Grothendieck construction for bicategories

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Abstract

In this article, we give the generalization of the Grothendieck construction for pseudo functors given in [5], which provides a biequivalence between the 2-categories of pseudofunctors and fibrations. We define the second Grothendieck construction which to any pseudo 2-functor associates a 2-fibration, defined by Hermida [6], and we prove the biequivalence between the 2-category of pseudo 2-functors and the 2-category of 2-fibrations.

1 Introduction

Throughout many areas of mathematics, we are repeatedly faced with two kinds of descriptions of mathematical objects of interest. First, such objects may be described by having certain property, and the formalism underlying this approach is given by the comprehension scheme of set theory, which says that given any property P, there is a set S, consisting exactly of the elements having the property P. The most advanced form of this formalism is the categorical comprehension scheme, given by Gray in [3], in order to avoid a clearly strong dependence of such approach on the axiomatization of the set theory. The second kind of description, is the one in which those properties are encoded as the part of the structure of mathematical objects. The transition between this two approaches is given by the axiom of choice. One of the central themes in the work of Grothendieck, deals with this paradigm, where objects are characterized by certain universal properties, and after the application of the axiom of choice, these properties give rise to coherence laws which become the part of the structure of derived objects.

To illustrate all this by an example, we consider the Grothendieck construction, given in [5], which for any category $\mathcal{C}$, provides the biequivalence

$$\int \colon PsF(\mathcal{C}) \to Fib(\mathcal{C})$$

between the 2-category $PsF(\mathcal{C})$ whose objects are pseudofunctors $P\colon \mathcal{C}^{op} \to Cat$, and the 2-category of fibrations (or fibered categories) $Fib(\mathcal{C})$ over $\mathcal{C}$, also introduced in [5]. The fibered category $F\colon \mathcal{E} \to \mathcal{C}$ is the functor, which is characterized by the property which associates to any pair (f, E), where $f\colon X \to Y$ is a morphism in $\mathcal{C}$ and E is an object in $\mathcal{E}$

such that $F(E) = X$, a cartesian morphism $\widetilde{f}\colon F \to E$, having certain universal property, such that $F(\widetilde{f}) = f$. After we choose a cartesian lifting for each such pair, their universal property gives rise to the coherence law for the natural transformations which are part of the data for the obtained pseudofunctor. Moreover, for any pseudofunctor $P\colon \mathcal{C}^{op} \to Cat$, this coherence laws are precisely responsible for the associativity and unit laws of the composition in the category $\int_{\mathcal{C}} P$ obtained from the Grothendieck construction.

In this article, we describe the generalization of the Grothendieck construction, for bicategories or weak 2-categories (in the language of Baez), which we therefore call the second Grothendieck construction. More precisely, for any category bicategory $\mathcal{B}$, we provide the triequivalence

$$\int\colon Ps2F(\mathcal{B}) \to 2Fib(\mathcal{B})$$

between the 3-category $Ps2F(\mathcal{B})$ whose objects are pseudo 2-functors $P\colon \mathcal{B}^{coop} \to 2-Cat$ (or equivalently homomorphisms of 3-categories given in [2]), and the 3-category of 2-fibrations (or fibered 2-categories) $2Fib(\mathcal{B})$ over $\mathcal{B}$. The objects of the 3-category $2Fib(\mathcal{B})$ are (weak) 2-functors $F\colon \mathcal{E} \to \mathcal{B}$, again characterized by the existence of cartesian liftings, now for both 1-morphisms and 2-morphisms. In the strict case, when both $\mathcal{E}$ and $\mathcal{B}$ are strict 2-categories, their definition is given by Hermida in [6]. Although he doesn't define explicitly 2-fibrations for bicategories, in his other article [7], he proposes their definition by using the coherence for the bireflection of bicategories and their homomorphisms (or weak 2-functors) into (strict) 2-categories and (strict) 2-functors. Here we give their explicit definition, using the properties of certain biadjunctions given by Gray, in his monumental work [4]. We prove that such 2-fibrations precisely arise by applying the second Grothendieck construction to a general pseudo 2-functor, and that strict 2-fibrations correspond to strict 2-functors 2-functors $P\colon \mathcal{B}^{coop} \to 2-Cat$.

One of the main motivations for this work, was to describe the second Grothendieck construction as the interpretation theorem for the third nonabelian cohomology $\mathcal{H}^3(\mathcal{B}, \mathcal{K})$ with the coefficients in the bundle of 2-groups $\mathcal{K}$ over the objects B_0 of the bigroupoid $\mathcal{B}$. In the case of groupoids, this approach was taken in [1], where the second nonabelian cohomology of groupoids, is defined by

$$\mathcal{H}^2(\mathcal{G}, \mathcal{K}) := [\mathcal{B}, AUT(\mathcal{K})]$$

in which $[\mathcal{G}, AUT(\mathcal{K})]$ is the set defined in terms of connected components of pseudofunctors $P\colon \mathcal{G}^{op} \to AUT(\mathcal{K})$ to the full sub 2-groupoid of the 2-groupoid Gpd of groupoids, induced naturally be the bundle of groups $\mathcal{K}$ over the objects G_0, of the groupoid $\mathcal{G}$. The Grothendieck construction here is an interpretation theorem for the classification of the groupoid fibrations, seen as short exact sequences of groupoids

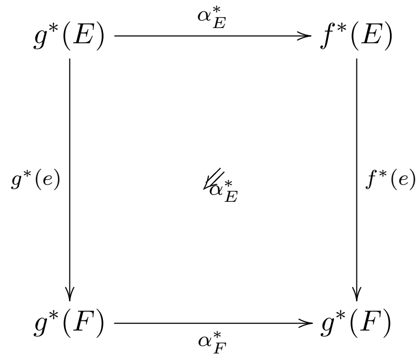
$$1 \longrightarrow K \longrightarrow G \longrightarrow B \longrightarrow 1$$

over the same underlying set of objects M, by means of the higher Schreier theory.

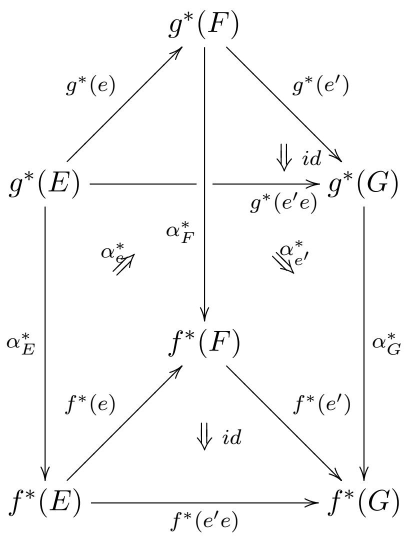
2 The 2-category of pseudo 2-functors

Definition 2.1. *A pseudo 2-functor is a weak 3-functor* $\mathcal{F}\colon \mathcal{B}^{coop} \to 2 - Cat$ *from the (strict) 2-category* $\mathcal{B}^{coop}$ *to the (strict) 3-category 2-Cat. It consists of the following data:*

- *a 2-category* $F(x)$*, for every object* x *in* $\mathcal{B}$*, which we also write* F_x*,*
- *a (weak) 2-functor* $F(f)\colon F(y) \to F(x)$*, for any 1-morphism* $f\colon x \to y$ *in* $\mathcal{B}$*, which we abbreviate by* $f^*\colon F_y \to F_x$*,*
- *a pseudo natural transformation* $F(\alpha)\colon F(g) \to F(f)$*, for any 2-morphism* $\alpha\colon f \Rightarrow g$ *in* $\mathcal{B}$*, which we also write as* $\alpha^*\colon g^* \to f^*$*, whose components are described by the square*



for any 1-morphism $e\colon E \to F$ *in* F_y*, whose coherence is given by the diagram*



for any composable pair of 1-morphisms $E \xrightarrow{e} F \xrightarrow{e'} G$ *in the 2-category* F_y*,*

- *pseudo natural transformation*

$$\begin{array}{ccc} \mathcal{B}_2 & \xrightarrow{F_2} & 2\mathcal{C}_2 \\ {\scriptstyle H}\Big\downarrow & \swarrow\!\!\!\!\swarrow_{\chi} & \Big\downarrow{\scriptstyle \otimes} \\ \mathcal{B}_1 & \xrightarrow[F_1]{} & 2\mathcal{C}_1 \end{array}$$

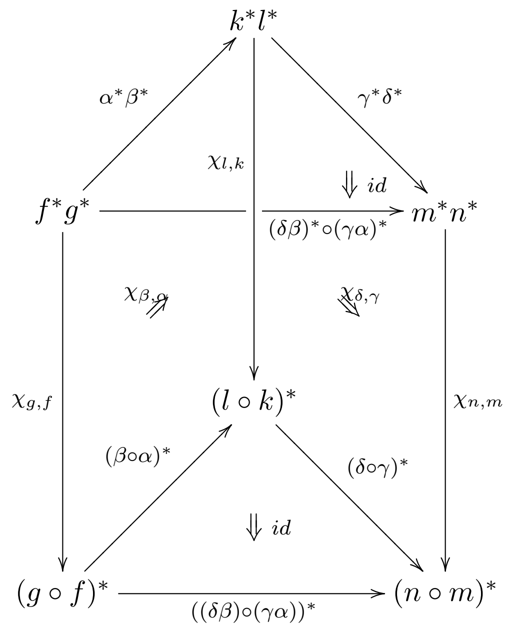
where $\mathcal{B}_2$ denotes the disjoint union of products of categories $\mathcal{B}(y,z) \times \mathcal{B}(x,y)$, indexed by all triples of objects (x,y,z) in $\mathcal{B}$, and the 2-functor $F_2 \colon \mathcal{B}_2 \to 2\mathcal{C}_2$ sends each product category $\mathcal{B}(y,z) \times \mathcal{B}(x,y)$ to the product of 2-categories $[F(y),F(z)] \times [F(x),F(y)]$, of 2-functors, pseudo natural transformations and modifications. We denoted by $\otimes \colon 2\mathcal{C}_2 \to 2\mathcal{C}_1$ the composition of 2-functors, which restricts to the 2-functor $\otimes \colon [F(y),F(z)] \times [F(x),F(y)] \to [F(x),F(z)]$, for any triple (x,y,z) of objects in $\mathcal{B}$. Thus, for any object in $\mathcal{B}_2$, which is just composable pair $x \xrightarrow{f} y \xrightarrow{g} z$ of 1-morphisms in $\mathcal{B}$, we have a component $\chi_{g,f} \colon f^*g^* \Rightarrow (gf)^*$ of the pseudo natural transformation χ, which is a 1-morphism in $2\mathcal{C}_1$, that is again a pseudo natural transformation. Also, for any morphism in $\mathcal{B}_2$, which is a just a horizontally composable pair

$$x \; \underset{k}{\overset{f}{\Downarrow\alpha}} \; y \; \underset{l}{\overset{g}{\Downarrow\beta}} \; z$$

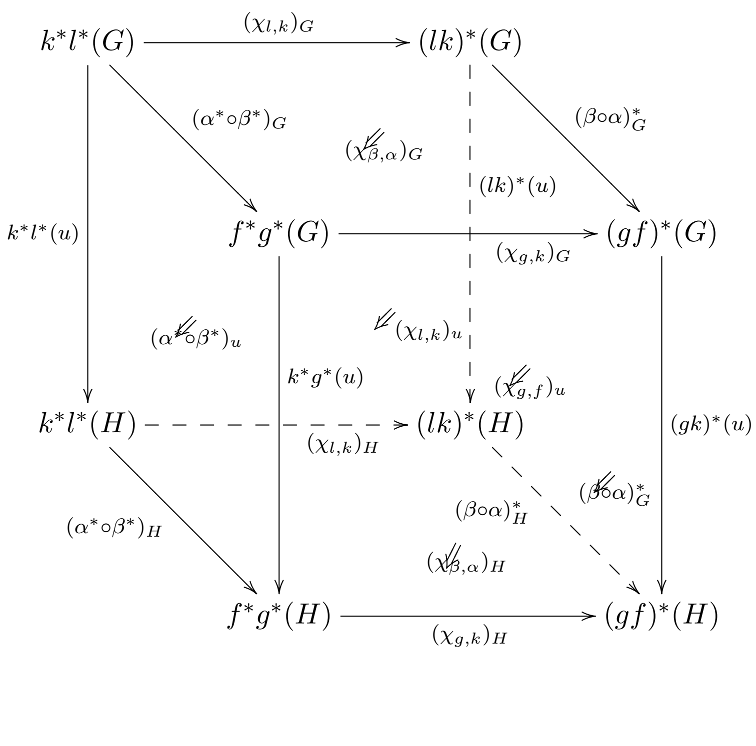
of 2-morphisms in $\mathcal{B}$, we have a component $\chi_{\beta,\alpha} \colon f^*g^* \Longrightarrow (gf)^*$ of the pseudo natural transformation χ, which is a 2-morphism in $2\mathcal{C}_1$, that is a modification. Thus, the data for the pseudo natural transformation χ, are given by the square

$$\begin{array}{ccc} k^*l^* & \xrightarrow{\chi_{l,k}} & (lk)^* \\ {\scriptstyle \alpha^*\beta^*}\Big\downarrow & \swarrow\!\!\!\!\swarrow_{\chi_{\beta,\alpha}} & \Big\downarrow{\scriptstyle (\beta\circ\alpha)^*} \\ f^*g^* & \xrightarrow[\chi_{g,f}]{} & (gf)^* \end{array}$$

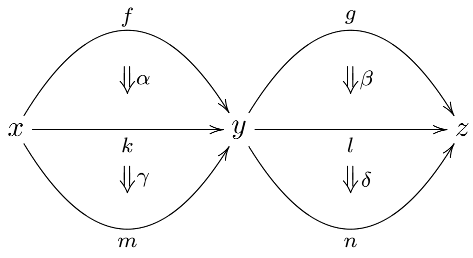
The coherence for the pseudo natural transformation χ is given by the diagram



and the coherence for the modification is given by the commutative cube

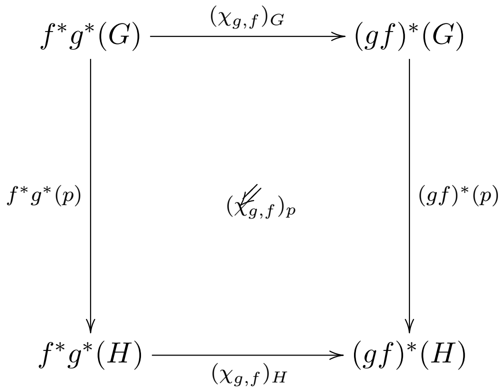


for any vertically composable pair of 2-morphisms

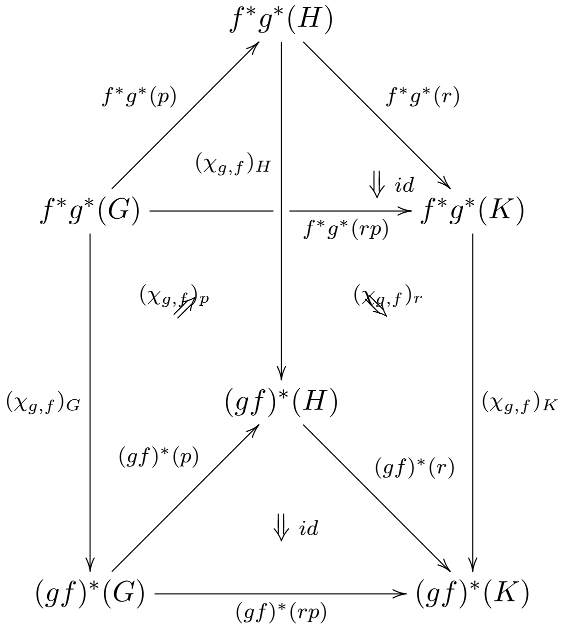


in the 2-category $\mathcal{B}_2$.

It has components given by the square

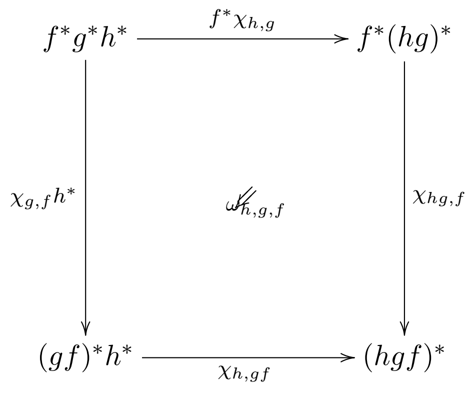


for any 1-morphism $p\colon G \to H$ *in the 2-category* F_z, *whose coherence is given by the* *diagram*

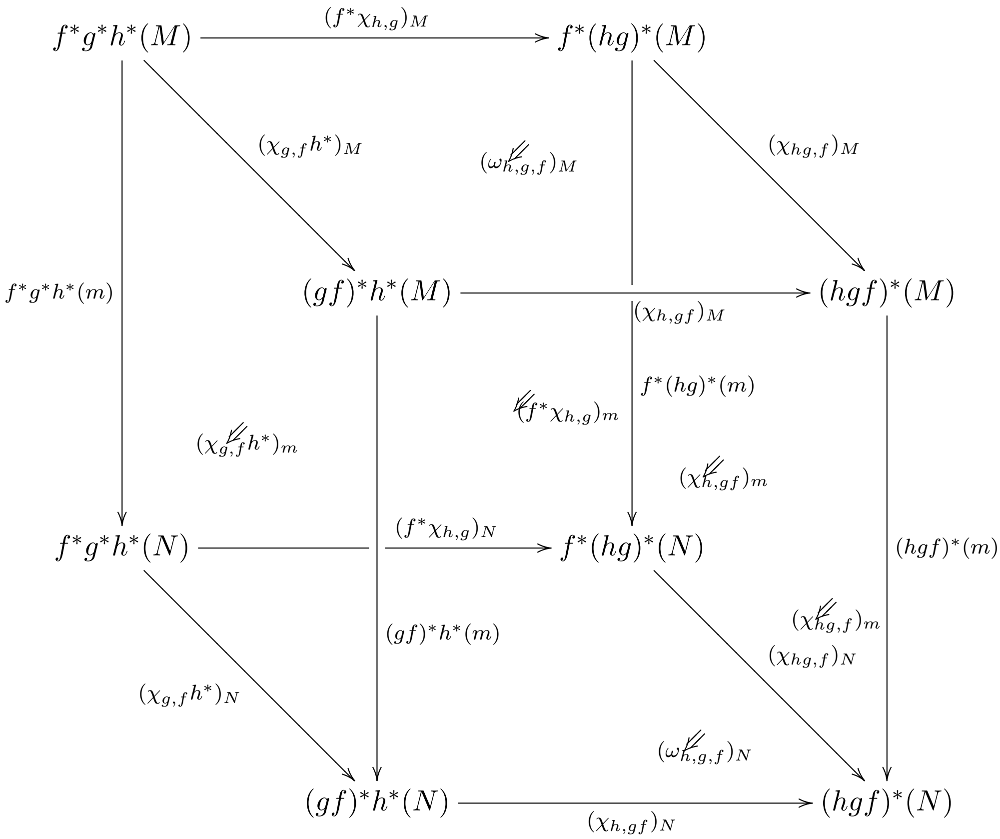


for any composable pair of 1-morphisms $H \xrightarrow{p} G \xrightarrow{r} K$ *in the 2-category* F_z*, which means that the pasting of upper two squares is equal to the bottom square given by the component* $(\chi_{g,f})_{rp}\colon (\chi_{g,f})_K f^*g^*(rp) \Rightarrow (\chi_{g,f})_G (gf)^*(rp)$ *(which we omitted from the diagram too avoid too many labels), since the two triangles are given just by identity 2-morphisms corresponding to two different strict 2-functors,*

- *for any composable triple* $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ *in* $\mathcal{B}$*, a modification*

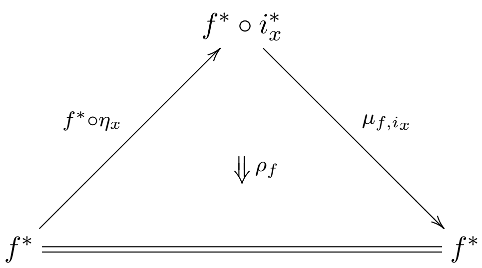


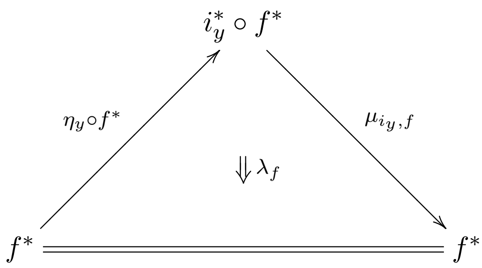
whose coherence is given by the commutative cube



for any 1-morphism $m\colon M \to N$ in the 2-category F_w,

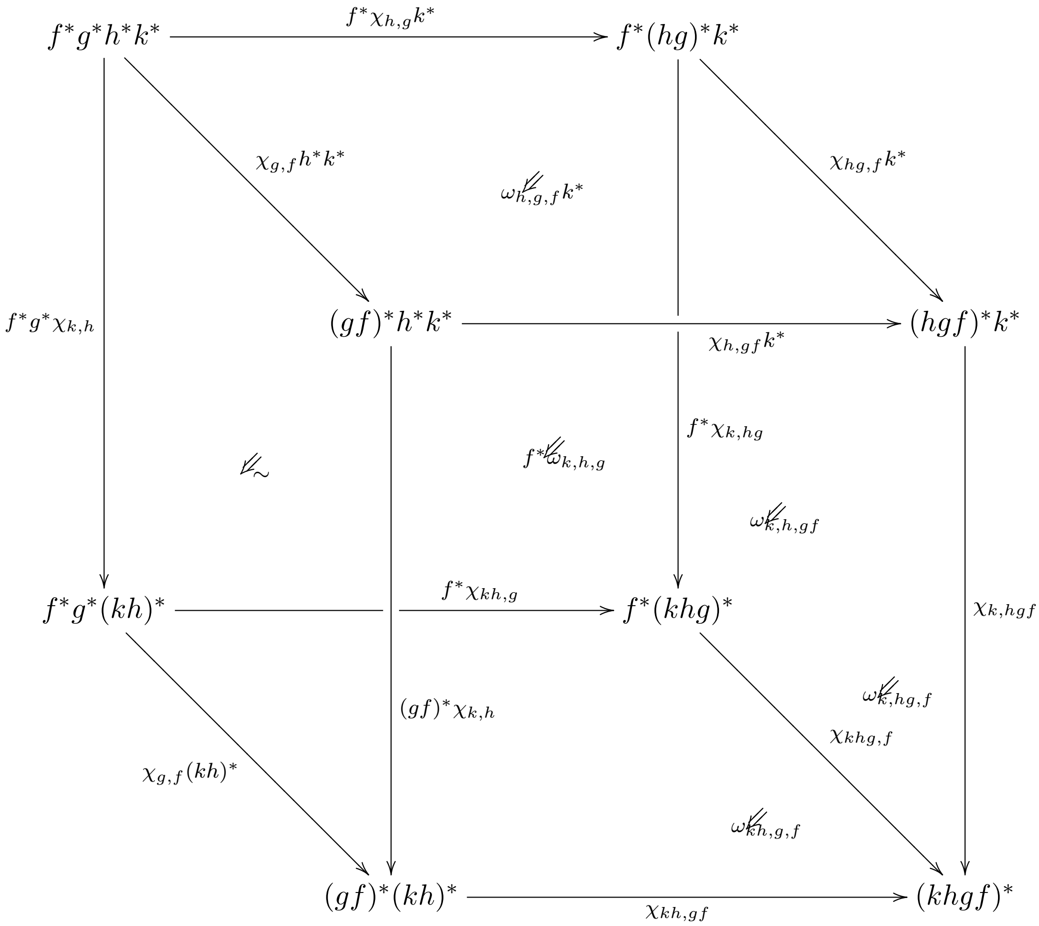
- *for any 1-morphism $f\colon x \to y$ in $\mathcal{B}$, two modifications*



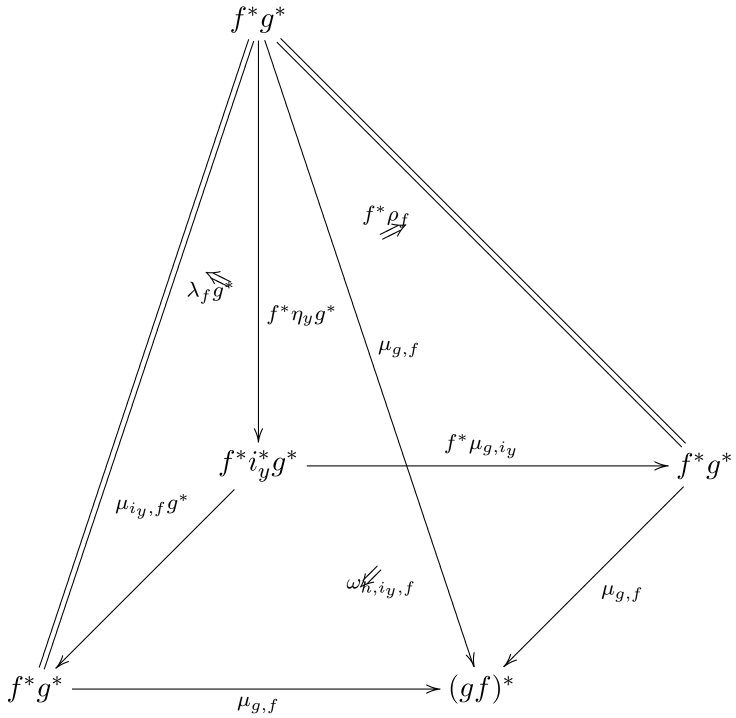


such that following axioms are satisfied:

- *the 3-cocycle condition given by a commutative cube*



- *commutative pyramid*


f^*g^*
$\lambda_f g^*$
$f^*\eta_y g^*$
$f^*\rho_f$
$\mu_{g,f}$
$f^*i_y^*g^*$
$f^*\mu_{g,i_y}$
f^*g^*
$\mu_{i_y,f}g^*$
$\omega_{h,i_y,f}$
$\mu_{g,f}$
f^*g^*
$\mu_{g,f}$
$(gf)^*$

3 2-fibrations

The following definition of biadjunction between two weak 2-functors is taken from [8].

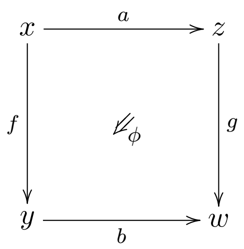
Definition 3.1. *A weak 2-functor* $F\colon \mathcal{B} \to \mathcal{C}$ *is called a left biadjoint to a weak 2-functor* $G\colon \mathcal{C} \to \mathcal{B}$ *if there is an equivalence*

$$\mathcal{C}(F(?), -) \simeq \mathcal{B}(?, G(-))$$

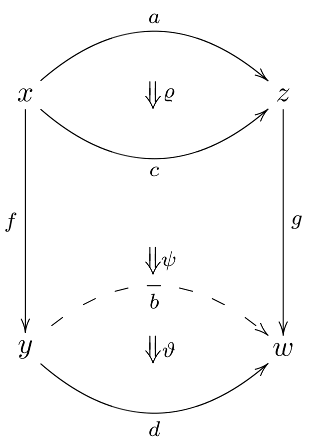
in a 2-category $Hom[\mathcal{B}^{op}, Hom[\mathcal{C}, \mathrm{Cat}]]$.

In the above definition, $\mathcal{C}(F(?), -)\colon \mathcal{B}^{op} \to Hom(\mathcal{C}, \mathrm{Cat})$ is a weak 2-functor sending each object x in $\mathcal{B}$ to the representable 2-functor $\mathcal{C}(F(x), -)\colon \mathcal{C} \to \mathrm{Cat}$, and similary $\mathcal{B}(?, G(-))\colon \mathcal{B}^{op} \to Hom(\mathcal{C}, \mathrm{Cat})$ is a weak 2-functor sending each object x in $\mathcal{B}$ to the weak 2-functor $\mathcal{B}(x, G(-))\colon \mathcal{C} \to \mathrm{Cat}$.

Let $\mathcal{E}$ be a 2-category. The 2-category $\mathcal{E}^{\to}$ of 1-morphisms, associated to $\mathcal{E}$ has 1-morphisms of $\mathcal{E}$ for objects, thus $\mathcal{E}_0^{\to} = E_1$. A 1-morphism from $f\colon x \to y$ to $g\colon z \to w$ is a triple (a, ϕ, b) consisting of 1-morphisms $a\colon x \to z$, $b\colon y \to w$ and a 2-morphism $\phi\colon g \circ a \Rightarrow b \circ f$ as in the diagram



and a 2-morphism from (a, ϕ, b) to (c, ψ, d) is a pair (ϱ, ϑ) of 2-morphisms $\varrho\colon a \Rightarrow c$ and $\vartheta\colon b \Rightarrow d$ such that the diagram



(in which we have omitted ϕ in order to avoid too many labels) commutes.

Definition 3.2. *A weak 2-functor $F\colon \mathcal{E} \to \mathcal{B}$ is called a 2-fibration if there exist a 2-functor $L\colon (\mathcal{B}, F) \to \mathcal{E}^{\to}$ which is right biadjoint right inverse to the weak 2-functor $S := (F^{[1]}, \mathcal{E}^{\delta_1})\colon \mathcal{E}^{\to} \to (\mathcal{B}, F)$.*

The 2-functor $S\colon \mathcal{E}^{\to} \to (\mathcal{B}, F)$ is then defined on any object $a\colon p \to q$ of $\mathcal{E}^{\to}$, by $S(a) = (F(p), F(a), q)$. The existence of its right biadjoint right inverse $L\colon (\mathcal{B}, F) \to \mathcal{E}^{\to}$ means that for any object (x, f, q) in $(\mathcal{B}, F)$, and any object g in $\mathcal{E}^{\to}$, we have an equivalence

$$Hom_{(\mathcal{B},F)}(S(g), (x, f, q)) \simeq Hom_{\mathcal{E}^{\to}}(g, L(x, f, q))$$

of categories, which is pseudo natural in both variables.

Thus, for any object (x, f, q) in $(\mathcal{B}, F)$ where $f\colon x \to F(q)$ is a 1-morphism in $\mathcal{B}$, we have $L(x, f, q) = \widetilde{f}$, where $\widetilde{f}\colon f^*(q) \to q$ is a biuniversal 1-morphism in $\mathcal{E}$,

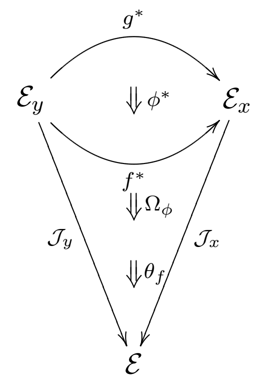
Definition 3.3. *Let $F\colon \mathcal{E} \to \mathcal{B}$ be a weak 2-functor between bicategories. A 2-cleavage consists of the following data:*

- *for each 1-morphism $f\colon x \to y$ in $\mathcal{B}$, a 2-functor $f^*\colon \mathcal{E}_y \to \mathcal{E}_x$ between the fibers, and a pseudo natural transformation $\theta_f\colon \mathcal{J}_x f^* \Longrightarrow \mathcal{J}_y$, where $\mathcal{J}_x\colon \mathcal{E}_x \to \mathcal{E}$ is an inclusion 2-functor of the fiber 2-category $\mathcal{E}_x$,*
- *for each 2-morphism $\phi\colon f \Rightarrow g$ in $\mathcal{B}$, a modification $\Omega_\phi\colon \theta_g \Longrightarrow \theta_f \phi^*$*

such that following axioms hold:

- *$F((\theta_f)_E) = f$ for each component of the pseudo natural transformation , indexed by the object E in $\mathcal{E}_y$,*
- *$F((\Omega_\phi)_E) = \phi$ for each component of the modification , indexed by the object E in $\mathcal{E}_y$.*

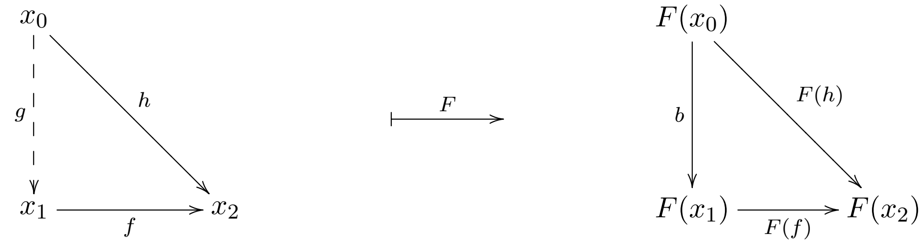
Remark 3.1. *The data for the 2-cleavage correspond precisely to the universal lax 3-cocone*



which represent the bicategory $\mathcal{E}$ as the lax 3-colimit of the lax 3-functor corresponding to the 2-cleavage (where we omitted the pseudo natural transformation θ_g from the back face).

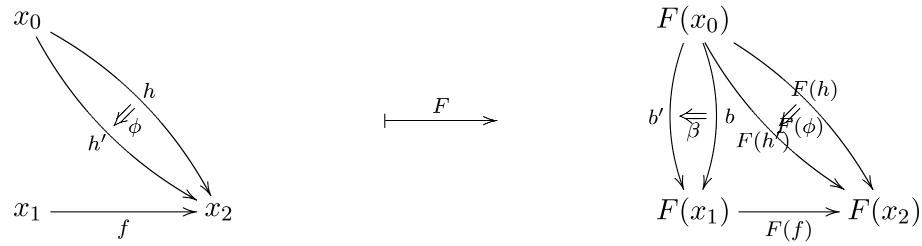
Definition 3.4. *Let $F\colon \mathcal{E} \to \mathcal{B}$ be a 2-functor.*

- *A 1-morphism $f\colon x_1 \to x_2$ in $\mathcal{E}$ is 1-cartesian if it is cartesian for the underlying functor of F. This means that for any 1-morphism $h\colon x_0 \to x_2$ in $\mathcal{E}$, such that there exists a 1-morphism $b\colon F(x_0) \to F(x_1)$ in $\mathcal{B}$, for which $F(h) = F(f) \circ g$, pictured by*



  then there exists a unique 1-morphism $g\colon x_0 \to x_1$, such that $h = f \circ g$ and $F(g) = b$.

- *A 1-morphism $f\colon x_1 \to x_2$ is 2-cartesian if it is 1-cartesian and if for any 2-morphism $\phi\colon h \Rightarrow h'$ in $\mathcal{E}$, and any 2-morphism $\beta\colon b \Rightarrow b'$ in $\mathcal{B}$, such that $F(\phi) = F(f) \circ \beta$,*

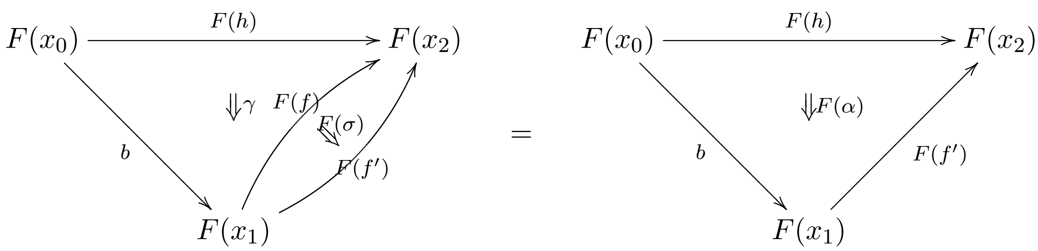


  there exists a unique 2-morphism $\gamma\colon g \Rightarrow g'$, such that $F(\phi) = f \circ \gamma$ and $F(\gamma) = \beta$.

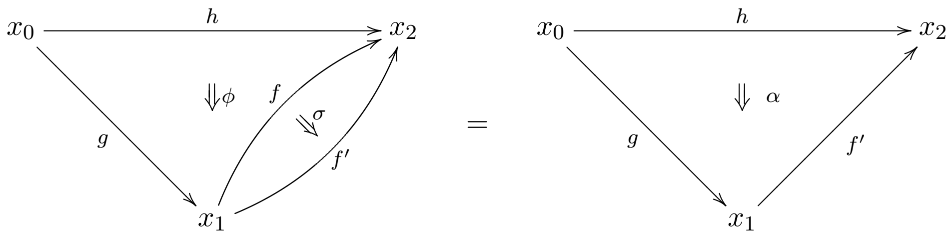
Remark 3.2. *The universal property for 2-cartesian 1-morphisms does not imply the universal property for 1-cartesian 1-morphisms. To see this, we take 2-cartesian $f\colon x_1 \to x_2$ 1-morphism as above, and we consider a 1-morphism $h\colon x_0 \to x_2$ as the identity 2-morphism. By the universal property for 2-morphisms, any 2-morphism $\beta\colon b \Rightarrow b'$ for which $F(h) = F(f) \circ \beta$ gives a unique 2-morphism $\gamma\colon g \Rightarrow g'$, such that $h = f \circ \gamma$ and $F(\gamma) = \beta$. But this gives two factorizations $h = f \circ g$ and $h = f \circ g'$ on the level of 1-morphisms.*

Definition 3.5. *Let $F\colon \mathcal{E} \to \mathcal{B}$ be a 2-functor.*

- *A 2-morphism $\sigma\colon f \to f'$ in $\mathcal{E}$ is 1-cartesian if for any 2-morphism $\alpha\colon F(h) \Rightarrow f' \circ g$ in $\mathcal{E}$, such that there exists a 2-morphism $\gamma\colon F(h) \to F(f) \circ b$ in $\mathcal{B}$, for which*



then there exists a unique 2-morphism $\phi\colon h \to f \circ g$, such that $\alpha = (\sigma \circ g)\phi$, as in



and $F(\phi) = \gamma$.

- *A 2-morphism $\sigma\colon f \to f'$ is 2-cartesian if it is 1-cartesian and if its 1-target $f'\colon x_1 \to x_2$ is 2-cartesian 1-morphism.*

The following definition is given by Hermida in [6].

Definition 3.6. *A (strict) 2-functor $F\colon \mathcal{E} \to \mathcal{B}$ is a 2-fibration if it satisfies following conditions:*

- *for any pair (f, E), where $f\colon x \to y$ is a 1-morphism in $\mathcal{B}$, and E is an object in $\mathcal{E}_y$, there exists a 2-cartesian 1-morphism $\widetilde{f}\colon f^*(E) \to E$, such that $F(\widetilde{f}) = f$,*
- *for any pair (ϕ, E), where $\phi\colon f \Rightarrow g$ is a 2-morphism in $\mathcal{B}$, and E is an object in $\mathcal{E}_y$, there exists a 2-cartesian 2-morphism $\widetilde{\phi}\colon \widetilde{f} \Rightarrow \widetilde{g}$, such that $F(\widetilde{\phi}) = \phi$.*

After we describe the universal properties of cartesian 1-morphisms and 2-morphisms corresponding to 2-fibrations of bicategories, we will prove the following important characterization.

Proposition 3.1. *Let $F\colon \mathcal{E} \to \mathcal{B}$ be a (weak) 2-functor between bicategories. Then F is a 2-fibration if and only if it has a 2-cleavage.*

4 The second Grothendieck construction

Given a pseudo 2-functor $\mathcal{F}\colon \mathcal{B}^{coop} \to 2-Cat$ as in the previous section, we will construct a 2-category $\mathcal{E}$, as a lax 2-colimit of F. It will be a total 2-category of the corresponding 2-fibration $P\colon \mathcal{E} \to \mathcal{B}$.

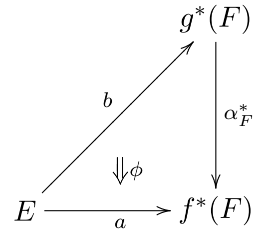
The objects of $\mathcal{E}$ are pairs (x, E) where x is an object of $\mathcal{B}$, and E is an object of the 2-category $F(x)$. A 1-morphism between two objects (x, E) and (y, F) is a pair

$$(f, a)\colon (x, E) \to (y, F)$$

such that $f\colon x \to y$ is a 1-morphisms in $\mathcal{B}$, and $m\colon E \to f^*(F)$ is a 1-morphism in $F(x)$. A 2-morphisms between two 1-morphisms (f, a) and (g, b) is a pair

$$(\alpha, \phi)\colon (f, a) \to (g, b)$$

such that $\alpha\colon f \Rightarrow g$ is a 2-morphisms in $\mathcal{B}$, and $\phi\colon \alpha^*_F b \Rightarrow a$ is a 2-morphism in $F(x)$

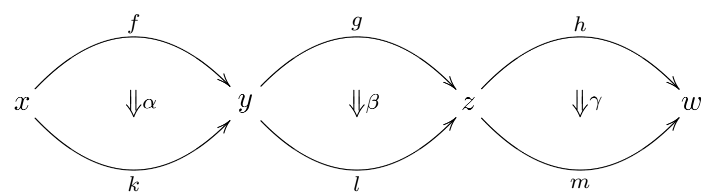


where $\alpha^*_F\colon g^*(F) \Rightarrow f^*(F)$ is the component indexed by the object F in $F(y)$, of the pseudo natural transformation $F(\alpha)\colon F(g) \Rightarrow F(f)$.

For any composable pair $(x, E) \xrightarrow{(f,a)} (y, F) \xrightarrow{(g,b)} (z, G)$ of 1-morphisms in $\mathcal{E}$, we define the composition $(gf, ba) = (g, b) \circ (f, a)$ where $gf\colon x \to y$ is the composition of 1-morphisms in $\mathcal{B}$, and $ba\colon E \to (gf)^*(G)$ is defined by the composition

$$E \xrightarrow{\quad a \quad} f^*(F) \xrightarrow{\quad f^*(b) \quad} f^*g^*(G) \xrightarrow{\quad (\chi_{g,f})_G \quad} (gf)^*(G)$$

and in the general case this composition will not be strictly associative (like in the oridnary Grothendieck construction), unless the pseudo natural transformation χ is strict natural transformation. Thus, this composition will be coherently associative, and we define associativity coherence for any three horizontally composable 2-morphisms in $\mathcal{B}$,



as following. For any composable triple $(x,E)\xrightarrow{(f,a)}(y,F)\xrightarrow{(g,b)}(z,G)\xrightarrow{(h,c)}(w,H)$ of 1-morphisms in $\mathcal{E}$, we have the composition $(h(gf),c(ba)) = (h,c)\circ((g,b)\circ(f,a))$ where $h(gf)\colon x\to y$ is the composition of 1-morphisms in $\mathcal{B}$, and $c(ba)\colon E\to(h(gf))^*(H)$ is defined by the composition

$$E\xrightarrow{a}f^*(F)\xrightarrow{f^*(b)}f^*g^*(G)\xrightarrow{(\chi_{g,f})_G}(gf)^*(G)\xrightarrow{(gf)^*(c)}(gf)^*h^*(H)\xrightarrow{(\chi_{h,gf})_H}(h(gf))^*(H)$$

of 1-morphisms in the 2-category F_x. Since $(hg,cb)=(h,c)\circ(g,b)$ where $cb\colon F\to(hg)^*(H)$ is given by the composition

$$F\xrightarrow{b}g^*(G)\xrightarrow{g^*(c)}g^*h^*(H)\xrightarrow{(\chi_{h,g})_H}(hg)^*(H)$$

we have that $((hg)f,(cb)a) = ((h,c)\circ(g,b))\circ(f,a) = (hg,cb)\circ(f,a)$ is given by the composition

$$E\xrightarrow{a}f^*(F)\xrightarrow{f^*(cb)}f^*(hg)^*(H)\xrightarrow{(\chi_{hg,f})_H}((hg)f)^*(H)$$

which is equal to the composition

$$E\xrightarrow{a}f^*(F)\xrightarrow{f^*(b)}f^*g^*(G)\xrightarrow{f^*g^*(c)}f^*g^*h^*(H)\xrightarrow{f^*((\chi_{h,g})_H)}f^*(hg)^*(H)\xrightarrow{(\chi_{hg,f})_H}((hg)f)^*(H)$$

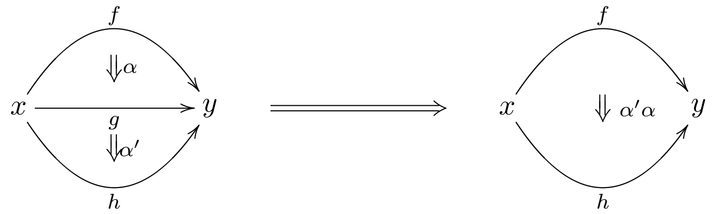
of 1-morphisms in the 2-category F_x. The resulting two compositions $((hg)f,(cb)a)$ and $(h(gf),c(ba))$, will generally not be equal, if two natural 2-morphisms in the diagram

$$\begin{array}{ccccccccc}
E & \xrightarrow{a} & f^*(F) & \xrightarrow{f^*(b)} & f^*g^*(G) & \xrightarrow{f^*g^*(c)} & f^*g^*h^*(H) & \xrightarrow{f^*((\chi_{h,g})_H)} & f^*(hg)^*(H) \\
 & & & & \downarrow{\scriptstyle (\chi_{g,f})_G} & \swarrow{\scriptstyle (\chi_{g,f})_c} & \downarrow{\scriptstyle (\chi_{g,f})_{h^*(H)}} & \swarrow{\scriptstyle (\omega_{h,g,f})_H} & \downarrow{\scriptstyle (\chi_{hg,f})_H} \\
 & & & & (gf)^*(G) & \xrightarrow[(gf)^*(c)]{} & (gf)^*h^*(H) & \xrightarrow[(\chi_{h,gf})_H]{} & (hgf)^*(H)
\end{array}$$

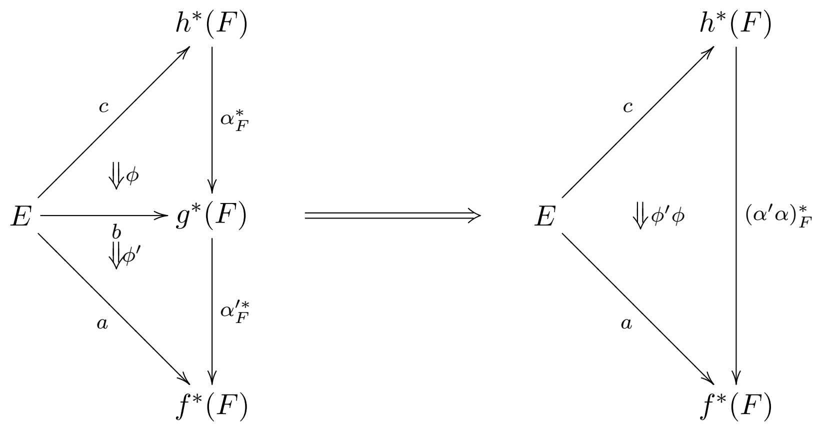
are not identities, that is if χ is not strict natural transformation and if the modification ω is not identity. The reason is that $((hg)f,(cb)a)$ is obtained by the composition of $E\xrightarrow{a}f^*(F)\xrightarrow{f^*(b)}f^*g^*(G)$ with the top and right edges of the diagram, while $(h(gf),c(ba))$ is obtained by the composition of the same 1-morphism with the left and bottom edges of the diagram. We define the 2-morphism $\alpha_{c,b,a}\colon((hg)f,(cb)a)\Rightarrow(h(gf),c(ba))$ by the pasting composite of the 1-morphism $E\xrightarrow{a}f^*(F)\xrightarrow{f^*(b)}f^*g^*(G)$ with the component 2-morphisms in the interiors of the above diagram, or more explicitly

$$\alpha_{c,b,a} := ((\chi_{h,gf})_H\circ(\chi_{g,f})_c\circ f^*(b)\circ a)((\omega_{h,g,f})_H\circ f^*g^*(c)\circ f^*(b)\circ a)$$

For vertical composition of any two 2-morphisms $f \overset{\alpha}{\Longrightarrow} g \overset{\alpha'}{\Longrightarrow} h$ in $\mathcal{B}$

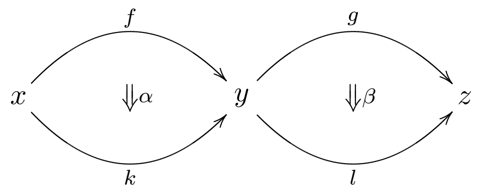


the vertical composition of two 2-morphisms $(f,a) \overset{(\alpha,\phi)}{\Longrightarrow} (g,b) \overset{(\alpha',\phi')}{\Longrightarrow} (h,c)$ in $\mathcal{E}$ is given

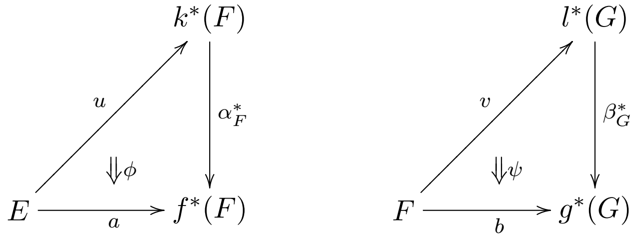


by $\phi'\phi := \phi'(\alpha'^*_F \circ \phi)$.

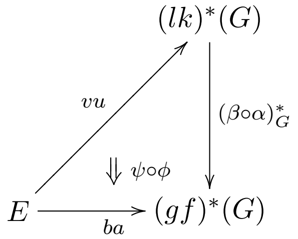
For any horizontally composable pair of 2-morphisms in $\mathcal{B}$,



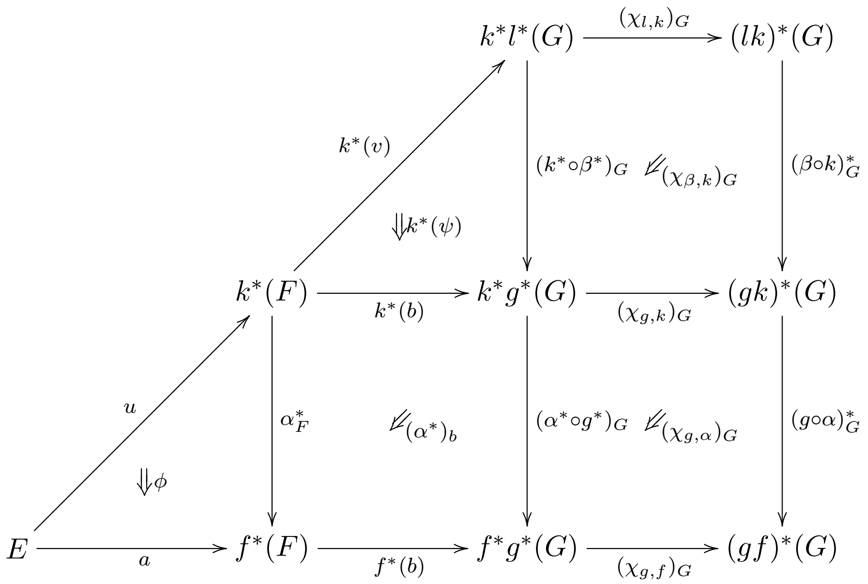
and for any two horizontally composable pair of 2-morphisms in $\mathcal{E}$ which covers them



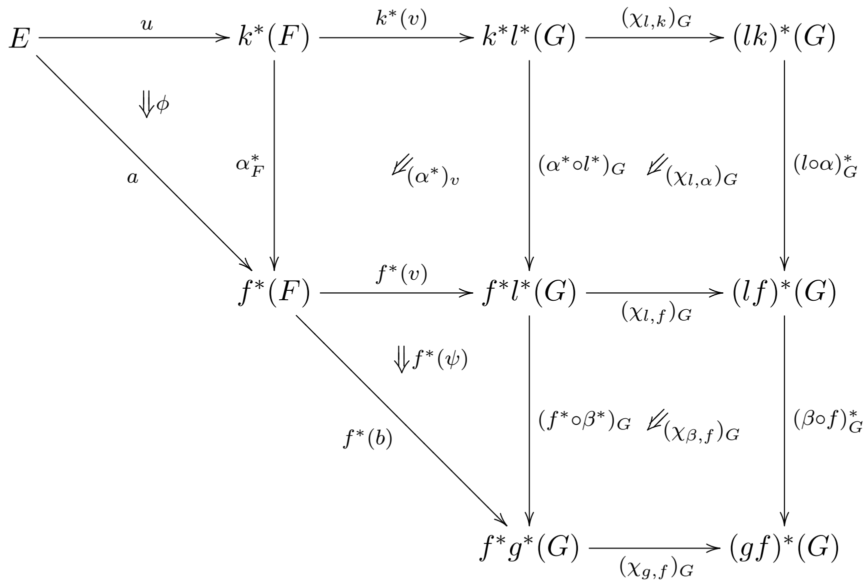
their horizontal composition $\psi\circ\phi\colon vu \Rightarrow ba$ is represented by the 2-morphism in the triangle



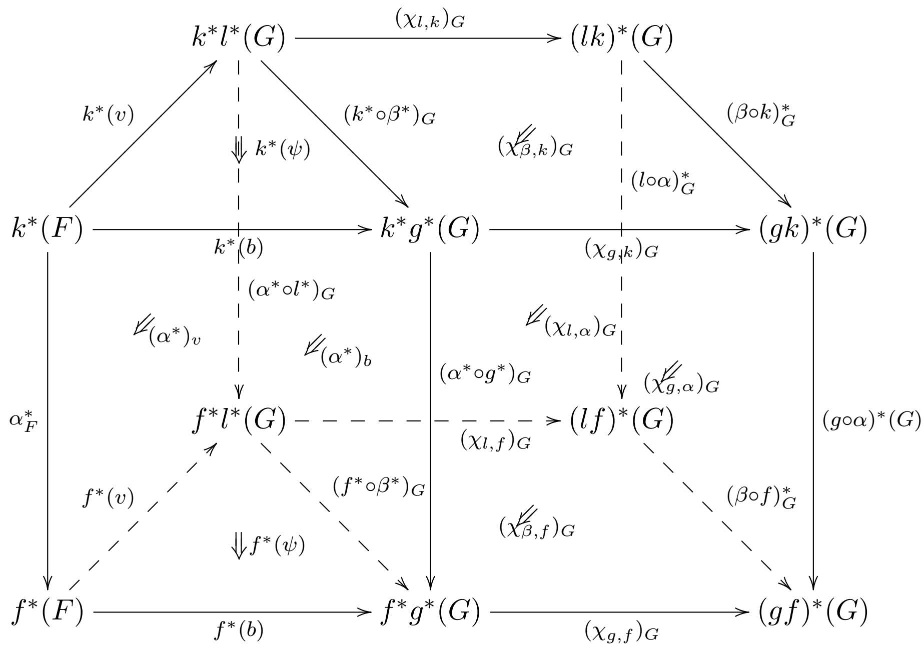
This horizontal composition is defined by the pasting composition of the diagram



which is equal to the pasting composition of the diagram

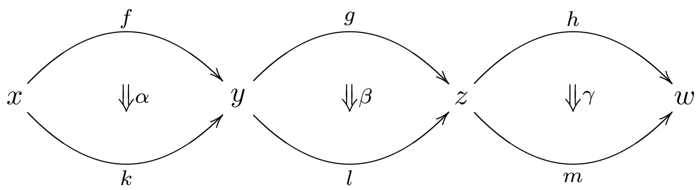


This two pasting composites are equal since the first is obtained by the pasting composite of the top and front faces of the diagram

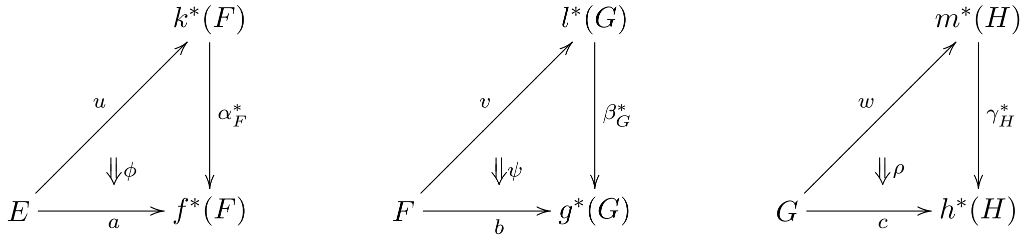


and the second one is obtained by pasting back and bottom faces, consisting of squares whose edges are given by broken arrows. This diagram commutes since the prism is a coherence, whose square which shares with the cube is an identity 2-morphism of the 1-morphism $(\alpha^* \circ \beta^*)_G \colon k^*l^*(G) \to f^*g^*(G)$, which is a component of the pseudo natural transformation $\alpha^* \circ \beta^* \colon k^*l^* \to f^*g^*$, indexed by the object G in the 2-category F_z. The cube is just the coherence for the modification χ which is a component indexed by the horizontal composition $\beta \circ \alpha$ of the pseudo natural transformation χ, so the vertical edge of the above horizontal composition is equal to the component $(\beta \circ \alpha)^*_G \colon (lk)^*(G) \to (gf)^*(G)$ indexed by the same horizontal composite. In order to prove that $\mathcal{E}$ is really a bicategory, we need to show that the Godement interchange law holds, and that the above horizontal composition is associative up to the coherent associativity isomorphism given by $\alpha_{c,b,a} \colon ((hg)f, (cb)a) \Rightarrow (h(gf), c(ba))$, for any three composable 1-morphisms.

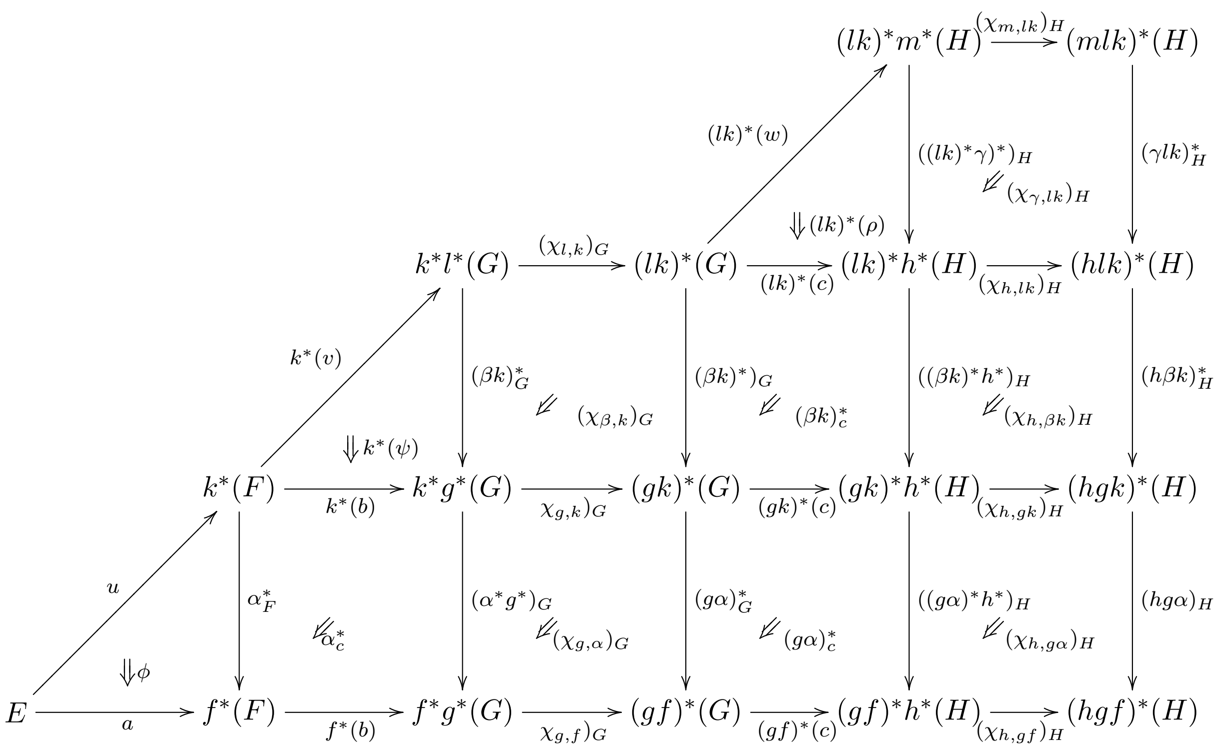
For any three horizontally composable 2-morphisms in $\mathcal{B}$,



and for any three horizontally composable 2-morphisms in $\mathcal{E}$ which covers them

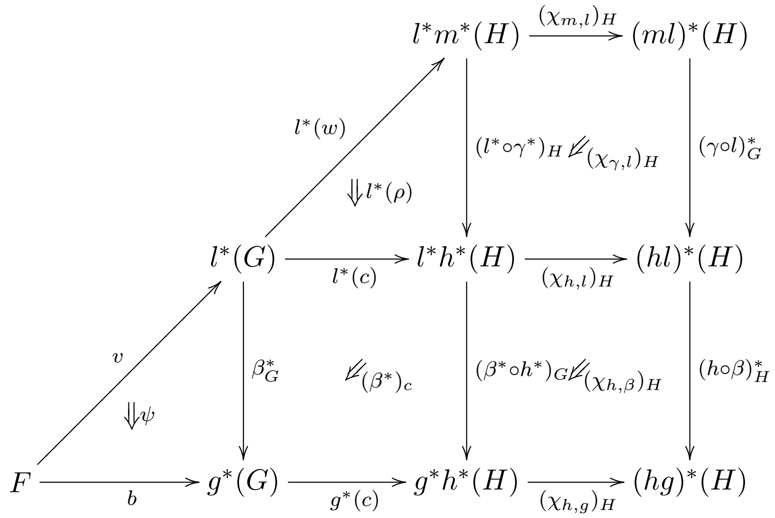


we need to show that $\alpha_{w,v,u} \circ ((\rho \circ \psi) \circ \phi) = (\rho \circ (\psi \circ \phi)) \circ \alpha_{c,b,a}$. The 2-morphism $\rho \circ (\psi \circ \phi)$ on right hand side is given by the pasting

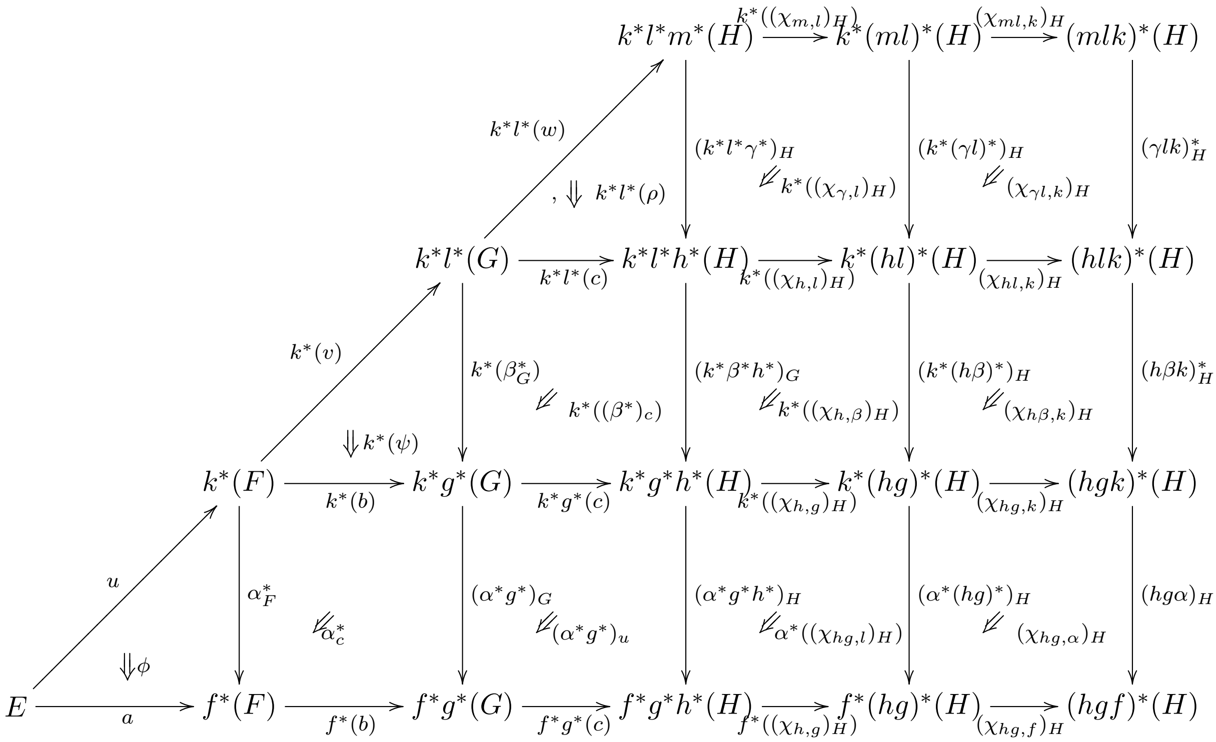


of the above diagram in which horizontal compositions are denoted by the concatenation.

By the definition, the horizontal composition $\rho \circ \psi$ is given by the pasting of the diagram



and the horizontal composition $(\rho \circ \psi) \circ \phi$ on the left hand side of the equation is given by



the pasting of the above diagram, in which all compositions are denoted by concatenation. By the above construction, we have the following main theorem.

Theorem 4.1. *For any pseudo 2-functor* $F\colon \mathcal{B}^{coop} \to 2 - Cat$*, the second Grothendieck construction gives a 2-fibration* $\mathcal{F}\colon \mathcal{E} \to \mathcal{B}$*.*

Proof. We define the 2-functor $\mathcal{F}\colon \mathcal{E} \to \mathcal{B}$ on any object (x, E) of $\mathcal{E}$ by $\mathcal{F}(x, E) := x$. Also for any 1-morphism $(f, a)\colon (x, E) \to (y, F)$ we define $\mathcal{F}(f, a) := f$, and for any 2-morphism $(\alpha, \phi)\colon (f, a) \to (g, b)$ we define $\mathcal{F}(\alpha, \phi) := \alpha$. It is easy to see that such 2-functor is really a 2-fibration. □

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