantum Computation. In *Handbook of Quantum Logic and Quantum Structures*; Engesser, K.; Gabbay, D.; Lehmann, D., Eds.; Elsevier: Amsterdam, 2007; Vol. *Quantum Structures*, pp. 751–787.
59. Megill, N.D.; Fresl, K.; Waegell, M.; Aravind, P.K.; Pavičić, M. Probabilistic Generation of Quantum Contextual Sets. *Phys. Lett. A* **2011**, **375**, 3419–3424.
60. Megill, N.D.; Pavičić, M. New Classes of Kochen-Specker Contextual Sets (Invited Talk). Proceedings of the 40th MIPRO Convention held on 22-26 May 2017. IEEE, 2017, IEEE Xplore Digital Library, pp. 195–200. ISBN: 9781509049691.