Bicategorical Yoneda lemma

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Abstract

We review the Yoneda lemma for bicategories and its connection to 2-descent and some universal constructions.

1 The 2-category of 2-presheaves

Definition 1.1. *Let $\mathcal{B}$ be a bicategory. A 2-presheaf P on a bicategory $\mathcal{B}$ is a homomorphism of bicategories $P\colon \mathcal{B}^{op} \to \mathrm{Cat}$.*

Let $P\colon \mathcal{B}^{op} \to \mathrm{Cat}$ be the 2-presheaf. It takes an object $B \in Ob(\mathcal{B})$ to the category $P(B)$, a 1-morphism $f\colon A \to B$ to the functor $P(f)\colon P(B) \to P(A)$, and any 2-morphism $\psi\colon f \to g$ to the natural transformation $P(\psi)\colon P(f) \to P(g)$. We also have natural transformations $\mu_{g,f}\colon P(f) \circ P(g) \to P(g \circ f)$ and $\eta_A\colon i_{P(A)} \to P(i_A)$, which satisfy following coherence laws

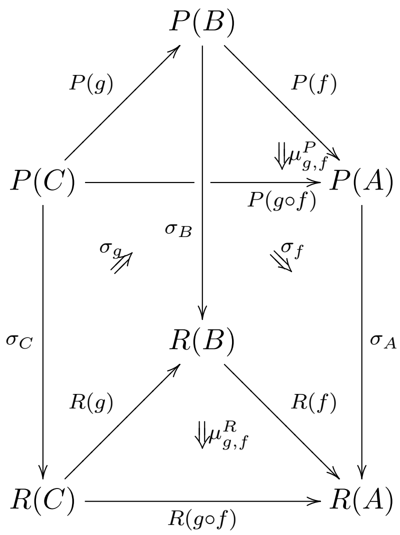
$$\begin{array}{ccccc}
(P(f) \circ P(g)) \circ P(h) & \xrightarrow{\mu_{g,f} \circ P(h)} & P(g \circ f) \circ P(h) & \xrightarrow{\mu_{h,g\circ f}} & P(h \circ (g \circ f)) \\
\Vert & & & & \uparrow{\scriptstyle P(a_{h,g,f})} \\
P(f) \circ (P(g) \circ P(h)) & \xrightarrow[P(f)\circ\mu_{h,g}]{} & P(f) \circ P(h \circ g) & \xrightarrow[\mu_{h\circ g,f}]{} & P((h \circ g) \circ f)
\end{array}$$

$$\begin{array}{ccc}
P(f) \circ P(i_B) & \xrightarrow{\mu_{i_B,f}} & P(i_B \circ f) \\
{\scriptstyle P(f)\circ\eta_B}\uparrow & & \downarrow{\scriptstyle P(\lambda_f)} \\
P(f) & = & P(f)
\end{array}
\qquad
\begin{array}{ccc}
P(i_A) \circ P(f) & \xrightarrow{\mu_{f,i_A}} & P(f \circ i_A) \\
{\scriptstyle \eta_A\circ P(f)}\uparrow & & \downarrow{\scriptstyle P(\rho_f)} \\
P(f) & = & P(f)
\end{array}$$

Definition 1.2. *Let $P, R\colon \mathcal{B}^{op} \to \mathrm{Cat}$ be two 2-presheaves on $\mathcal{B}$. A (left) pseudo natural transformation $\sigma\colon P \Rightarrow R$ consists of the functors $\sigma_A\colon P(A) \to R(A)$ for each object $A \in \mathcal{B}$ and is given by the square*

$$\begin{array}{ccc} P(B) & \xrightarrow{\sigma_B} & R(B) \\ {\scriptstyle P(f)}\downarrow & \swarrow\!\!\!\swarrow \sigma_f & \downarrow{\scriptstyle R(f)} \\ P(A) & \xrightarrow[\sigma_A]{} & R(A) \end{array}$$

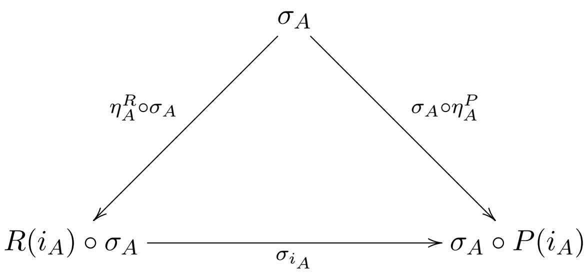
which represents a natural transformation $\sigma_f\colon R(f) \circ \sigma_B \Rightarrow \sigma_A \circ P(f)$ for any 1-morphism $f\colon A \to B$ in $\mathcal{B}$. These natural transformations satisfy the coherence law given by the diagram



which becomes a commutative diagram

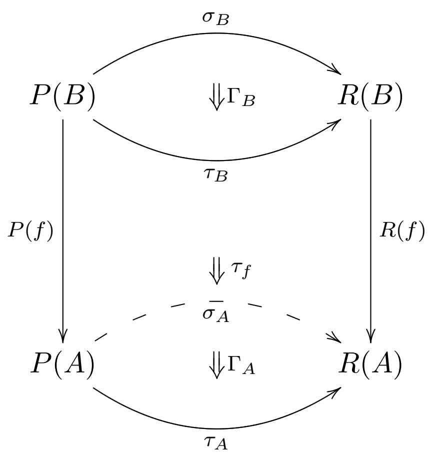
$$\begin{array}{ccc} R(f) \circ R(g) \circ \sigma_C & \xrightarrow{R(f)\circ\sigma_g} & R(f) \circ \sigma_B \circ P(g) \\ {\scriptstyle \mu^R_{g,f}\circ\sigma_C}\downarrow & & \\ R(g \circ f) \circ \sigma_C & & \downarrow{\scriptstyle \sigma_f \circ P(g)} \\ {\scriptstyle \sigma_{g\circ f}}\downarrow & & \\ \sigma_A \circ P(g \circ f) & \xleftarrow[\sigma_z\circ\mu^P_{g,f}]{} & \sigma_A \circ P(f) \circ P(g) \end{array}$$

for any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ *of 1-morphisms in* $\mathcal{B}$*. The second coherence is given by the commutative diagram*



of natural transformations.

Definition 1.3. *A modification* $\Gamma\colon \sigma \to \tau$ *consists of the natural transformation* $\Gamma_A\colon \sigma_A \to \tau_A$ *for each object* A *in* $\mathcal{B}$ *such that the following diagram*



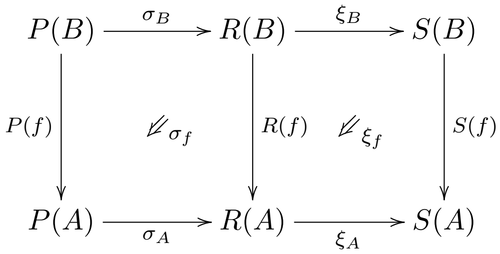
commutes. This means that the above diagram becomes a commutative diagram

$$\begin{array}{ccc} R(f)\circ\sigma_B & \xrightarrow{R(f)\circ\Gamma_B} & R(f)\circ\tau_B \\ \Big\downarrow{\scriptstyle \sigma_f} & & \Big\downarrow{\scriptstyle \tau_f} \\ \sigma_A\circ P(f) & \xrightarrow[\Gamma_A\circ P(f)]{} & \tau_A\circ P(f) \end{array}$$

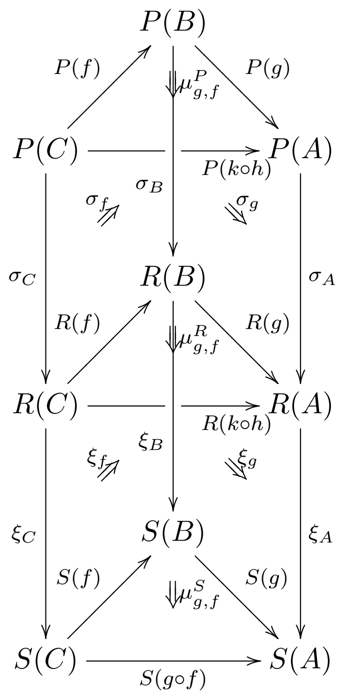
of 2-morphisms in $\mathcal{B}$*.*

Proposition 1.1. *Let $\mathcal{B}$ be a bicategory. Then 2-preshevaes are the objects of the 2-category $\mathcal{P} := Hom_{Bicat}(\mathcal{B}^{op}, \mathrm{Cat})$ in which 1-morphisms are pseudo natural transformation and 2-morphisms are modifications.*

Proof. For any two pseudo natural transformations $P \overset{\sigma}{\Longrightarrow} R \overset{\xi}{\Longrightarrow} S$ the composition is defined by $(\xi \circ \sigma)_A := \xi_A \circ \sigma_A$ and $(\xi \circ \sigma)_f := (\xi_A \circ \sigma_f)(\xi_f \circ \sigma_B)$ which is just the pasting



composite of the above diagram. That this is well defined composition we deduce from the pasting composite



which gives a coherence for a pseudo natural transformation. This composition is associative since for any three pseudo natural transformations $P \overset{\sigma}{\Longrightarrow} R \overset{\xi}{\Longrightarrow} S \overset{\omega}{\Longrightarrow} T$ we have the associativity

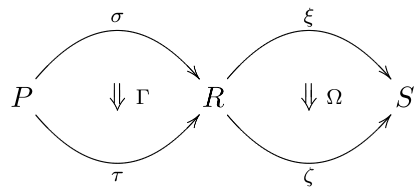
$$[(\omega \circ \xi) \circ \sigma]_A = (\omega \circ \xi)_A \circ \sigma_A = (\omega_A \circ \xi_A) \circ \sigma_A = \omega_A \circ (\xi_A \circ \sigma_A) = \omega_A \circ (\xi \circ \sigma)_A = [\omega \circ (\xi \circ \sigma)]_A$$

for components indexed by the object B in $\mathcal{B}$, and

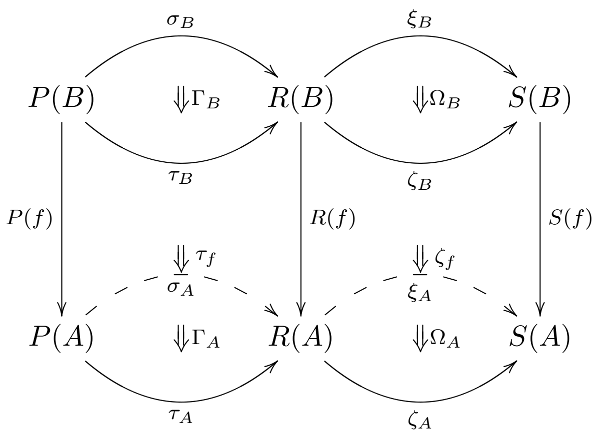
$$[(\omega \circ \xi) \circ \sigma]_f = [(\omega \circ \xi)_A \circ \sigma_f][(\omega \circ \xi)_f \circ \sigma_B] = [(\omega_A \circ \xi_A) \circ \sigma_f][((\omega_A \circ \xi_f)(\omega_f \circ \xi_B)) \circ \sigma_B] =$$
$$= [(\omega_A \circ \xi_A) \circ \sigma_f][(\omega_A \circ \xi_f) \circ \sigma_B][(\omega_f \circ \xi_B) \circ \sigma_B] = [\omega_A \circ (\xi_A \circ \sigma_f)][\omega_A \circ (\xi_f \circ \sigma_B)][\omega_f \circ (\xi_B \circ \sigma_B)] =$$
$$= [(\omega_A \circ (\xi_A \circ \sigma_f)(\xi_f \circ \sigma_B)][(\omega_f \circ (\xi_B \circ \sigma_B))] = [\omega_A \circ (\xi \circ \sigma)_f][\omega_f \circ (\xi \circ \sigma)_B] = [\omega \circ (\xi \circ \sigma)]_f$$

for components indexed by the morphism $f\colon A \to B$ in $\mathcal{B}$.

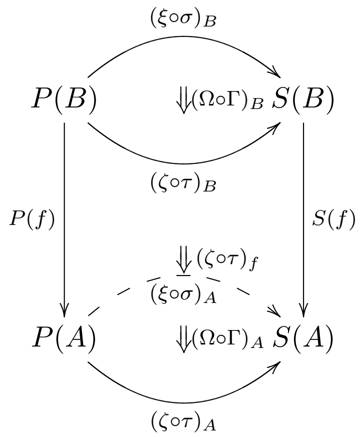
The vertical composition for any two modifications $\sigma \overset{\Gamma}{\Longrightarrow} \tau \overset{\Pi}{\Longrightarrow} \pi$ is defined simply by $(\Pi\Gamma)_A := \Pi_A \Gamma_A$. The horizontal composition for any two modifications



is defined by $(\Omega \circ \Gamma)_A := \Omega_B \circ \Gamma_B$. We need to show that pasting of the diagram



gives the coherence diagram for the composite modification $\Omega \circ \Gamma \colon \xi \circ \sigma \to \zeta \circ \tau$.

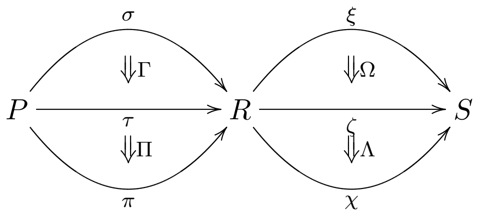


The commutativity of the above diagram is deduced from the commutative diagram

$$
\begin{array}{ccccc}
S(f)\circ(\xi_B\circ\sigma_B) & \xrightarrow{S(f)\circ(\xi_B\circ\Gamma_B)} & S(f)\circ(\xi_B\circ\tau_B) & \xrightarrow{S(f)\circ(\Omega_B\circ\tau_B)} & S(f)\circ(\zeta_B\circ\tau_B) \\
\downarrow{\scriptstyle \alpha_{S(f),\xi_B,\sigma_B}} & & \downarrow{\scriptstyle \alpha_{S(f),\xi_B,\tau_B}} & & \downarrow{\scriptstyle \alpha_{S(f),\zeta_B,\tau_B}} \\
(S(f)\circ\xi_B)\circ\sigma_B & \xrightarrow{(S(f)\circ\xi_B)\circ\Gamma_B} & (S(f)\circ\xi_B)\circ\tau_B & \xrightarrow{(S(f)\circ\Omega_B)\circ\tau_B} & (S(f)\circ\zeta_B)\circ\tau_B \\
\downarrow{\scriptstyle \xi_f\circ\sigma_B} & & \downarrow{\scriptstyle \xi_f\circ\tau_B} & & \downarrow{\scriptstyle \zeta_f\circ\tau_B} \\
(\xi_A\circ R(f))\circ\sigma_B & \xrightarrow{(\xi_A\circ R(f))\circ\Gamma_B} & (\xi_A\circ R(f))\circ\tau_B & \xrightarrow{(\Omega_A\circ R(f))\circ\tau_B} & (\zeta_A\circ R(f))\circ\tau_B \\
\downarrow{\scriptstyle \alpha_{\xi_A,R(f),\sigma_B}} & & \downarrow{\scriptstyle \alpha_{\xi_A,R(f),\tau_B}} & & \downarrow{\scriptstyle \alpha_{\zeta_A,R(f),\tau_B}} \\
\xi_A\circ(R(f)\circ\sigma_B) & \xrightarrow{\xi_A\circ(R(f)\circ\Gamma_B)} & \xi_A\circ(R(f)\circ\tau_B) & \xrightarrow{\Omega_A\circ(R(f)\circ\tau_B)} & \zeta_A\circ(R(f)\circ\tau_B) \\
\downarrow{\scriptstyle \xi_A\circ\sigma_f} & & \downarrow{\scriptstyle \xi_A\circ\tau_f} & & \downarrow{\scriptstyle \zeta_A\circ\tau_f} \\
\xi_A\circ(\sigma_A\circ P(f)) & \xrightarrow{\xi_A\circ(\Gamma_A\circ P(f))} & \xi_A\circ(\tau_A\circ P(f)) & \xrightarrow{\Omega_A\circ(\tau_A\circ P(f))} & \zeta_A\circ(\tau_A\circ P(f) \\
\downarrow{\scriptstyle \alpha^{-1}_{\xi_A,\sigma_A,P(f)}} & & \downarrow{\scriptstyle \alpha^{-1}_{\xi_A,\tau_A,P(f)}} & & \downarrow{\scriptstyle \alpha^{-1}_{\zeta_A,\tau_A,P(f)}} \\
(\xi_A\circ\sigma_A)\circ P(f) & \xrightarrow[(\xi_A\circ\Gamma_A)\circ P(f)]{} & (\xi_A\circ\tau_A)\circ P(f) & \xrightarrow[(\Omega_A\circ\tau_A)\circ P(f)]{} & (\zeta_A\circ\tau_A)\circ P(f)
\end{array}
$$

where the first square in the second row and the second square in the the fourth row commute because they define horizontal compositions $\xi_f \circ \Gamma_B$ and $\Omega_A \circ \tau_f$, respectively. The commutativity of the second square in the second row and the first square in the fourth row follows from the coherence for modifications $\Omega \colon \xi \to \zeta$ and $\Gamma \colon \sigma \to \tau$, respectively and squares in the first, third and fifth row are associativity coherence. But the pasting composition of the top and front faces of the coherence for the modification $\Omega \circ \Gamma$ is equal to the composition of the top and right edges of the above diagram, and the pasting composition of the back and bottom faces of the coherence is equal to the composition of the left and bottom edges of the diagram.

The associativity and identities for horizontal composition follows immediately from the associativity of horizontal composition of natural transformations in Cat. Also, for any diagram of modifications



we have Godement interchange law

$$(\Lambda \circ \Pi)(\Omega \circ \Gamma) = (\Lambda\Omega) \circ (\Pi\Gamma)$$

since it holds for natural transformations and the composition were defined componentwise.
□

2 The Yoneda embedding

We will now take a more closer look to the representable 2-presheaves. Each object C in $\mathcal{B}$ gives rise to the 2-presheaf $\mathcal{Y}_C\colon \mathcal{B}^{op} \to \mathrm{Cat}$ on $\mathcal{B}$, defined by $\mathcal{Y}_C(B) := \mathcal{B}(B, C)$, for each object B in $\mathcal{B}$, and for any two objects $A, B \in \mathcal{B}^{op}$ by the functor

$$\mathcal{Y}_C^{A,B}\colon \mathcal{B}(A, B) \to \mathrm{Cat}(\mathcal{Y}_C(B), \mathcal{Y}_C(A)),$$

for which we will usually omit indexing objects in the superscript. The value of this functor at any object $f \in \mathcal{B}(A, B)$ is a functor $\mathcal{Y}_C(f)\colon \mathcal{Y}_C(B) \to \mathcal{Y}_C(A)$ defined by

$$\mathcal{Y}_C(f)(g) := g \circ f, \qquad \mathcal{Y}_C(f)(\phi) := \psi \circ f$$

for any 1-morphism $g \in \mathcal{B}(B, C)$ and any 2-morphism $\psi \in \mathcal{B}(B, C)$. The value of the functor $\mathcal{Y}_C^{A,B}\colon \mathcal{B}(A, B) \to \mathrm{Cat}(\mathcal{Y}^C(B), \mathcal{Y}^C(A))$ for any 2-morphism $\phi\colon f \to f'$ (viewed as a morphism of the category $\mathcal{B}(A, B)$), is the natural transformation

$$\mathcal{Y}_C(\phi)\colon \mathcal{Y}_C(f) \to \mathcal{Y}_C(f')$$

as in the diagram

$$\begin{array}{ccc}
\mathcal{Y}_C(f)(g) & \xrightarrow{\mathcal{Y}_C(\phi)_g} & \mathcal{Y}_C(f')(g) \\
{\scriptstyle \mathcal{Y}_C(f)(\nu)}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}_C(f')(\nu)} \\
\mathcal{Y}_C(f)(g') & \xrightarrow[\mathcal{Y}_C(\phi)_{g'}]{} & \mathcal{Y}_C(f')(g')
\end{array}$$

for any 2-morphism $\nu\colon g \to g'$ in $\mathcal{B}(B, C)$, whose component for any object $g \in \mathcal{B}(B, C)$ is defined by $\mathcal{Y}_C(\phi)_g := g \circ \phi$. The commutativity of the diagram is assured by the definition of the horizontal composition of 2-morphisms and the Godement interchange law

$$(g' \circ \phi)(\nu \circ f) = \nu \circ \psi = (\nu \circ f')(g \circ \phi).$$

Proposition 2.1. *Let C be an object of the bicategory $\mathcal{B}$. Then the above construction defines a 2-presheaf $\mathcal{Y}_C\colon \mathcal{B}^{op} \to \mathrm{Cat}$.*

Proof. The components of the natural transformation $\mu_{g,f}\colon \mathcal{Y}_C(f) \circ \mathcal{Y}_C(g) \to \mathcal{Y}_C(g \circ f)$ are given by $\mu_{g,f}(h) := \alpha_{h,g,f}\colon (\mathcal{Y}_C(f) \circ \mathcal{Y}_C(g))(h) \to \mathcal{Y}_C(g \circ f)(h)$ and the diagram for an associativity coherence

$$\begin{array}{ccc}
((k \circ h) \circ g) \circ f & = & ((k \circ h) \circ g) \circ f \\
\downarrow{\scriptstyle a_{k\circ h,g,f}} & & \downarrow{\scriptstyle a_{k,h,g}\circ f} \\
(k \circ h) \circ (g \circ f) & & (k \circ (h \circ g)) \circ f \\
\downarrow{\scriptstyle a_{k,h,g\circ f}} & & \downarrow{\scriptstyle a_{k,h\circ g,f}} \\
k \circ (h \circ (g \circ f)) & \xleftarrow[k\circ a_{h,g,f}]{} & k \circ ((h \circ g) \circ f)
\end{array}$$

becomes the commutative diagram

$$\begin{array}{ccc}
((\mathcal{Y}_C(f) \circ \mathcal{Y}_C(g)) \circ \mathcal{Y}_C(h))(k) & = & (\mathcal{Y}_C(f) \circ (\mathcal{Y}_C(g) \circ \mathcal{Y}_C(h)))(k) \\
\downarrow{\scriptstyle \mu_{g,f}(\mathcal{Y}_C(h)(k))} & & \downarrow{\scriptstyle \mathcal{Y}_C(f)(\mu_{h,g}(k))} \\
\mathcal{Y}_C(g \circ f) \circ \mathcal{Y}_C(h)(k) & & (\mathcal{Y}_C(f) \circ \mathcal{Y}_C(h \circ g))(k) \\
\downarrow{\scriptstyle \mu_{h,g\circ f}(k)} & & \downarrow{\scriptstyle \mu_{h\circ g,f}(k)} \\
\mathcal{Y}_C(h \circ (g \circ f))(k) & \xleftarrow[\mathcal{Y}_C(a_{h,g,f})(k)]{} & \mathcal{Y}_C((h \circ g) \circ f)(k)
\end{array}$$

and since the last diagram is commutative for any morphism $k\colon D \to E$ (viewed as an object of the category $\mathcal{B}(D, E)$), we obtain the coherence diagram

$$\begin{array}{ccc}
(\mathcal{Y}_C(f) \circ \mathcal{Y}_C(g)) \circ \mathcal{Y}_C(h) & = & \mathcal{Y}_C(f) \circ (\mathcal{Y}_C(g) \circ \mathcal{Y}_C(h)) \\
\downarrow{\scriptstyle \mu_{g,f}\circ \mathcal{Y}_C(h)} & & \downarrow{\scriptstyle \mathcal{Y}_C(f)\circ \mu_{h,g}} \\
\mathcal{Y}_C(g \circ f) \circ \mathcal{Y}_C(h) & & \mathcal{Y}_C(f) \circ \mathcal{Y}_C(h \circ g) \\
\downarrow{\scriptstyle \mu_{h,g\circ f}} & & \downarrow{\scriptstyle \mu_{h\circ g,f}} \\
\mathcal{Y}_C(h \circ (g \circ f)) & \xleftarrow[\mathcal{Y}_C(a_{h,g,f})]{} & \mathcal{Y}_C((h \circ g) \circ f)
\end{array}$$

for the horizontal composition. The coherence for units follows also from the left and right identity coherence. The components of the identity natural transformation $\eta_B\colon I_{\mathcal{Y}_C(B)} \to \mathcal{Y}_C(i_B)$ are defined for each morphism $g\colon B \to C$ (again seen as an object of $\mathcal{B}(B, C)$) by $\eta_g := \rho_g^{-1}\colon g \to g \circ i_B$. For any pair of 1-morphisms $f\colon A \to B$ and $g\colon B \to C$ the deformed triangle diagram for identity coherence

$$\begin{array}{ccc}
(g \circ i_B) \circ f & \xrightarrow{\alpha_{g,i_B,f}} & g \circ (i_B \circ f) \\
\downarrow{\scriptstyle \rho_g\circ f} & & \downarrow{\scriptstyle g\circ \lambda_f} \\
g \circ f & = & g \circ f
\end{array}$$

becomes a commutative diagram

$$\begin{array}{ccc}
(\mathcal{Y}_C(f)\circ\mathcal{Y}_C(i_B))(g) & \xrightarrow{\mu_{i_B,f}(g)} & \mathcal{Y}_C(i_B\circ f)(g) \\
{\scriptstyle \mathcal{Y}_C(f)(\rho_g^{-1})}\uparrow & & \downarrow{\scriptstyle \mathcal{Y}_C(\lambda_f)(g)} \\
(\mathcal{Y}_C(f)\circ I_{\mathcal{Y}_C(B)})(g) & = & \mathcal{Y}_C(f)(g)
\end{array}$$

and since the last diagram is commutative for any morphism $g\colon B\to C$ (viewed as an object of the category $\mathcal{B}(B,C)$), we obtain the coherence diagram

$$\begin{array}{ccc}
\mathcal{Y}_C(f)\circ\mathcal{Y}_C(i_B) & \xrightarrow{\mu_{i_B,f}} & \mathcal{Y}_C(i_B\circ f) \\
{\scriptstyle \mathcal{Y}_C(f)\circ\eta_B}\uparrow & & \downarrow{\scriptstyle \mathcal{Y}_C(\lambda_f)} \\
\mathcal{Y}_C(f)\circ I_{\mathcal{Y}_C(B)} & = & \mathcal{Y}_C(f)
\end{array}$$

□

For any 1-morphism $h\colon C\to D$ in $\mathcal{B}$, there is a pseudo natural transformation of corresponding 2-presheaves

$$\mathcal{Y}(h)\colon\mathcal{Y}_C\to\mathcal{Y}_D$$

whose component for any object $B\in Ob(\mathcal{B})$ is a functor $\mathcal{Y}(h)_B\colon\mathcal{B}(B,C)\to\mathcal{B}(B,D)$ defined by

$$\mathcal{Y}(h)_B(g):=h\circ g,\qquad \mathcal{Y}(h)_B(\nu):=h\circ\nu$$

for any 1-morphism $g\colon B\to C$, and any 2-morphism $\nu\colon g\to g'$ (viewed as an object and a morphism in $\mathcal{B}(B,C)$, respectively). The component of the pseudo natural transformation is given by the diagram

$$\begin{array}{ccc}
\mathcal{Y}_C(B) & \xrightarrow{\mathcal{Y}(h)_B} & \mathcal{Y}_D(B) \\
{\scriptstyle \mathcal{Y}_C(f)}\downarrow & \swArrow\,{\scriptstyle \mathcal{Y}(h)_f} & \downarrow{\scriptstyle \mathcal{Y}_D(f)} \\
\mathcal{Y}_C(A) & \xrightarrow[\mathcal{Y}(h)_A]{} & \mathcal{Y}_D(A)
\end{array}$$

for any 1-morphism $f\colon A \to B$. It is a natural transformation whose components are defined by the associativity isomorphism as $\mathcal{Y}(h)_f(g) := \alpha_{h,g,f}\colon (h \circ g) \circ f \to h \circ (g \circ f)$. Again, the associativity coherence gives a coherence for a pseudo natural transformation.

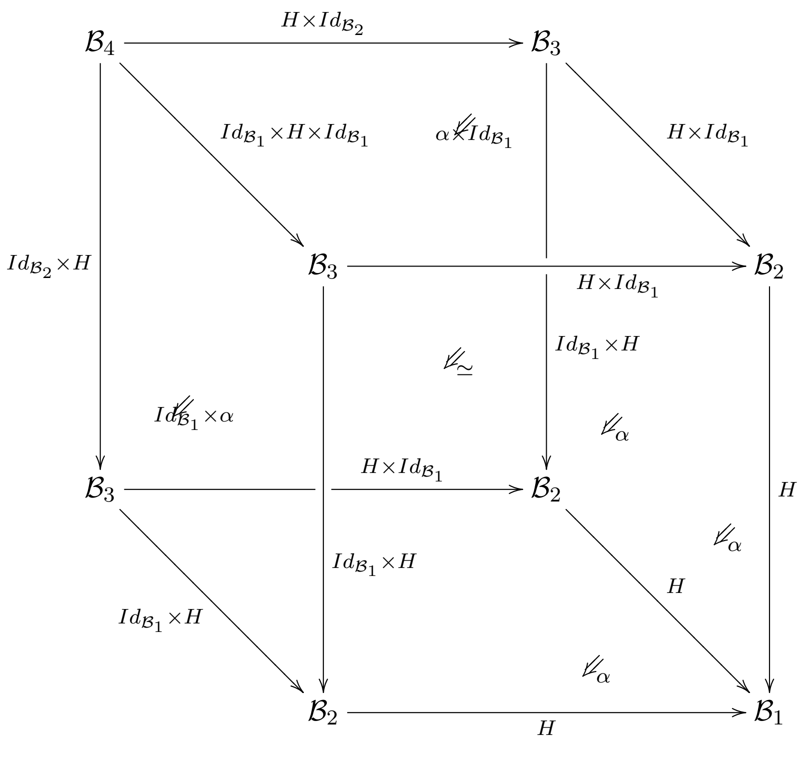
Also for any 2-morphism $\beta\colon h \Rightarrow h'$ we have a modification $\mathcal{Y}(\beta)\colon \mathcal{Y}(h) \to \mathcal{Y}(h')$ whose component indexed by the object B in $\mathcal{B}$ is given by the natural transformation $\mathcal{Y}(\beta)_B\colon \mathcal{Y}(h)_B \to \mathcal{Y}(h')_B$ defined by $\mathcal{Y}(\beta)_B := \beta \circ g$ for any 1-morphism $g\colon B \to C$ (viewed as the object in $\mathcal{B}(B, C)$).

Proposition 2.2. *The above construction gives rise to the homomorphism*

$$\mathcal{Y}\colon \mathcal{B} \to \mathrm{Cat}^{\mathcal{B}^{op}}$$

from the bicategory $\mathcal{B}$ to the 2-category $\mathcal{P} = \mathrm{Cat}^{\mathcal{B}^{op}}$ of 2-presheaves which we call Yoneda embedding.

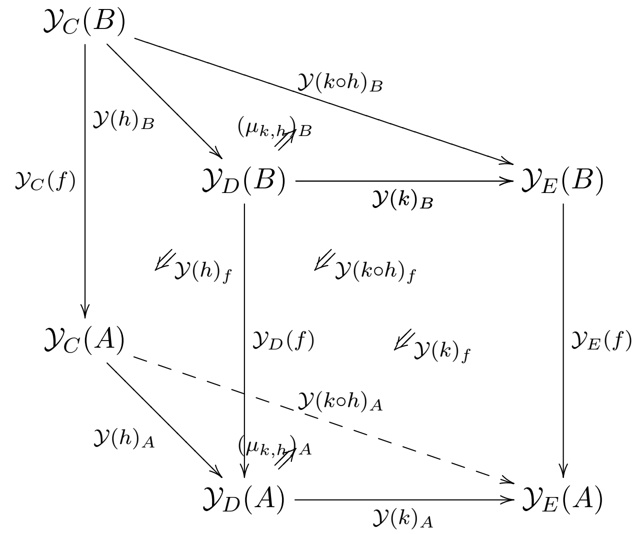
Proof. The above homomorphism sent an object C in $\mathcal{B}$ to the representable 2-presheaf $\mathcal{Y}_C$, and any 1-morphism $h\colon C \to D$ to the pseudo natural transformation $\mathcal{Y}(h)\colon \mathcal{Y}_C \to \mathcal{Y}_D$. For any composable pair of 1-morphisms $C \xrightarrow{h} D \xrightarrow{k} E$ in $\mathcal{B}$ we define a modification $\mu_{k,h}\colon \mathcal{Y}(k) \circ \mathcal{Y}(h) \to \mathcal{Y}(k \circ h)$ whose component $(\mu_{k,h})_g$ indexed by the object $g \in \mathcal{Y}_C(B)$ is given by the associativity isomorphism $\alpha^{-1}_{k,h,g}\colon k \circ (h \circ g) \to (k \circ h) \circ g$. The coherence for the associativity in $\mathcal{B}$



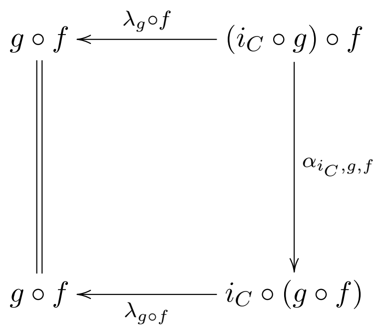
is the equality

$$\alpha_{k,h,g\circ f}\alpha_{k\circ h,g,f} = (k \circ \alpha_{h,g,f})\alpha_{k,h\circ g,f}\alpha_{k,h\circ g,f}$$

of natural transformations indexed by composable quadruple of 1-morphisms in $\mathcal{B}$. For fixed choice of 1-morphisms f, h, k the above cube becomes a coherence for the modification $\mu_{k,h}$



in which the back and right face of the cube became the back face of the prism. The component of the modification $\eta_C \colon I_{\mathcal{Y}_C} \to \mathcal{Y}(i_C)$ indexed by the object B in $\mathcal{B}$ is the natural transformation $(\eta_C)_B \colon (I_{\mathcal{Y}_C})_B \to (\mathcal{Y}(i_C))_B$, whose component indexed by the 1-morphism $g \colon B \to C$ (seen as an object of the category $\mathcal{Y}_C(B)$) is defined by $(\eta_C)_B(g) := \lambda_g^{-1} \colon g \to i_C \circ g$. The commutative diagram



obtained from the coherence for left and right identities, becomes a diagram

$$\begin{array}{ccc}
(\mathcal{Y}_C(f)\circ(I_{\mathcal{Y}_C})_B)(g) & \xrightarrow{\mathcal{Y}_C(f)\circ(\eta_C)_B(g)} & (\mathcal{Y}_C(f)\circ\mathcal{Y}(i_C)_B)(g) \\
\Big\downarrow{\scriptstyle (I_{\mathcal{Y}_C})_f} & & \Big\downarrow{\scriptstyle (\mathcal{Y}(i_C)_f)_g} \\
((I_{\mathcal{Y}_C})_A\circ\mathcal{Y}_C(f))(g) & \xrightarrow[(\eta_C)_A\circ(I_{\mathcal{Y}_C})_f(g)]{} & (\mathcal{Y}(i_C)_A\circ\mathcal{Y}_C(f))(g).
\end{array}$$

Since the above diagram is commutative for any 1-morphism $g\colon B\to C$ (seen as an object of the category $\mathcal{Y}_C(B)$) it gives a coherence

$$\begin{array}{ccc}
\mathcal{Y}_C(f)\circ(I_{\mathcal{Y}_C})_B & \xrightarrow{\mathcal{Y}_C(f)\circ(\eta_C)_B} & \mathcal{Y}_C(f)\circ\mathcal{Y}(i_C)_B \\
\Big\downarrow{\scriptstyle (I_{\mathcal{Y}_C})_f} & & \Big\downarrow{\scriptstyle \mathcal{Y}(i_C)_f} \\
(I_{\mathcal{Y}_C})_A\circ\mathcal{Y}_C(f) & \xrightarrow[(\eta_C)_A\circ(I_{\mathcal{Y}_C})_f]{} & \mathcal{Y}(i_C)_A\circ\mathcal{Y}_C(f)
\end{array}$$

for the modification $\eta_C\colon I_{\mathcal{Y}_C}\to\mathcal{Y}(i_C)$. □

3 Bicategorical Yoneda lemma

An object of the category $Hom_{\mathcal{P}}(\mathcal{Y}_C, P)$ is a pseudo natural transformation $\gamma\colon \mathcal{Y}_C \to P$ which consists of a family of functors $\gamma_B\colon \mathcal{Y}_C(B) \to P(B)$, indexed by objects $B \in \mathcal{B}$, and of a natural transformations

$$\begin{array}{ccc} \mathcal{Y}_C(B) & \xrightarrow{\gamma_B} & P(B) \\ {\scriptstyle \mathcal{Y}_C(f)}\downarrow & \swarrow \gamma_f & \downarrow{\scriptstyle P(f)} \\ \mathcal{Y}_C(A) & \xrightarrow[\gamma_A]{} & P(A) \end{array}$$

for each morphism $f\colon A \to B$ in $\mathcal{B}$, such that the coherence diagram

$$\begin{array}{ccc} P(f)\circ P(g)\circ \gamma_C & \xrightarrow{P(f)\circ\gamma_g} & P(f)\circ\gamma_B\circ\mathcal{Y}_C(g) \\ {\scriptstyle \mu^P_{g,f}\circ\gamma_C}\downarrow & & \\ P(g\circ f)\circ\gamma_C & & \downarrow{\scriptstyle \gamma_f\circ\mathcal{Y}_C(g)} \\ {\scriptstyle \gamma_{g\circ f}}\downarrow & & \\ \gamma_A\circ\mathcal{Y}_C(g\circ f) & \xleftarrow[\gamma_A\circ\mu^{\mathcal{Y}_C}_{g,f}]{} & \gamma_A\circ\mathcal{Y}_C(f)\circ\mathcal{Y}_C(g) \end{array}$$

for any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ of 1-morphisms in $\mathcal{B}$, commutes. Any modification $\Gamma\colon \gamma \to \delta$ consists of the family of natural transformations $\Gamma_B\colon \gamma_B \to \delta_B$ indexed by the objects $B \in \mathcal{B}$.

For any 2-presheaf $P\colon \mathcal{B}^{op} \to \mathrm{Cat}$, we will define a new 2-presheaf $\mathcal{S}P\colon \mathcal{B}^{op} \to \mathrm{Cat}$ by $\mathcal{S}P(C) := \mathcal{P}(\mathcal{Y}_C, P)$ for any object $C \in \mathcal{B}$, and for any two objects $C, D \in \mathcal{B}$ we have a functor $\mathcal{S}P_{C,D}\colon \mathcal{B}(C,D) \to [\mathcal{S}P(D), \mathcal{S}P(C)]$, which takes a morphism $h\colon C \to D$ to the functor $\mathcal{S}P(h) := \mathcal{P}(\mathcal{Y}(h), P)\colon \mathcal{S}P(D) \to \mathcal{S}P(C)$, (where we omitted subscripts for convenience) defined by $\mathcal{S}P(h)(\phi) := \phi \circ \mathcal{Y}(h)$ and $\mathcal{S}P(h)(\Omega) := \Omega \circ \mathcal{Y}(h)$ for any pseudo natural transformation $\phi\colon \mathcal{Y}_D \to P$ and any modification $\Omega\colon \phi \to \psi$, seen as an object and morphism of the category $\mathcal{P}(\mathcal{Y}_D, P)$, respectively. Moreover, the functor $\mathcal{S}P_{C,D}\colon \mathcal{B}(C,D) \to [\mathcal{S}P(D), \mathcal{S}P(C)]$ takes a 2-morphism $\beta\colon h \to h'$ to a natural transformation $\mathcal{S}P(\beta)\colon \mathcal{S}P(h) \to \mathcal{S}P(h')$, whose component at the object $\phi \in \mathcal{S}P(D)$ is given by $\mathcal{S}P(\beta)(\phi) := \phi \circ \mathcal{Y}(\beta)\colon \mathcal{S}P(h)(\phi) \to \mathcal{S}P(h')(\phi)$.

Proposition 3.1. *For any 2-presheaf $P\colon \mathcal{B}^{op} \to \mathrm{Cat}$, the above construction defines a new 2-presheaf $\mathcal{S}P\colon \mathcal{B}^{op} \to \mathrm{Cat}$.*

Proof. The component of the natural transformation $\mu^{\mathcal{S}P}_{h,g}\colon \mathcal{S}P(g) \circ \mathcal{S}P(h) \to \mathcal{S}P(h \circ g)$ indexed by the pseudo natural transformation $\phi\colon \mathcal{Y}_D \to P$ is the modification $\mu^{\mathcal{S}P}_{h,g}(\phi)$ from the pseudo natural transformation

$$[\mathcal{S}P(g) \circ \mathcal{S}P(h)](\phi) = \mathcal{S}P(g)(\phi \circ \mathcal{Y}(h)) = (\phi \circ \mathcal{Y}(h)) \circ \mathcal{Y}(g) = \phi \circ (\mathcal{Y}(h) \circ \mathcal{Y}(g))\colon \mathcal{Y}_B \to P$$

to the pseudo natural transformation

$$\mathcal{S}P(h \circ g)(\phi) = \phi \circ \mathcal{Y}(h \circ g)\colon \mathcal{Y}_B \to P$$

in $\mathcal{S}P(B)$. Its component indexed by the object A in $\mathcal{B}$ is a natural transformation $\mu^{\mathcal{S}P}_{h,g}(\phi)_A\colon [\phi \circ (\mathcal{Y}(h) \circ \mathcal{Y}(g))]_A \to [\phi \circ (\mathcal{Y}(h \circ g))]_A$ whose component indexed by 1-morphism $f\colon A \to B$ (seen as an object of the category $\mathcal{Y}_B(A)$) is given by $\mu^{\mathcal{S}P}_{h,g}(\phi)_A(f) := \phi_A(\alpha^{-1}_{h,g,f})$. The corresponding coherence condition

$$\begin{array}{ccccc}
(\mathcal{S}P(g) \circ \mathcal{S}P(h)) \circ \mathcal{S}P(k) & \xrightarrow{\mu_{h,g} \circ \mathcal{S}P(k)} & \mathcal{S}P(h \circ g) \circ \mathcal{S}P(k) & \xrightarrow{\mu_{k,h\circ g}} & \mathcal{S}P(k \circ (h \circ g)) \\
\Big\Vert & & & & \Big\uparrow {\scriptstyle \mathcal{S}P(\alpha_{k,h,g})} \\
\mathcal{S}P(g) \circ (\mathcal{S}P(h) \circ \mathcal{S}P(k)) & \xrightarrow[\mathcal{S}P(g) \circ \mu_{k,h}]{} & \mathcal{S}P(g) \circ \mathcal{S}P(k \circ h) & \xrightarrow[\mu_{k \circ h, g}]{} & \mathcal{S}P((k \circ h) \circ g)
\end{array}$$

evaluated by the pseudo natural transformation $\epsilon\colon \mathcal{Y}_E \to P$ is the diagram of modifications

$$\begin{array}{ccccc}
(\epsilon \circ \mathcal{Y}(k)) \circ (\mathcal{Y}(h) \circ \mathcal{Y}(g)) & \xrightarrow{\mu_{h,g}(\epsilon \circ \mathcal{Y}(k))} & (\epsilon \circ \mathcal{Y}(k)) \circ \mathcal{Y}(h \circ g) & \xrightarrow{\mu_{k,h\circ g}(\epsilon)} & \epsilon \circ \mathcal{Y}(k \circ (h \circ g)) \\
\Big\Vert & & & & \Big\uparrow {\scriptstyle \epsilon \circ \mathcal{Y}(\alpha_{k,h,g})} \\
[\epsilon \circ (\mathcal{Y}(k) \circ \mathcal{Y}(h))] \circ \mathcal{Y}(g) & \xrightarrow[\mu_{k,h}(\epsilon) \circ \mathcal{Y}(g)]{} & (\epsilon \circ \mathcal{Y}(k \circ h)) \circ \mathcal{Y}(g) & \xrightarrow[\mu_{k\circ h,g}(\epsilon)]{} & \epsilon \circ \mathcal{Y}((k \circ h) \circ g)
\end{array}$$

which becomes an image by the functor $\epsilon_A\colon \mathcal{Y}_E(A) \to P(A)$

$$\begin{array}{ccccc}
\epsilon_A(k \circ [h \circ (g \circ f)]) & \xrightarrow{\epsilon_A(k \circ \alpha^{-1}_{h,g,f})} & \epsilon_A(k \circ [(h \circ g) \circ f]) & \xrightarrow{\epsilon_A(\alpha^{-1}_{k,h\circ g,f})} & \epsilon_A([k \circ (h \circ g)] \circ f) \\
\Big\Vert & & & & \Big\uparrow {\scriptstyle \epsilon_A(\alpha_{k,h,g} \circ f)} \\
\epsilon_A(k \circ [h \circ (g \circ f)]) & \xrightarrow[\epsilon_A(\alpha^{-1}_{k,h,g\circ f})]{} & \epsilon_A([k \circ h] \circ [g \circ f]) & \xrightarrow[\epsilon_A(\alpha^{-1}_{k\circ h,g,f})]{} & \epsilon_A([(k \circ h) \circ g] \circ f)
\end{array}$$

of the pentagonal coherence for associativity when evaluated for any morphism $f\colon A \to B$.

The natural transformation $\eta_C\colon I_{\mathcal{S}P(C)} \to \mathcal{S}P(i_C)$ indexed by the pseudo natural transformation $\gamma\colon \mathcal{Y}_C \to P$ is given by the modification $\eta_C(\gamma)\colon \gamma \to \gamma \circ \mathcal{Y}(i_C)$, whose component indexed by the object B in $\mathcal{B}$ is a natural transformation $\eta_C(\gamma)_B\colon \gamma_B \to \gamma_B \circ \mathcal{Y}(i_C)_B$. Its component indexed by the 1-morphism $g\colon B \to C$ (seen as an object of the category $\mathcal{Y}_C(B)$) is given by the morphism $\eta_C(\gamma)_B(g) := \gamma_B(\lambda_g)\colon \gamma_B(i_C \circ g) \to \gamma_B(g)$ in $P(B)$. Two coherence conditions for natural transformation $\eta_C\colon I_{\mathcal{S}P(C)} \to \mathcal{S}P(i_C)$

$$\begin{array}{ccc} \mathcal{S}P(g)\circ\mathcal{S}P(i_C) & \xrightarrow{\mu_{i_C,g}} & \mathcal{S}P(i_C\circ g) \\ {\scriptstyle \mathcal{S}P(g)\circ\eta_C}\uparrow & & \downarrow{\scriptstyle \mathcal{S}P(\lambda_g)} \\ \mathcal{S}P(g) & = & \mathcal{S}P(g) \end{array} \qquad \begin{array}{ccc} \mathcal{S}P(i_B)\circ\mathcal{S}P(g) & \xrightarrow{\mu_{g,i_B}} & \mathcal{S}P(g\circ i_B) \\ {\scriptstyle \eta_B\circ\mathcal{S}P(g)}\uparrow & & \downarrow{\scriptstyle \mathcal{S}P(\rho_g)} \\ \mathcal{S}P(g) & = & \mathcal{S}P(g) \end{array}$$

are satisfied since when evaluated for pseudo natural transformation $\gamma\colon \mathcal{Y}_C \to P$, we have

$$\begin{array}{ccc} (\gamma\circ Y(i_C))\circ\mathcal{Y}(g) & \xrightarrow{\mu_{i_C,g}(\gamma)} & \gamma\circ\mathcal{Y}(i_C\circ g) \\ {\scriptstyle \eta_C(\gamma)\circ Y(g)}\uparrow & & \downarrow{\scriptstyle \gamma\circ\mathcal{Y}(\lambda_g)} \\ \gamma\circ\mathcal{Y}(g) & = & \gamma\circ\mathcal{Y}(g) \end{array} \qquad \begin{array}{ccc} (\gamma\circ Y(g))\circ\mathcal{Y}(i_B) & \xrightarrow{\mu_{g,i_B}(\gamma)} & \gamma\circ Y(g\circ i_B) \\ {\scriptstyle \eta_B(\gamma\circ\mathcal{Y}(g))}\uparrow & & \downarrow{\scriptstyle \gamma\circ\mathcal{Y}(\rho_g)} \\ \mathcal{S}P(g) & = & \mathcal{S}P(g) \end{array}$$

and this two diagrams for any 1-morphism $f\colon A \to B$ give the image by the functor $\gamma_A\colon \mathcal{Y}_C(A) \to P(A)$

$$\begin{array}{ccc} \gamma_A(i_C\circ(g\circ f)) & \xrightarrow{\gamma_A(\alpha^{-1}_{i_C,g,f})} & \gamma_A((i_C\circ g)\circ f) \\ {\scriptstyle \gamma_A(\lambda^{-1}_{g\circ f})}\uparrow & & \downarrow{\scriptstyle \gamma_A(\lambda_g\circ f)} \\ \gamma_A(g\circ f) & = & \gamma_A(g\circ f) \end{array} \qquad \begin{array}{ccc} \gamma_A(g\circ(i_B\circ f)) & \xrightarrow{\gamma_A(\alpha^{-1}_{i_C,g,f})} & \gamma_A((g\circ i_B)\circ f) \\ {\scriptstyle \gamma_A(g\circ\lambda^{-1}_f)}\uparrow & & \downarrow{\scriptstyle \gamma_A(\rho_g\circ f)} \\ \gamma_A(g\circ f) & = & \gamma_A(g\circ f) \end{array}$$

of two coherence diagrams for left and right identities. □

This allows us to state the categorification of the Yoneda lemma.

Theorem 3.1. *(Bicategorical Yoneda lemma) There is an equivalence*

$$\theta\colon \mathcal{S}P \to P$$

in $\mathcal{P}$, (which is just a pseudo natural equivalence $\theta\colon \mathcal{P}(\mathcal{Y}, P) \to P$ between 2-presheaves).

Proof. The component of the pseudo natural transformation θ indexed by an object $C \in \mathcal{B}$ is a functor $\theta_C\colon \mathcal{P}(\mathcal{Y}_C, P) \to P(C)$, defined by $\theta_C(\gamma) := \gamma_C(id_C)$ for any pseudo natural transformation $\gamma\colon \mathcal{Y}_C \to P$, and $\theta_C(\Gamma) := \Gamma_C(id_C)\colon \gamma_C(id_C) \Rightarrow \delta_C(id_C)$ for any modification $\Gamma\colon \gamma \to \delta$ in $\mathcal{P}(\mathcal{Y}_C, P)$. This 2-morphism in $P(C)$ is just a component indexed by the object $id_C \in \mathcal{Y}_C(C)$ of the natural transformation $\Gamma_C\colon \gamma_C \to \delta_C$, which is in turn component of the modification $\Gamma\colon \gamma \to \delta$, indexed by the object $C \in \mathcal{B}$. For any morphism $h\colon C \to D$, the natural transformation $\theta_h\colon P(h) \circ \theta_D \to \theta_C \circ \mathcal{S}P(h)$, given by the square

$$\begin{array}{ccc} \mathcal{S}P(D) & \xrightarrow{\theta_D} & P(D) \\ {\scriptstyle \mathcal{S}P(h)}\downarrow & \swarrow_{\theta_h} & \downarrow{\scriptstyle P(h)} \\ \mathcal{S}P(C) & \xrightarrow[\theta_C]{} & P(C) \end{array}$$

has the morphism $\theta_h(\phi)\colon (P(h) \circ \theta_D)(\phi) \to (\theta_C \circ \mathcal{S}P(h))(\phi)$ in the category $P(C)$ as the component indexed by the object $\phi\colon \mathcal{Y}_D \to P$ of the category $\mathcal{S}P(D)$. This morphism is defined by the composition

$$P(h)(\phi_D(id_D)) \xrightarrow{\phi_h(id_D)} \phi_C(id_D \circ h) \xrightarrow{\phi_C(\lambda_h)} \phi_C(h) \xrightarrow{\phi_C(\rho_h^{-1})} \phi_C(h \circ id_C)$$

where the morphism $\phi_h(id_D)\colon P(h)(\phi_D(id_D)) \to \phi_C(id_D \circ h)$ is defined by the component of

$$\begin{array}{ccc} \mathcal{Y}_D(D) & \xrightarrow{\phi_D} & P(D) \\ {\scriptstyle \mathcal{Y}_D(h)}\downarrow & \swarrow_{\phi_h} & \downarrow{\scriptstyle P(h)} \\ \mathcal{Y}_D(C) & \xrightarrow[\phi_C]{} & P(C) \end{array}$$

given by $\phi_h(id_D)\colon (P(h) \circ \phi_D)(id_D) \to (\phi_C \circ \mathcal{Y}_D(h))(id_D)$, which is just the morphism $\phi_h(id_D)\colon P(h)(\phi_D(id_D)) \to \phi_C(id_D \circ h)$ in $P(C)$.

To show that this really defines a pseudo natural transformation $\theta\colon \mathcal{SP} \to P$, we need to check that the coherence diagram

$$
\begin{array}{ccc}
P(f) \circ P(g) \circ \theta_C & \xrightarrow{P(f)\circ\theta_g} & P(f) \circ \theta_B \circ \mathcal{S}P(g) \\
\Big\downarrow{\scriptstyle \mu^P_{g,f}\circ\theta_C} & & \\
P(g \circ f) \circ \theta_C & & \Big\downarrow{\scriptstyle \theta_f\circ\mathcal{S}P(g)} \\
\Big\downarrow{\scriptstyle \theta_{g\circ f}} & & \\
\theta_A \circ \mathcal{S}P(g \circ f) & \xleftarrow[\theta_A\circ\mu^{\mathcal{S}P}_{g,f}]{} & \theta_A \circ \mathcal{S}P(f) \circ \mathcal{S}P(g)
\end{array}
$$

for any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ of 1-morphisms in $\mathcal{B}$, commutes. But for any pseudo natural transformation $\gamma\colon \mathcal{Y}_C \to P$, we obtain the diagram

$$
\begin{array}{ccc}
(P(f) \circ P(g) \circ \theta_C)(\gamma) & \xrightarrow{(P(f)\circ\theta_g)(\gamma)} & (P(f) \circ \theta_B \circ \mathcal{S}P(g))(\gamma) \\
\Big\downarrow{\scriptstyle (\mu^P_{g,f}\circ\theta_C)(\gamma)} & & \\
(P(g \circ f) \circ \theta_C)(\gamma) & & \Big\downarrow{\scriptstyle (\theta_f\circ\mathcal{S}P(g))(\gamma)} \\
\Big\downarrow{\scriptstyle (\theta_{g\circ f})(\gamma)} & & \\
(\theta_A \circ \mathcal{S}P(g \circ f))(\gamma) & \xleftarrow[(\theta_A\circ\mu^{\mathcal{S}P}_{g,f})(\gamma)]{} & (\theta_A \circ \mathcal{S}P(f) \circ \mathcal{S}P(g))(\gamma)
\end{array}
$$

and this diagram transforms into the diagram

$$
\begin{array}{ccccccc}
(P(f) \circ P(g) \circ \gamma_C)(id_C) & \xrightarrow{(P(f)\circ\gamma_g)(id_C)} & (P(f) \circ \gamma_B)(id_C \circ g) & \xrightarrow{(P(f)\circ\gamma_B)(\lambda_g)} & (P(f) \circ \gamma_B)(g) & \xrightarrow{(P(f)\circ\gamma_B)(\rho_g^{-1})} & (P(f) \circ \gamma_B)(g \circ id_B) \\
\Big\downarrow{\scriptstyle \mu^P_{g,f}(\gamma_C(id_C))} & & & & & & \Big\downarrow{\scriptstyle \gamma_f(g\circ id_B)} \\
(P(g \circ f) \circ \gamma_C)(id_C) & & & & & & \gamma_A((g \circ id_B) \circ f) \\
\Big\downarrow{\scriptstyle \gamma_{g\circ f}(id_C)} & & & & & & \Big\downarrow{\scriptstyle \gamma_A(\alpha_{g,id_B,f})} \\
\gamma_A(id_C \circ (g \circ f)) & & & & & & \gamma_A(g \circ (id_B \circ f)) \\
\Big\downarrow{\scriptstyle \gamma_A(\lambda_{g\circ f})} & & & & & & \Big\downarrow{\scriptstyle \gamma_A(g\circ\lambda_f)} \\
\gamma_A(g \circ f) & \xrightarrow[\gamma_A(\rho^{-1}_{g\circ f})]{} & \gamma_A((g \circ f) \circ id_A) & \xleftarrow[\gamma_A(\alpha^{-1}_{g,f,id_A})]{} & \gamma_A(g \circ (f \circ id_A)) & \xleftarrow[\gamma_A(g\circ\rho_f^{-1})]{} & \gamma_A(g \circ f)
\end{array}
$$

whose bottom edge is the identity. This enable us to put it in the form of the diagram

$$
\begin{array}{ccccc}
(P(f)\circ P(g)\circ\gamma_C)(id_C) & \xrightarrow{(P(f)\circ\gamma_g)(id_C)} & (P(f)\circ\gamma_B)(id_C\circ g) & \xrightarrow{(P(f)\circ\gamma_B)(\lambda_g)} & (P(f)\circ\gamma_B)(g) \\
{\scriptstyle \mu^P_{g,f}(\gamma_C(id_C))}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma_f(id_C\circ g)} & & \Big\downarrow{\scriptstyle (P(f)\circ\gamma_B)(\rho_g^{-1})} \\
(P(g\circ f)\circ\gamma_C)(id_C) & & \gamma_A((id_C\circ g)\circ f) & & (P(f)\circ\gamma_B)(g\circ id_B) \\
{\scriptstyle \gamma_{g\circ f}(id_C)}\Big\downarrow & \swarrow{\scriptstyle \gamma_A(\alpha_{id_C,g,f})} & \Big\downarrow{\scriptstyle \gamma_A(\lambda_g\circ f)} & & \Big\downarrow{\scriptstyle \gamma_f(g\circ id_B)} \\
\gamma_A(id_C\circ(g\circ f)) & \xrightarrow[\gamma_A(\lambda_{g\circ f})]{} & \gamma_A(g\circ f) & \xleftarrow[\gamma_A(\rho_g\circ f)]{} & \gamma_A((g\circ id_B)\circ f)
\end{array}
$$

and this diagram commutes since the pentagon is the coherence

$$
\begin{array}{ccc}
(P(f)\circ P(g)\circ\gamma_C)(id_C) & \xrightarrow{(P(f)\circ\gamma_g)(id_C)} & (P(f)\circ\gamma_B)(id_C\circ g) \\
{\scriptstyle (\mu^P_{g,f}\circ\gamma_C)(id_C)}\Big\downarrow & & \Big| \\
(P(g\circ f)\circ\gamma_C)(id_C) & & \Big\downarrow{\scriptstyle \gamma_f(id_C\circ g)} \\
{\scriptstyle (\gamma_{g\circ f})(id_C)}\Big\downarrow & & \\
\gamma_A(id_C\circ(g\circ f)) & \xleftarrow[\gamma_A(\alpha_{id_C,g,f})]{} & \gamma_A(id_C\circ g)\circ f)
\end{array}
$$

for the pseudo natural transformation $\gamma\colon \mathcal{Y}_C \to P$, evaluated by the object id_C in $\mathcal{Y}_C(C)$. The commutativity of the triangle is the consequence of the triangle coherence for left and right identities and the hexagon is the deformed square expressing the naturality of $\gamma_f\colon P(f)\circ\gamma_B \Rightarrow \gamma_A\circ\mathcal{Y}_C(f)$ for the 2-morphism $\rho_g^{-1}\circ\lambda_g\colon id_C\circ g \Rightarrow g\circ id_B$. □

The assertion of the theorem means that each component functor $\theta_C\colon \mathcal{S}P \to P(C)$ of the pseudo natural transformation is an equivalence and that each component natural transformation $\theta_h\colon P(h)\circ\theta_D \Rightarrow \theta_C\circ\mathcal{S}P(h)$ is a natural isomorphism. In order to prove this, we will use the following result.

Lemma 3.1. *There exist a pseudo natural transformation $\omega\colon P \to \mathcal{S}P$ in $\mathcal{P}$, which is a weak inverse to $\theta\colon \mathcal{S}P \to P$.*

Proof. The component of the pseudo natural transformation $\omega\colon P \to \mathcal{S}P$ indexed by the object $D \in \mathcal{B}$, is the functor $\omega_D\colon P(D) \to \mathcal{S}P(D)$, which sends any object $Z \in P(D)$ to the pseudo natural transformation $\omega_D^Z\colon \mathcal{Y}_D \to P$, whose component $(\omega_D^Z)_C\colon \mathcal{Y}_D(C) \to P(C)$ indexed by the object $C \in \mathcal{B}$, is a functor which sends any 1-morphism $h\colon C \to D$ (viewed as an object of the category $\mathcal{B}(C,D)$), to the object $(\omega_D^Z)_C(h) := P(h)(Z)$ in the category $P(C)$, and which sends any 2-morphism $\epsilon\colon h \to h'$ (viewed as a morphism of the category $\mathcal{B}(C,D)$), to a morphism $(\omega_D^Z)_C(\epsilon) := P(\epsilon)_Z\colon P(h)(Z) \to P(h')(Z)$ in $P(C)$, which is just the component indexed by the object $Z \in P(D)$ of the natural transformation $P(\epsilon)\colon P(h) \to P(h')$ between the two functors $P(h), P(h')\colon P(D) \to P(C)$. The component of the pseudo natural transformation $\omega_D^Z\colon \mathcal{Y}_D \to P$ indexed by the morphism $g\colon B \to C$ in $\mathcal{B}$, is the natural transformation given by the diagram

$$\begin{array}{ccc} \mathcal{Y}_D(C) & \xrightarrow{(\omega_D^Z)_C} & P(C) \\ {\scriptstyle \mathcal{Y}_D(g)}\big\downarrow & \swarrow_{(\omega_D^Z)_g} & \big\downarrow{\scriptstyle P(g)} \\ \mathcal{Y}_D(B) & \xrightarrow[(\omega_D^Z)_B]{} & P(B) \end{array}$$

whose component indexed by the 1-morphism $h\colon C \to D$ (viewed as an object of the category $\mathcal{B}(C,D)$), is the morphism $(\omega_D^Z)_g(h)\colon (P(g)\circ(\omega_D^Z)_C)(h) \to ((\omega_D^Z)_B \circ \mathcal{Y}_D(g))(h)$ in the category $P(B)$, defined by $(\omega_D^Z)_g(h) := (\mu^P_{h,g})_Z\colon (P(g)\circ P(h))(Z) \to P(h\circ g)(Z)$. For any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$ of 1-morphisms in $\mathcal{B}$, the coherence diagram for the pseudo natural transformation $\omega_D^Z\colon \mathcal{Y}_D \to P$

$$\begin{array}{ccc} P(f)\circ P(g)\circ(\omega_D^Z)_C & \xrightarrow{P(f)\circ(\omega_D^Z)_g} & P(f)\circ(\omega_D^Z)_B\circ\mathcal{Y}_D(g) \\ {\scriptstyle \mu^P_{g,f}\circ(\omega_D^Z)_C}\big\downarrow & & \\ P(g\circ f)\circ(\omega_D^Z)_C & & \big\downarrow{\scriptstyle (\omega_D^Z)_f\circ\mathcal{Y}_D(g)} \\ {\scriptstyle (\omega_D^Z)_{g\circ f}}\big\downarrow & & \\ (\omega_D^Z)_A\circ\mathcal{Y}_D(g\circ f) & \xleftarrow[(\omega_D^Z)_A\circ\mu^{\mathcal{Y}_D}_{g,f}]{} & (\omega_D^Z)_A\circ\mathcal{Y}_D(f)\circ\mathcal{Y}_D(g) \end{array}$$

evaluated by the morphism $h\colon C \to D$ in $\mathcal{B}$, gives a diagram

$$\begin{array}{ccc} (P(f)\circ P(g)\circ(\omega_D^Z)_C)(h) & \xrightarrow{(P(f)\circ(\omega_D^Z)_g)(h)} & (P(f)\circ(\omega_D^Z)_B\circ\mathcal{Y}_D(g))(h) \\ {\scriptstyle (\mu^P_{g,f}\circ(\omega_D^Z)_C)(h)}\big\downarrow & & \\ (P(g\circ f)\circ(\omega_D^Z)_C)(h) & & \big\downarrow{\scriptstyle ((\omega_D^Z)_f\circ\mathcal{Y}_D(g))(h)} \\ {\scriptstyle ((\omega_D^Z)_{g\circ f})(h)}\big\downarrow & & \\ ((\omega_D^Z)_A\circ\mathcal{Y}_D(g\circ f))(h) & \xleftarrow[((\omega_D^Z)_A\circ\mu^{\mathcal{Y}_D}_{g,f})(h)]{} & ((\omega_D^Z)_A\circ\mathcal{Y}_D(f)\circ\mathcal{Y}_D(g))(h) \end{array}$$

which is equal to the coherence diagram

$$\begin{array}{ccc}
(P(f)\circ P(g)\circ P(h))(Z) & \xrightarrow{(P(f)\circ\mu^P_{h,g})_Z} & (P(f)\circ P(h\circ g))(Z) \\
\downarrow{\scriptstyle (\mu^P_{g,f}\circ P(h))_Z} & & \\
(P(g\circ f)\circ P(h))(Z) & & \downarrow{\scriptstyle (\mu^P_{h,g\circ f})_Z} \\
\downarrow{\scriptstyle (\mu^P_{h,g\circ f})_Z} & & \\
P(h\circ(g\circ f))(Z) & \xleftarrow[P(\alpha_{h,g,f})_Z]{} & P((h\circ g)\circ f)(Z)
\end{array}$$

for the 2-presheaf $P\colon \mathcal{B}^{op}\to \mathrm{Cat}$, evaluated by the object Z in $P(D)$.

The functor $\omega_D\colon P(D)\to \mathcal{S}P(D)$ sends any morphism $z\colon Z\to Z'$ in the category $P(D)$ to the modification $\omega^z_D\colon \omega^Z_D\to\omega^{Z'}_D$ whose component indexed by the object $C\in\mathcal{B}$ is a natural transformation $(\omega^z_D)_C\colon (\omega^Z_D)_C\Rightarrow(\omega^{Z'}_D)_C$. The component of the natural transformation indexed by the 1-morphism $h\colon C\to D$ (viewed as an object of the category $\mathcal{B}(C,D)$), is the morphism $(\omega^Z_D)_C(h)\colon(\omega^Z_D)_C(h)\to(\omega^{Z'}_D)_C(h)$ in the category $P(C)$, given by $P(h)(z)\colon P(h)(Z)\to P(h)(Z')$. The coherence for this modification

$$\begin{array}{ccc}
P(g)\circ(\omega^Z_D)_C & \xrightarrow{P(g)\circ(\omega^z_D)_C} & P(g)\circ(\omega^{Z'}_D)_C \\
\downarrow{\scriptstyle (\omega^Z_D)_g} & & \downarrow{\scriptstyle (\omega^{Z'}_D)_g} \\
(\omega^Z_D)_B\circ\mathcal{Y}_D(g) & \xrightarrow[(\omega^z_D)_B\circ\mathcal{Y}_D(g)]{} & (\omega^{Z'}_D)_B\circ\mathcal{Y}_D(g)
\end{array}$$

evaluated by the morphism $h\colon C\to D$ gives a diagram

$$\begin{array}{ccc}
(P(g)\circ(\omega^Z_D)_C)(h) & \xrightarrow{(P(g)\circ(\omega^z_D)_C)(h)} & (P(g)\circ(\omega^{Z'}_D)_C)(h) \\
\downarrow{\scriptstyle ((\omega^Z_D)_g)(h)} & & \downarrow{\scriptstyle (\omega^{Z'}_D)_g(h)} \\
((\omega^Z_D)_B\circ\mathcal{Y}_D(g))(h) & \xrightarrow[((\omega^z_D)_B\circ\mathcal{Y}_D(g))(h)]{} & ((\omega^{Z'}_D)_B\circ\mathcal{Y}_D(g))(h)
\end{array}$$

which is equal to the naturality diagram

$$\begin{array}{ccc} (P(g)\circ P(h))(Z) & \xrightarrow{(P(g)\circ P(h))(z)} & (P(g)\circ P(h))(Z') \\ \Big\downarrow{\scriptstyle (\mu^P_{h,g})_Z} & & \Big\downarrow{\scriptstyle (\mu^P_{h,g})_{Z'}} \\ P(h\circ g)(Z) & \xrightarrow[P(h\circ g)(z)]{} & P(h\circ g)(Z') \end{array}$$

for the natural transformation $\mu^P_{h,g}\colon P(g)\circ P(h)\to P(h\circ g)$, and the morphism $z\colon Z\to Z'$ in the category $P(D)$.

The value of the pseudo natural transformation ω indexed by the morphism $h\colon C\to D$ is given by the natural transformation

$$\begin{array}{ccc} P(D) & \xrightarrow{\omega_D} & \mathcal{S}P(D) \\ \Big\downarrow{\scriptstyle P(h)} & \swarrow_{\omega_h} & \Big\downarrow{\scriptstyle \mathcal{S}P(h)} \\ P(C) & \xrightarrow[\omega_C]{} & \mathcal{S}P(C) \end{array}$$

whose component $\omega^Z_h\colon (\mathcal{S}P(h)\circ\omega_D)(Z)\to(\omega_C\circ P(h))(Z)$ indexed by the object $Z\in P(D)$, is a modification $\omega^Z_h\colon \omega^Z_D\circ\mathcal{Y}(h)\to\omega_C^{P(h)(Z)}$, and its component indexed by the object $B\in\mathcal{B}$ is the natural transformation $(\omega^Z_h)_B\colon(\omega^Z_D\circ\mathcal{Y}(h))_B\to(\omega_C^{P(h)(Z)})_B$. Its component $(\omega^Z_h)_B(g)\colon(\omega^Z_D\circ\mathcal{Y}(h))_B(g)\to(\omega_C^{P(h)(Z)})_B(g)$ indexed by the 1-morphism $g\colon B\to C$ (viewed as an object of the category $\mathcal{B}(B,C)$), is a morphism given by the inverse of component of the coherence $(\omega^Z_h)_B(g):=(\mu^P_{h,g})^{-1}_Z\colon P(h\circ g)(Z)\to(P(g)\circ P(h))(Z)$ for the composition.

The composition $\omega\circ\theta\colon\mathcal{S}P\to\mathcal{S}P$ is a pseudo natural transformation whose component indexed by any object D in $\mathcal{B}$, is a functor $(\omega\circ\theta)_D\colon\mathcal{S}P(D)\to\mathcal{S}P(D)$ which takes any pseudo natural transformation $\phi\colon\mathcal{Y}_D\to P$ to the pseudo natural transformation $(\omega_D\circ\theta_D)(\phi)=\omega_D(\phi_D(id_D))=\omega_D^{\phi_D(id_D)}\colon\mathcal{Y}_D\to P$. Its component $(\omega_D^{\phi_D(id_D)})_C\colon\mathcal{Y}_D(C)\to P(C)$ indexed by the object $C\in\mathcal{B}$, is a functor which sends any 1-morphism $h\colon C\to D$ (viewed as an object of the category $\mathcal{B}(C,D)$), to the object $(\omega_D^{\phi_D(id_D)})_C(h)=P(h)(\phi_D(id_D))=(P(h)\circ\phi_D)(id_D)$ in the category $P(C)$, and which sends any 2-morphism $\epsilon\colon h\to h'$ to a morphism $(\omega_D^{\phi_D(id_D)})_C(\epsilon)=P(\epsilon)_{\phi_D(id_D)}\colon P(h)(\phi_D(id_D))\to P(h')(\phi_D(id_D))$ in $P(C)$, which is just the component indexed by the object $id_D\in\mathcal{Y}_D(D)$ of the natural transformation $P(\epsilon)\circ\phi_D\colon P(h)\circ\phi_D\Rightarrow P(h')\circ\phi_D$. The value of the functor $(\omega\circ\theta)_D\colon\mathcal{S}P(D)\to\mathcal{S}P(D)$ for any modification $\Omega\colon \phi \to \psi$ in $\mathcal{S}P(D)$ is the modification $(\omega_D \circ \theta_D)(\Omega) = \omega_D(\Omega_C(id_D)) = \omega_D^{\Omega_D(id_D)}\colon \omega_D^{\phi_D(id_D)} \to \omega_D^{\psi_D(id_D)}$, whose component indexed by the object $C \in \mathcal{B}$ is a natural transformation $(\omega_D^{\Omega_D(id_D)})_C\colon (\omega_D^{\phi_D(id_D)})_C \to (\omega_D^{\psi_D(id_D)})_C$. The component of this natural transformation indexed by the 1-morphism $h\colon C \to D$ (viewed as an object of the category $\mathcal{B}(C, D)$), is the morphism $(\omega_D^{\Omega_D(id_D)})_C(h)\colon (\omega_D^{\phi_D(id_D)})_C(h) \to (\omega_D^{\psi_D(id_D)})_C(h)$ in the category $P(C)$, given by $P(h)(\Omega_D(id_D))\colon P(h)(\phi_D(id_D)) \to P(h)(\psi_D(id_D))$.

The component of the pseudo natural transformation $\omega \circ \theta\colon \mathcal{S}P \to \mathcal{S}P$ indexed by the 1-morphism $h\colon C \to D$ in $\mathcal{B}$, is the natural transformation $(\omega \circ \theta)_h$ given by the diagram

$$\begin{array}{ccc} \mathcal{S}P(D) & \xrightarrow{(\omega\circ\theta)_D} & \mathcal{S}P(D) \\ {\scriptstyle \mathcal{S}P(h)}\downarrow & \swArrow_{(\omega\circ\theta)_h} & \downarrow{\scriptstyle \mathcal{S}P(h)} \\ \mathcal{S}P(C) & \xrightarrow[(\omega\circ\theta)_C]{} & \mathcal{S}P(C) \end{array}$$

and defined by the composition $(\omega_C \circ \theta_h)(\omega_h \circ \theta_D)\colon \mathcal{S}P(h) \circ (\omega \circ \theta)_D \to (\omega \circ \theta)_C \circ \mathcal{S}P(h)$. The component indexed by the pseudo natural transformation $\phi\colon \mathcal{Y}_D \to P$ is the modification

$$(\omega \circ \theta)_h(\phi) = [(\omega_C \circ \theta_h)(\omega_h \circ \theta_D)](\phi) = [\omega_C \circ \theta_h](\phi)[\omega_h \circ \theta_D](\phi) = \omega_C^{\phi_C(\rho_h^{-1} \circ \lambda_h)\phi_h(id_D)} \omega_h^{\phi_D(id_D)}$$

between pseudo natural transformation $\omega_D^{\phi_D(id_D)} \circ \mathcal{Y}(h)$ and $\omega_C^{(\phi \circ \mathcal{Y}(h))_C(id_C)}$ in $\mathcal{S}P(C)$. The component of the modification $(\omega \circ \theta)_h(\phi)_B\colon \mathcal{Y}_C(B) \to P(B)$ indexed by the object B in $\mathcal{B}$, is natural transformation $(\omega \circ \theta)_h(\phi)_B\colon (\omega_D^{\phi_D(id_D)} \circ \mathcal{Y}(h))_B \to (\omega_C^{(\phi \circ \mathcal{Y}(h))_C(id_C)})_B$. The equality

$$\begin{aligned}(\omega \circ \theta)_h(\phi)_B(g) &= (\omega_C^{\phi_C(\rho_h^{-1}\lambda_h)} \omega_C^{\phi_h(id_D)} \omega_h^{\phi_D(id_D)})_B(g) = [(\omega_C^{\phi_C(\rho_h^{-1}\lambda_h)})_B(\omega_C^{\phi_h(id_D)})_B(\omega_h^{\phi_D(id_D)})_B](g) = \\ &= [(\omega_C^{\phi_C(\rho_h^{-1}\lambda_h))})_B](g)[(\omega_C^{\phi_h(id_D)})_B](g)[(\omega_h^{\phi_D(id_D)})_B](g) = P(g)(\phi_C(\rho_h^{-1}\lambda_h)\phi_h(id_D))(\mu_{h,g}^P)^{-1}_{\phi_D(id_D)}\end{aligned}$$

gives the component of the natural transformation $(\omega \circ \theta)_h(\phi)_B$ indexed by $g\colon B \to C$

$$(\omega \circ \theta)_h(\phi)_B(g)\colon (\omega_D^{\phi_D(id_D)} \circ \mathcal{Y}(h))_B(g) \to (\omega_C^{(\phi \circ \mathcal{Y}(h))_C(id_C)})_B(g)$$

which is given by the composition

$$[P(g)(\phi_h(id_D))][(\mu_{h,g}^P)^{-1}_{\phi_D(id_D)}]\colon P(h \circ g)(\phi_D(id_D)) \to P(g)(\phi_C(h \circ id_C))$$

of morphisms in the category $P(B)$. $\square$

Lemma 3.2. *There exists an invertible modification* $\Upsilon\colon \omega\circ\theta \to \iota$ *to the identity pseudo natural transformation* $\iota\colon \mathcal{S}P \to \mathcal{S}P$*.*

Proof. Its component indexed by the object D in $\mathcal{B}$ is a natural transformation $\Upsilon_D\colon (\omega\circ\theta)_D \to \iota_D$ whose component indexed by the pseudo natural transformation $\phi\colon \mathcal{Y}_D \to P$ is a modification $\Upsilon_D(\phi)\colon \omega_D^{\phi_D(id_D)} \to \phi$. Its component indexed by the object C in $\mathcal{B}$ is the natural transformation $\Upsilon_D(\phi)_C\colon (\omega_D^{\phi_D(id_D)})_C \to \phi_C$, such that morphism

$$[\Upsilon_D(\phi)_C](h)\colon P(h)(\phi_D(id_D)) \to \phi_C(h)$$

defines which is its component indexed by the morphism $h\colon C \to D$, is defined by

$$[\Upsilon_D(\phi)_C](h) := \phi_C(\lambda_h)\phi_h(id_D)$$

.

The coherence for modification $\Upsilon\colon \omega\circ\theta \to \iota$

$$\begin{array}{ccc} \mathcal{S}P(h)\circ(\omega\circ\theta)_D & \xrightarrow{\mathcal{S}P(h)\circ\Upsilon_D} & \mathcal{S}P(h)\circ\iota_D \\ \Big\downarrow{\scriptstyle (\omega\circ\theta)_h} & & \Big\downarrow{\scriptstyle \iota_h} \\ (\omega\circ\theta)_C\circ\mathcal{S}P(h) & \xrightarrow[\Upsilon_C\circ\mathcal{S}P(h)]{} & \iota_C\circ\mathcal{S}P(h) \end{array}$$

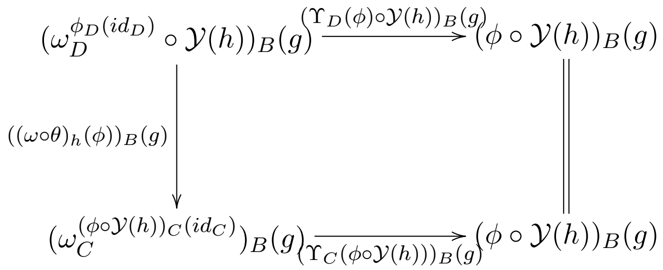
is satisfied since by evaluating the above diagram for any $\phi\colon \mathcal{Y}_D \to P$, the diagram

$$\begin{array}{ccc} (\mathcal{S}P(h)\circ(\omega\circ\theta)_D)(\phi) & \xrightarrow{(\mathcal{S}P(h)\circ\Upsilon_D)(\phi)} & (\mathcal{S}P(h)\circ\iota_D)(\phi) \\ \Big\downarrow{\scriptstyle (\omega\circ\theta)_h(\phi)} & & \Big\downarrow{\scriptstyle \iota_h(\phi)} \\ (\omega\circ\theta)_C\circ\mathcal{S}P(h))(\phi) & \xrightarrow[(\Upsilon_C\circ\mathcal{S}P(h))(\phi)]{} & (\iota_C\circ\mathcal{S}P(h))(\phi) \end{array}$$

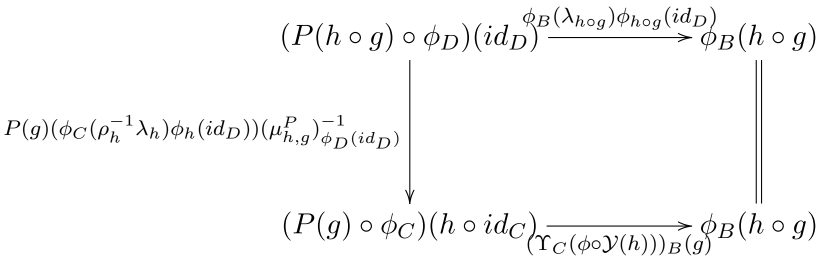
becomes the diagram of modifications

$$\begin{array}{ccc} \omega_D^{\phi_D(id_D)}\circ\mathcal{Y}(h) & \xrightarrow{\Upsilon_D(\phi)\circ\mathcal{Y}(h)} & \phi\circ\mathcal{Y}(h) \\ \Big\downarrow{\scriptstyle (\omega\circ\theta)_h(\phi)} & & \Big\| \\ \omega_C^{(\phi\circ\mathcal{Y}(h))_C(id_C)} & \xrightarrow[\Upsilon_C(\phi\circ\mathcal{Y}(h))]{} & \phi\circ\mathcal{Y}(h) \end{array}$$

Evaluating this diagram by the morphism $g\colon B \to C$



we obtain the diagram

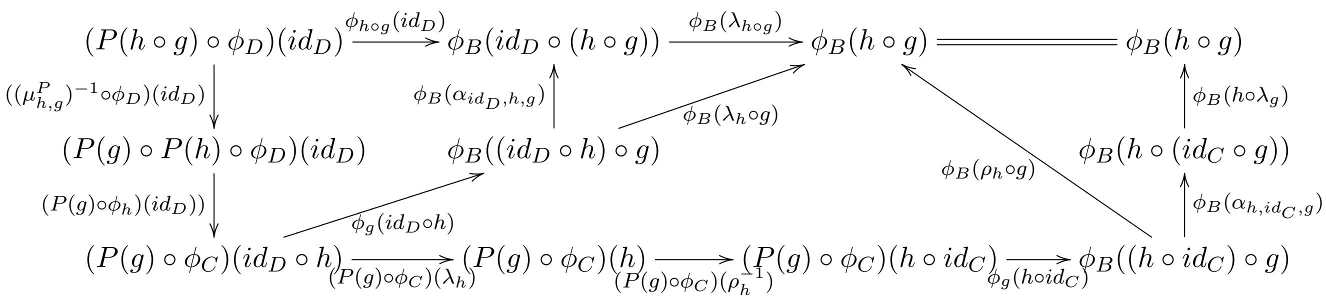


Since we have the identities

$$[\Upsilon_D(\phi)_C](h) := \phi_C(\lambda_h)\phi_h(id_D)$$

$$(\Upsilon_C(\phi\circ\mathcal{Y}(h)))_B(g) = (\phi\circ\mathcal{Y}(h))_B(\lambda_g)(\phi\circ\mathcal{Y}(h))_g(id_C) = \phi_B(h\circ\lambda_g)(\phi_B\circ\mathcal{Y}(h)_g)(id_C)(\phi_g\circ\mathcal{Y}(h)_C)(id_C) =$$
$$= \phi_B(h\circ\lambda_g)\phi_B(\alpha_{h,id_C,g})\phi_g(h\circ id_C)$$

the last diagram is equal to the edge of the diagram



in which the first pentagonal diagram commute since it is the coherence diagram for the pseudo natural transformation $\phi\colon \mathcal{Y}_D \to P$

$$
\begin{array}{ccc}
(P(g)\circ P(h)\circ\phi_D)(id_D) & \xrightarrow{(P(g)\circ\phi_h)(id_D)} & (P(g)\circ\phi_C)(id_D\circ h) \\
\Big\downarrow{\scriptstyle (\mu^P_{h,g}\circ\phi_D)(id_D)} & & \\
(P(h\circ g)\circ\phi_D)(id_D) & & \Big\downarrow{\scriptstyle \phi_g(id_D\circ h)} \\
\Big\downarrow{\scriptstyle \phi_{h\circ g}(id_D)} & & \\
\phi_B(id_D\circ(h\circ g)) & \xleftarrow[\phi_B(\alpha_{id_D,h,g})]{} & \phi_B((id_D\circ h)\circ g)
\end{array}
$$

evaluated by the id_D. The commutativity of two triangle diagrams is the consequence of the triangle coherence for left and right identities and the hexagonal diagram is the deformed square expressing the naturality of $\phi_g\colon P(g)\circ\phi_C \Rightarrow \phi_B\circ\mathcal{Y}_D(g)$ for the 2-morphism $\rho_h^{-1}\lambda_h\colon id_D\circ h\Rightarrow h\circ id_C$. $\square$