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Noncommutative gerbes and deformation quantization

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ABSTRACT

We define noncommutative gerbes using the language of star products. Quantized twisted Poisson structures are discussed as an explicit realization in the sense of deformation quantization. Our motivation is the noncommutative description of D-branes in the presence of topologically non-trivial background fields.

1. Introduction

Abelian gerbes, more precisely gerbes with an abelian band [1–5], are the next step up from a line bundle on the geometric ladder in the following sense: A unitary line bundle can be represented by a 1-cocycle in Čech cohomology, i.e., a collection of smooth transition functions $g_{\alpha\beta}$ on the intersections $U_\alpha \cap U_\beta$ of an open cover $\{U_\alpha\}$ of a manifold M satisfying $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Similarly, an abelian gerbe can be represented by a 2-cocycle in Čech cohomology, i.e., by a collection $\lambda = \{\lambda_{\alpha\beta\gamma}\}$ of maps $\lambda_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to U(1)$, valued in the abelian group $U(1)$, satisfying

$$\lambda_{\alpha\beta\gamma} = \lambda^{-1}_{\beta\alpha\gamma} = \lambda^{-1}_{\alpha\gamma\beta} = \lambda^{-1}_{\gamma\beta\alpha} \tag{1}$$

and the 2-cocycle condition

$$\delta\lambda = \lambda_{\beta\gamma\delta}\, \lambda^{-1}_{\alpha\gamma\delta}\, \lambda_{\alpha\beta\delta}\, \lambda^{-1}_{\alpha\beta\gamma} = 1 \tag{2}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$. The collection $\lambda = \{\lambda_{\alpha\beta\gamma}\}$ of maps with the stated properties represents a gerbe in the same sense as a collection of transition functions represents a line bundle. In the special case where λ is a Čech 2-coboundary with $\lambda = \delta h$, i.e., $\lambda_{\alpha\beta\gamma} = h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha}$, we say that the collection $h = \{h_{\alpha\beta}\}$ of functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ represents a trivialization of a gerbe. Considering the "difference" of two 2-coboundaries $\{h_{\alpha\beta}\}$, $\{h'_{\alpha\beta}\}$ representing two trivializations of a gerbe we step down the geometric ladder again and obtain a 1-cocycle: $g_{\alpha\beta} \equiv h_{\alpha\beta}/h'_{\alpha\beta}$ satisfies the 1-cocycle condition $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$.

There exists a local trivialization of a 2-cocycle for any particular open set U_0 of the covering: Defining $h_{\beta\gamma} \equiv \lambda_{0\beta\gamma}$ with $\beta, \gamma \neq 0$ we find from the 2-cocycle condition that $\lambda_{\alpha\beta\gamma} = h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha}$. This observation leads to a definition of an abelian

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gerbe (more precisely "gerbe data") á la Hitchin [3] in terms of line bundles on the double overlaps of the cover. The only difference with respect to line bundles from this point of view is that we step up the geometric ladder, in the sense that now line bundles on $U_\alpha \cap U_\beta$ are used as replacements for transition functions. A gerbe á la Hitchin is then a collection of line bundles $L_{\alpha\beta}$ for each double overlap $U_\alpha \cap U_\beta$, such that:

G1. There is an isomorphism $L_{\alpha\beta} \cong L^{-1}_{\beta\alpha}$.
G2. There is a trivialization $\lambda_{\alpha\beta\gamma}$ of $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.
G3. The trivialization $\lambda_{\alpha\beta\gamma}$ satisfies $\delta\lambda = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$.

In this paper we will use the term gerbe for an (abelian) gerbe á la Hitchin. In this respect, we should notice that in general the term gerbe is used to name a locally non-empty and locally connected stack in groupoids [1,2,4,5]. We will use the term standard gerbe in order to name such gerbes.

Gerbes are interesting in physics for several reasons: One motivation is the interpretation of D-brane charges in terms of K-theory in the presence of a topologically non-trivial B-field, when the gauge fields living on D-branes become connections on certain noncommutative algebras rather than on a vector bundle [6–14]. Azumaya algebras appear to be a natural choice and give the link to gerbes. Gerbes, rather than line bundles, are the structure that arises in the presence of closed 3-form backgrounds as, e.g., in WZW models and Poisson sigma models with WZW term [11,15,16]. Gerbes help illuminate the geometry of mirror symmetry of 3-dimensional Calabi–Yau manifolds [3] and they provide a language to formulate duality transformations with higher order antisymmetric fields [17]. Our motivation is the noncommutative description of D-branes in the presence of topologically non-trivial background fields.

The paper is organized as follows: In Section 2, we recall the local description of noncommutative line bundles in the framework of deformation quantization. Instead of repeating that construction we shall take the properties that were derived in [18,19] as a formal definition of a noncommutative line bundle. In the same spirit we define noncommutative gerbes in Section 3, using the language of star products and complement this definition, in Section 4, with an explicit realization of noncommutative gerbes as quantizations of twisted Poisson structures as introduced in [21] and further discussed in [22].

Notice that we will use the term noncommutative gerbe to describe a specific non-abelian 2-cocycle. By the correspondence (in the sense of 2-categories, see [5] for details) between degree two non-abelian cohomology classes and equivalence classes of (standard) gerbes understood as locally non-empty and locally connected stack in groupoids there is such a (standard) gerbe corresponding to this specific non-abelian 2-cocycle. Hence our definition of a noncommutative gerbe leads to a non-abelian gerbe in the standard sense of Giraud, Deligne, Breen and Brylinski [1,2,4,5]. We will discuss this shortly in Section 5.

Since the first version of this paper was posted on the arXiv, see hep-th/0206101v1 (where Section 5 was not present), some related work appeared in [23–27].

2. Noncommutative line bundles

Here we collect some facts on noncommutative line bundles [28,18] that we will need in what follows.[1] Let (M, θ) be a general Poisson manifold, and let $\star$ be the Kontsevich's deformation quantization of the Poisson tensor θ. Further let us consider a good covering $\{U^i\}$ of M. For the purposes of this paper a noncommutative line bundle $\mathcal{L}$ is defined by a collection of $\mathbb{C}[[\hbar]]$-valued local transition functions $G^{ij} \in C^\infty(U^i \cap U^j)[[\hbar]]$ (that can be thought valued in the enveloping algebra of $U(1)$, see [29]), and a collection of maps $\mathcal{D}^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$, formal power series in $\hbar$, starting with the identity, and with coefficients being differential operators, such that

$$G^{ij} \star G^{jk} = G^{ik} \tag{3}$$

on $U^i \cap U^j \cap U^k$, $G^{ii} = 1$ on U^i, and

$$\mathrm{Ad}_\star G^{ij} = \mathcal{D}^i \circ (\mathcal{D}^j)^{-1} \tag{4}$$

on $U^i \cap U^j$ or, equivalently, $\mathcal{D}^i(f) \star G^{ij} = G^{ij} \star \mathcal{D}^j(f)$ for all $f \in C^\infty(U^i \cap U^j)[[\hbar]]$. Obviously, with this definition the local maps $\mathcal{D}^i$ can be used to define *globally* a new star product $\star'$ (because the inner automorphisms $\mathrm{Ad}_\star G^{ij}$ do not affect $\star'$)

$$\mathcal{D}^i(f \star' g) = \mathcal{D}^i f \star \mathcal{D}^i g. \tag{5}$$

We say that two line bundles $\mathcal{L}_1 = \{G_1^{ij}, \mathcal{D}_1^i, \star\}$ and $\mathcal{L}_2 = \{G_2^{ij}, \mathcal{D}_2^i, \star\}$ are equivalent if there exists a collection of invertible local functions $H^i \in C^\infty(U^i)[[\hbar]]$ such that

$$G_1^{ij} = H^i \star G_2^{ij} \star (H^j)^{-1} \tag{6}$$

and

$$\mathcal{D}_1^i = \mathrm{Ad}_\star H^i \circ \mathcal{D}_2^i. \tag{7}$$

[1] A noncommutative line bundle is a finite projective module. In the present context it can be understood as a quantization of a line bundle over a compact manifold in the sense of deformation quantization. Here we shall take the properties of quantized line bundles as derived in [18,19] as a formal definition of a noncommutative line bundle.

The tensor product of two line bundles $\mathcal{L}_1 = \{G_1^{ij}, \mathcal{D}_1^i, \star_1\}$ and $\mathcal{L}_2 = \{G_2^{ij}, \mathcal{D}_2^i, \star_2\}$ is well defined if $\star_2 = \star_1'$ (or $\star_1 = \star_2'$). Then the corresponding tensor product is a line bundle $\mathcal{L}_2 \otimes \mathcal{L}_1 = \mathcal{L}_{21} = \{G_{12}^{ij}, \mathcal{D}_{12}^{ij}, \star_1\}$ defined as

$$G_{12}^{ij} = \mathcal{D}_1^i(G_2^{ij}) \star_1 G_1^{ij} = G_1^{ij} \star_1 \mathcal{D}_1^j(G_2^{ij}) \tag{8}$$

and

$$\mathcal{D}_{12}^i = \mathcal{D}_1^i \circ \mathcal{D}_2^i. \tag{9}$$

The order of indices of $\mathcal{L}_{21}$ indicates the bimodule structure of the corresponding space of sections to be defined later, whereas the first index on the G_{12}'s and $\mathcal{D}_{12}$'s indicates the star product (here: $\star_1$) by which the objects multiply.

A section $\Psi = (\Psi^i)$ is a collection of functions $\Psi^i \in C_{\mathbb{C}}^\infty(U^i)[[\hbar]]$ satisfying consistency relations

$$\Psi^i = G^{ij} \star \Psi^i \tag{10}$$

on all intersections $U^i \cap U^j$. With this definition the space of sections $\mathcal{E}$ is a right $\mathfrak{A} = (C^\infty(M)[[\hbar]], \star)$ module. We shall use the notation $\mathcal{E}_\mathfrak{A}$ for it. The right action of the function $f \in \mathfrak{A}$ is the regular one

$$\Psi.f = (\Psi^k \star f). \tag{11}$$

Using the maps $\mathcal{D}^i$ it is easy to turn $\mathcal{E}$ also into a left $\mathfrak{A}' = (C^\infty(M)[[\hbar]], \star')$ module ${}_{\mathfrak{A}'}\mathcal{E}$. The left action of $\mathfrak{A}'$ is given by

$$f.\Psi = (\mathcal{D}^i(f) \star \Psi^i). \tag{12}$$

It is easy to check, using (4), that the left action (12) is compatible with (10). From the property (5) of the maps $\mathcal{D}^i$ we find

$$f.(g.\Psi) = (f \star' g).\Psi. \tag{13}$$

Together we have a bimodule structure ${}_{\mathfrak{A}'}\mathcal{E}_\mathfrak{A}$ on the space of sections.

There is an obvious way of tensoring sections. The section

$$\Psi_{12}^i = \mathcal{D}_1^i(\Psi_2^i) \star_1 \Psi_1^i \tag{14}$$

is a section of the tensor product line bundle (8) and (9). Tensoring of line bundles naturally corresponds to tensoring of bimodules.

Using the Hochschild complex we can introduce a natural differential calculus on the algebra $\mathfrak{A}$.[2] The p-cochains, elements of $C^p = \mathrm{Hom}_\mathbb{C}(\mathfrak{A}^{\otimes p}, \mathfrak{A})$, play the role of p-forms and the derivation $\mathrm{d} : C^p \to C^{p+1}$ is given on $C \in C^p$ as

$$\begin{aligned} \mathrm{d}C(f_1, f_2, \ldots, f_{p+1}) = {} & f_1 \star C(f_2, \ldots, f_{p+1}) - C(f_1 \star f_2, \ldots, f_{p+1}) + C(f_1, f_2 \star f_3, \ldots, f_{p+1}) - \cdots \\ & + (-1)^p C(f_1, f_2, \ldots, f_p \star f_{p+1}) + (-1)^{p+1} C(f_1, f_2, \ldots, f_p) \star f_{p+1}. \end{aligned} \tag{15}$$

A (contravariant) connection $\nabla : \mathcal{E} \otimes_\mathfrak{A} C^p \to \mathcal{E} \otimes_\mathfrak{A} C^{p+1}$ can now be defined by a formula similar to (15) using the natural extension of the left and right module structure of $\mathcal{E}$ to $\mathcal{E} \otimes_\mathfrak{A} C^p$. Namely, for a $\Phi \in \mathcal{E} \otimes_\mathfrak{A} C^p$ we have

$$\begin{aligned} \nabla\Phi(f_1, f_2, \ldots, f_{p+1}) = {} & f_1.\Phi(f_2, \ldots, f_{p+1}) - \Phi(f_1 \star f_2, \ldots, f_{p+1}) + \Phi(f_1, f_2 \star f_3, \ldots, f_{p+1}) - \cdots \\ & + (-1)^p \Phi(f_1, f_2, \ldots, f_p \star f_{p+1}) + (-1)^{p+1} \Phi(f_1, f_2, \ldots, f_p).f_{p+1}. \end{aligned} \tag{16}$$

We also have the cup product $C_1 \cup C_2$ of two cochains $C_1 \in C^p$ and $C_2 \in C^q$;

$$(C_1 \cup C_2)(f_1, \ldots, f_{p+q}) = C_1(f_1, \ldots, f_p) \star C_2(f_{p+1}, \ldots, f_q). \tag{17}$$

The cup product extends to a map from $(\mathcal{E} \otimes_\mathfrak{A} C^p) \otimes_\mathfrak{A} C^q$ to $\mathcal{E} \otimes_\mathfrak{A} C^{p+q}$. The connection ∇ satisfies the graded Leibniz rule with respect to the cup product and thus defines a bona fide connection on the module $\mathcal{E}_\mathfrak{A}$. On the sections, the connection ∇ introduced here is simply the difference between the left and right action of $C^\infty(M)[[\hbar]]$ on $\mathcal{E}$:

$$\nabla\Psi(f) = f.\Psi - \Psi.f = \left(\nabla^i \Psi^i(f)\right) = \left(\mathcal{D}^i(f) \star \Psi^i - \Psi^i \star f\right). \tag{18}$$

As in [19] we define the gauge potential $\mathcal{A} = (\mathcal{A}^i)$, where the $\mathcal{A}^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$ are local 1-cochains, by

$$\mathcal{A}^i \equiv \mathcal{D}^i - \mathrm{id}. \tag{19}$$

Then we have for a section $\Psi = (\Psi^i)$, where the $\Psi^i \in C_\mathbb{C}^\infty(U^i)[[\hbar]]$ are local 0-cochains,

$$\nabla^i \Psi^i(f) = \mathrm{d}\Psi^i(f) + \mathcal{A}^i(f) \star \Psi^i, \tag{20}$$

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2 Other choices for the differential calculus are of course possible, e.g., the Lie algebra complex.

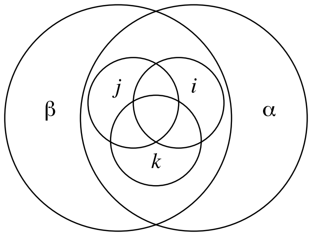

Fig. 1. Double intersection $U_\alpha \cap U_\beta$ equipped with a NC line bundle $G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta}$.

and more generally $\nabla^i \Phi^i = \mathrm{d}\Phi^i + \mathcal{A}^i \cup \Phi^i$ with $\Phi = (\Phi^i) \in \mathcal{E} \otimes_{\mathfrak{A}} C^p$. In the intersections $U^i \cap U^j$ we have the gauge transformation (cf. (4))

$$\mathcal{A}^i = \mathrm{Ad}_\star G^{ij} \circ \mathcal{A}^j + G^{ij} \star \mathrm{d}(G^{ij})^{-1}. \tag{21}$$

The curvature $K_\nabla \equiv \nabla^2 : \mathcal{E} \otimes_{\mathfrak{A}} C^p \to \mathcal{E} \otimes_{\mathfrak{A}} C^{p+2}$ corresponding to the connection ∇, measures the difference between the two star products $\star'$ and $\star$. On a section Ψ, it is given by

$$(K_\nabla \Psi)(f, g) = \left(\mathcal{D}^i(f \star' g - f \star g) \star \Psi^i\right). \tag{22}$$

The connection for the tensor product line bundle (8) is given on sections as

$$\nabla_{12} \Psi^i_{12} = \mathcal{D}^i_1(\nabla_2 \Psi^i_2) \star_1 \Psi^i_1 + \mathcal{D}^i_1(\Psi_2) \star_1 \nabla_1 \Psi^i_1. \tag{23}$$

Symbolically,

$$\nabla_{12} = \nabla_1 + \mathcal{D}_1(\nabla_2). \tag{24}$$

Let us note that if we assume the base manifold M to be compact, then the space of sections $\mathcal{E}$ as a right $\mathfrak{A}$-module is projective of finite type. Of course, the same holds if $\mathcal{E}$ is considered as a left $\mathfrak{A}'$ module. Also let us note that the two algebras $\mathfrak{A}$ and $\mathfrak{A}'$ are Morita equivalent. Up to a global isomorphism they must be related by an action of the Picard group $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ as follows. Let $L \in \mathrm{Pic}(M)$ be a (complex) line bundle on M and c its Chern class. Let F be a curvature two form on M whose cohomology class $[F]$ is (the image in $\mathbb{R}$ of) the Chern class c. Consider the formal Poisson structure θ' given by the geometric series

$$\theta' = \theta(1 + \hbar F \theta)^{-1}. \tag{25}$$

In this formula θ and F are understood as maps $\theta : T^*M \to TM$, $F : TM \to T^*M$ and θ' is the result of the indicated map compositions. Then $\star'$ must (up to a global isomorphism) be the deformation quantization of θ' corresponding to $c \in H^2(M, \mathbb{Z})$. This construction depends only on the integer cohomology class c, indeed if c is the trivial class then $F = \mathrm{d}a$ and the corresponding quantum line bundle is trivial, i.e.,

$$G^{ij} = (H^i)^{-1} \star H^j. \tag{26}$$

In this case the linear map

$$\mathcal{D} = \mathrm{Ad}_\star H^i \circ \mathcal{D}^i \tag{27}$$

defines a global equivalence (a stronger notion than Morita equivalence) of $\star$ and $\star'$.

3. Noncommutative gerbes

Now let us consider any covering $\{U_\alpha\}$ (not necessarily a good one) of a manifold M. Here we switch from upper Latin to lower Greek indices to label the local patches. The reason for the different notation will become clear soon. Consider each local patch equipped with its own star product $\star_\alpha$ the deformation quantization of a local Poisson structure θ_α. We assume that on each double intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the local Poisson structures θ_α and θ_β are related similarly as in the previous section via some integral closed two form $F_{\beta\alpha}$, which is the curvature of a line bundle $L_{\beta\alpha} \in \mathrm{Pic}(U_{\alpha\beta})$

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha}\theta_\beta)^{-1}. \tag{28}$$

Let us now consider, as in Fig. 1, a good covering $U^i_{\alpha\beta}$ of each double intersection $U_\alpha \cap U_\beta$[3] with a noncommutative line bundle $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha\}$

$$G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta}, \qquad G^{ii}_{\alpha\beta} = 1, \tag{29}$$

$$\mathcal{D}^i_{\alpha\beta}(f) \star_\alpha G^{ij}_{\alpha\beta} = G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(f) \tag{30}$$

[3] At this point we use so called hypercovers. Hypercovers are defined, e.g., in [20]. A thorough treatment with non-abelian cohomology classes is given in [5]. The hypercovers are not necessary if M is paracompact.

and

$$\mathcal{D}^i_{\alpha\beta}(f \star_\beta g) = \mathcal{D}^i_{\alpha\beta}(f) \star_\alpha \mathcal{D}^i_{\alpha\beta}(g). \tag{31}$$

The opposite order of indices labelling the line bundles and the corresponding transition functions and equivalences simply reflects a choice of convention. As in the previous section the order of indices of $\mathcal{L}_{\alpha\beta}$ indicates the bimodule structure of the corresponding space of sections, whereas the order of Greek indices on G's and D's indicates the star product in which the objects multiply. The product always goes with the first index of the multiplied objects.

A noncommutative gerbe is characterised by the following axioms:

Axiom 1. $\mathcal{L}_{\alpha\beta} = \{G^{ij}_{\beta\alpha}, \mathcal{D}^i_{\beta\alpha}, \star_\beta\}$ and $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha\}$ are related as follows

$$\{G^{ij}_{\beta\alpha}, \mathcal{D}^i_{\beta\alpha}, \star_\beta\} = \{(\mathcal{D}^j_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta}), (\mathcal{D}^i_{\alpha\beta})^{-1}, \star_\beta\} \tag{32}$$

i.e. $\mathcal{L}_{\alpha\beta} = \mathcal{L}^{-1}_{\beta\alpha}$. (Notice also that $(\mathcal{D}^j_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta}) = (\mathcal{D}^i_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta})$.)

Axiom 2. On the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ the tensor product $\mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ is equivalent to the line bundle $\mathcal{L}_{\gamma\alpha}$. Explicitly

$$G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) = \Lambda^i_{\alpha\beta\gamma} \star_\alpha G^{ij}_{\alpha\gamma} \star_\alpha (\Lambda^j)^{-1}_{\alpha\beta\gamma}, \tag{33}$$

$$\mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma} = \mathrm{Ad}_{\star_\alpha} \Lambda^i_{\alpha\beta\gamma} \circ \mathcal{D}^i_{\alpha\gamma}. \tag{34}$$

Axiom 3. On the quadruple intersection $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$\Lambda^i_{\alpha\beta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\delta} = \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta}, \tag{35}$$

$$\Lambda^i_{\alpha\beta\gamma} = (\Lambda^i_{\alpha\gamma\beta})^{-1} \quad \text{and} \quad \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\alpha}) = \Lambda^i_{\alpha\beta\gamma}. \tag{36}$$

With slight abuse of notation we have used Latin indices $\{i, j, \ldots\}$ to label both the good coverings of the intersection of the local patches U_α and the corresponding transition functions of the consistent restrictions of line bundles $\mathcal{L}_{\alpha\beta}$ to these intersections. A short comment on the consistency of Axiom 3 is in order. Let us define

$$\mathcal{D}^i_{\alpha\beta\gamma} = \mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma} \circ \mathcal{D}^i_{\gamma\alpha}. \tag{37}$$

Then it is easy to see that

$$\mathcal{D}^i_{\alpha\beta\gamma} \circ \mathcal{D}^i_{\alpha\gamma\delta} \circ \mathcal{D}^i_{\alpha\delta\beta} = \mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma\delta} \circ \mathcal{D}^i_{\beta\alpha}. \tag{38}$$

In view of (34) this implies that

$$\Lambda^i_{\alpha\beta\gamma\delta} \equiv \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \star_\alpha \Lambda^i_{\alpha\delta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\beta}$$

is central. Using this and the associativity of $\star_\alpha$ together with (33) applied to the triple tensor product $\mathcal{L}_{\delta\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ transition functions

$$G^{ij}_{\alpha\beta\gamma} \equiv G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) \star_\alpha \mathcal{D}^j_{\alpha\beta}(\mathcal{D}^j_{\beta\gamma}(G^{ij}_{\gamma\delta})) \tag{39}$$

reveals that $\Lambda^i_{\alpha\beta\gamma\delta}$ is independent of i. It is therefore consistent to set $\Lambda^i_{\alpha\beta\gamma\delta}$ equal to 1. A similar consistency check works also for (36). If we replace all noncommutative line bundles $\mathcal{L}_{\alpha\beta}$ in Axioms 1–3 by equivalent ones, we get by definition an equivalent noncommutative gerbe.

There is a natural (contravariant) connection on a noncommutative gerbe. It is defined using the (contravariant) connections $\nabla_{\alpha\beta} = (\nabla^i_{\alpha\beta})$ (cf. (16) and (18)) on quantum line bundles $\mathcal{L}_{\beta\alpha}$. Let us denote by $\nabla_{\alpha\beta\gamma}$ the contravariant connection formed on the triple tensor product $\mathcal{L}_{\alpha\gamma\beta} \equiv \mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ with maps $\mathcal{D}^i_{\alpha\beta\gamma}$ and transition functions (39) according to the rule (24). Axiom 2 states that $\Lambda^i_{\alpha\beta\gamma}$ is a trivialization of $\mathcal{L}_{\alpha\gamma\beta}$ and that

$$\nabla^i_{\alpha\beta\gamma} \Lambda^i_{\alpha\beta\gamma} = 0. \tag{40}$$

Using Axiom 2 one can show that the product bundle

$$\mathcal{L}_{\alpha\beta\gamma\delta} = \mathcal{L}_{\alpha\beta\gamma} \otimes \mathcal{L}_{\alpha\gamma\delta} \otimes \mathcal{L}_{\alpha\delta\beta} \otimes \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\delta\gamma} \otimes \mathcal{L}_{\beta\alpha} \tag{41}$$

is trivial: it has transition functions $G^{ij}_{\alpha\beta\gamma\delta} = 1$ and maps $\mathcal{D}^i_{\alpha\beta\gamma\delta} = \mathrm{id}$. The constant unit section is thus well defined on this bundle. On $\mathcal{L}_{\alpha\beta\gamma\delta}$ we also have the section $(\Lambda^i_{\alpha\beta\gamma\delta})$. Axiom 3 implies $(\Lambda^i_{\alpha\beta\gamma\delta})$ to be the unit section. If two of the indices $\alpha, \beta, \gamma, \delta$ are equal, triviality of the bundle $\mathcal{L}_{\alpha\beta\gamma\delta}$ implies (36). Using for example the first relation in (36) one can show that (35) written in the form $\mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \star_\alpha \Lambda^i_{\alpha\delta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\beta} = 1$ is invariant under cyclic permutations of any three of the four factors appearing on the l.h.s.

If we now assume that all line bundles $\mathcal{L}_{\beta\alpha}$ are trivial (this is for example the case when $\{U_\alpha\}$ is a good covering) then $F_{\alpha\beta} = \mathrm{d}a_{\alpha\beta}$ for each $U_\alpha \cap U_\beta$ and

$$G^{ij}_{\alpha\beta} = (H^i_{\alpha\beta})^{-1} \star_\alpha H^j_{\alpha\beta}$$

$$\mathcal{D}_{\alpha\beta} = \mathrm{Ad}_{\star_\alpha} H^i_{\alpha\beta} \circ \mathcal{D}^i_{\alpha\beta}.$$

It then easily follows that

$$\Lambda_{\alpha\beta\gamma} \equiv H^i_{\alpha\beta} \star_\alpha \mathcal{D}^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha \mathcal{D}^i_{\alpha\beta}\mathcal{D}^i_{\beta\gamma}(H^i_{\gamma\alpha}) \star_\alpha \Lambda^i_{\alpha\beta\gamma} \tag{42}$$

defines a global function on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. $\Lambda_{\alpha\beta\gamma}$ is just the quotient of the two sections $\left(H^i_{\alpha\beta} \star_\alpha \mathcal{D}^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha \mathcal{D}^i_{\alpha\beta}\mathcal{D}^i_{\beta\gamma}(H^i_{\gamma\alpha})\right)^{-1}$ and $\Lambda^i_{\alpha\beta\gamma}$ of the triple tensor product $\mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$. On the quadruple overlap $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ it satisfies conditions analogous to (35) and (36)

$$\Lambda_{\alpha\beta\gamma} \star_\alpha \Lambda_{\alpha\gamma\delta} = \mathcal{D}_{\alpha\beta}(\Lambda_{\beta\gamma\delta}) \star_\alpha \Lambda_{\alpha\beta\delta}, \tag{43}$$

$$\Lambda_{\alpha\beta\gamma} = (\Lambda_{\alpha\gamma\beta})^{-1} \quad \text{and} \quad \mathcal{D}_{\alpha\beta}(\Lambda_{\beta\gamma\alpha}) = \Lambda_{\alpha\beta\gamma}. \tag{44}$$

Also

$$\mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \mathrm{Ad}_{\star_\alpha} \Lambda_{\alpha\beta\gamma}. \tag{45}$$

So we can take formulas (43)–(45) as a definition of a noncommutative gerbe in the case of a good covering $\{U_\alpha\}$. We say that the gerbe is defined by the local data $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$.

From now on we shall consider only good coverings. A noncommutative gerbe defined by $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ is said to be trivial if there exists a global star product $\star$ on M and a collection of "twisted" transition functions $G_{\alpha\beta}$ defined on each overlap $U_\alpha \cap U_\beta$ and a collection $\mathcal{D}_\alpha$ of local equivalences between the global product $\star$ and the local products $\star_\alpha$

$$\mathcal{D}_\alpha(f) \star \mathcal{D}_\alpha(g) = \mathcal{D}_\alpha(f \star_\alpha g) \tag{46}$$

satisfying the following two conditions:

$$G_{\alpha\beta} \star G_{\beta\gamma} = \mathcal{D}_\alpha(\Lambda_{\alpha\beta\gamma}) \star G_{\alpha\gamma} \tag{47}$$

and

$$\mathrm{Ad}_\star G_{\alpha\beta} \circ \mathcal{D}_\beta = \mathcal{D}_\alpha \circ \mathcal{D}_{\alpha\beta}. \tag{48}$$

Locally, every noncommutative gerbe is trivial as is easily seen from (43), (44) and (45) by fixing the index α. Defining as in (19), $\mathcal{A}_\alpha = \mathcal{D}_\alpha - \mathrm{id}$, $\mathcal{A}_{\alpha\beta} = \mathcal{D}_{\alpha\beta} - \mathrm{id}$ we obtain the "twisted" gauge transformations

$$\mathcal{A}_\alpha = \mathrm{Ad}_\star G_{\alpha\beta} \circ \mathcal{A}_\beta + G_{\alpha\beta} \star \mathrm{d}(G_{\alpha\beta})^{-1} - \mathcal{D}_\alpha \circ \mathcal{A}_{\alpha\beta}. \tag{49}$$

Two noncommutative gerbes defined[4] by their corresponding local data $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ and $\{\star'_\alpha, \mathcal{D}'_{\alpha\beta}, \Lambda'_{\alpha\beta\gamma}\}$ are equivalent if there exist local equivalences $\mathcal{D}_\alpha$ of star products $\star_\alpha$ and $\star'_\alpha$, i.e.,

$$\mathcal{D}_\alpha(f) \star'_\alpha \mathcal{D}_\alpha(g) = \mathcal{D}_\alpha(f \star_\alpha g) \tag{50}$$

and local functions $\Lambda_{\alpha\beta}$ such that

$$\mathrm{Ad}_{\star'_\alpha} \Lambda_{\alpha\beta} \circ \mathcal{D}'_{\alpha\beta} \circ \mathcal{D}_\beta = \mathcal{D}_\alpha \circ \mathcal{D}_{\alpha\beta} \tag{51}$$

and

$$\mathcal{D}_\alpha(\Lambda_{\alpha\beta\gamma}) \star'_\alpha \Lambda_{\alpha\gamma} = \Lambda_{\alpha\beta} \star'_\alpha \mathcal{D}'_{\alpha\beta}(\Lambda_{\beta\gamma}) \star'_\alpha \Lambda'_{\alpha\beta\gamma}. \tag{52}$$

The classical limit of a noncommutative gerbe is the (classical) Hitchin gerbe defined by considering the classical limit (in the deformation quantization sense) of the structures in Axioms 1–3. Correspondingly the classical limit of the local data $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ gives the local data $\{\cdot_\alpha, \mathrm{id}_\alpha, \lambda_{\alpha\beta\gamma}\}$, where $\cdot_\alpha$ is the restriction to U_α of the (globally defined) ordinary point-wise product of functions on the base manifold M, and $\lambda_{\alpha\beta\gamma}$ is the 2-cocycle of the underlying classical Hitchin gerbe. We say that the noncommutative gerbe $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ is a trivial deformation quantization of this classical Hitchin gerbe if it is equivalent to it in the sense of (50)–(52). In particular we have a non-trivial deformation quantization of a Hitchin gerbe, whenever the local products $\star_\alpha$ are not a trivial deformation of (i.e. are not equivalent to) the ordinary commutative point-wise product (cf. (46) with $\star$ replaced by $\cdot$). This is the case considered in the next section, concerning quantization of twisted Poisson structures.

[4] After passing to a common refinement of respective trivializing coverings, if necessary.

We conclude this section with the following remark concerning the role of local functions $\Lambda_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta}$ satisfying relations (43)–(45). These represent a honest non-abelian 2-cocycle, as defined for example in [5]. It follows from the discussion of Section 2, that each $\mathcal{D}_{\alpha\beta}$ defines an equivalence, in the sense of deformation quantization, of star products $\star_\alpha$ and $\star_\beta$ on $U_\alpha \cap U_\beta$. The non-triviality of the non-abelian 2-cocycle (43)–(45) can therefore be seen as an obstruction to gluing the collection of local star products $\{\star_\alpha\}$, i.e., the collection of local rings $C^\infty(U_\alpha)[[\hbar]]$, into a global one. We also mention that in [30] a 2-cocycle similar to that of (43)–(45) represents an obstruction to gluing together certain local rings appearing in quantization of contact manifolds.

4. Quantization of twisted Poisson structures

Let $H \in H^3(M, \mathbb{Z})$ be a closed integral three form on M. Such a form is known to define a gerbe on M. We can find a good covering $\{U_\alpha\}$ and local potentials B_α with $H = \mathrm{d}B_\alpha$ for H. On $U_\alpha \cap U_\beta$ the difference of the two local potentials $B_\alpha - B_\beta$ is closed and hence exact: $B_\alpha - B_\beta = \mathrm{d}a_{\alpha\beta}$. On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = -\mathrm{i}\lambda_{\alpha\beta\gamma}\mathrm{d}\lambda^{-1}_{\alpha\beta\gamma}. \tag{53}$$

The collection of local functions $\{\lambda_{\alpha\beta\gamma}\}$ represents a gerbe.

Let us also assume the existence of a formal antisymmetric bivector field $\theta = \theta^{(0)} + \hbar\theta^{(1)} + \cdots$ on M such that

$$[\theta, \theta] = \hbar\, \theta^* H, \tag{54}$$

where $[\,,\,]$ is the Schouten–Nijenhuis bracket and θ^* denotes the natural map sending n-forms to n-vector fields by "using θ to raise indices". Explicitly, in local coordinates, $\theta^* H^{ijk} = \theta^{im}\theta^{jn}\theta^{ko}H_{mno}$. We call θ a Poisson structure twisted by H [21,11,15]. On each U_α we can introduce a local formal Poisson structure $\theta_\alpha = \theta(1 - \hbar B_\alpha\theta)^{-1}$, $[\theta_\alpha, \theta_\alpha] = 0$. The Poisson structures θ_α and θ_β are related on the intersection $U_\alpha \cap U_\beta$ as in (28)

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha}\theta_\beta)^{-1}, \tag{55}$$

with an exact $F_{\beta\alpha} = da_{\beta\alpha}$. Now we can use Kontsevich's formality [31] to obtain local star products $\star_\alpha$ and to construct for each intersection $U_\alpha \cap U_\beta$ the corresponding equivalence maps $\mathcal{D}_{\alpha\beta}$. See [19,18] for an explicit formula for the equivalence maps. According to our discussion in the previous section these $\mathcal{D}_{\alpha\beta}$, supplemented by trivial transition functions, define a collection of trivial line bundles $\mathcal{L}_{\beta\alpha}$. On each triple intersection we then have

$$\mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \mathrm{Ad}_{\star_\alpha}\Lambda_{\alpha\beta\gamma}. \tag{56}$$

It follows from the discussion after formula (36) that the collection of local functions $\{\Lambda_{\alpha\beta\gamma}\}$ represents a noncommutative gerbe (a deformation quantization of the classical gerbe represented by $\{\lambda_{\alpha\beta\gamma}\}$) if each of the central functions $\Lambda_{\alpha\beta\gamma\delta}$ introduced there can be chosen to be equal to 1. See [22, Section 5] and [32] that this is really the case. As mentioned at the end of the previous section, the non-triviality of the non-abelian 2-cocycle (43)–(45) can be seen as an obstruction to gluing the collection of local star products $\{\star_\alpha\}$, i.e., the collection of local rings $C^\infty(U_\alpha)[[\hbar]]$, into a global one. Hence, in the context of this section, this obstruction comes as a deformation quantization of the classical obstruction to gluing together local formal Poisson structures $\{\,,\,\}_\alpha$ into a global one.

5. Relation to [30] and to gerbes in the sense of Giraud, Deligne, Breen, and Brylinski

In the paper of Kashiwara [30] a 2-cocycle, similar to that of (43)–(45), represents an obstruction to gluing together certain local rings appearing in quantization of contact manifolds. In order to make closer contact with [30], and apply its results, we consider for each open U_α the corresponding sheaf of local rings $C^\infty(U_\alpha)[[\hbar]]$. The prestack of left $\star_\alpha$-modules $\mathfrak{M}_\alpha$ on U_α is actually a stack [33]. It follows from [18,34] that $\star_\alpha$ and $\star_\beta$ are Morita equivalent on $U_\alpha \cap U_\beta$. The Morita equivalence is given by the bimodule ${}_\alpha\mathcal{E}_\beta$ of sections of the noncommutative line bundle $\mathcal{L}_{\alpha\beta}$. Therefore, we have a functor (an equivalence of stacks) $\varphi_{\alpha\beta} : \mathfrak{M}_\alpha|U_\alpha \cap U_\beta \to \mathfrak{M}_\beta|U_\alpha \cap U_\beta$ defined by $\mathcal{L}_{\alpha\beta}$. Because of the Axiom 2 of Section 3, $\Lambda_{\alpha\beta\gamma}$ defines on $U_\alpha \cap U_\beta \cap U_\gamma$ an isomorphism of functors $\phi_{\alpha\beta\gamma} : \varphi_{\alpha\beta}\varphi_{\beta\gamma} \to \varphi_{\alpha\gamma}$, which due to Axiom 3 of Section 3 satisfies the associativity condition on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$. Then, according to [30] there exists, up to equivalence, a unique stack $\mathfrak{M}$ such that the stacks $\mathfrak{M}|U_\alpha$ and $\mathfrak{M}_\alpha$ are equivalent.

We now show that our noncommutative gerbe can be seen as a "standard gerbe" in the sense of [1,2,4,5], i.e., a gerbe understood as locally non-empty and locally connected stacks in groupoids. As already mentioned, the local functions $\Lambda_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta}$ satisfying relations (43)–(45) represent a honest non-abelian 2-cocycle as defined, e.g., in [5]. Due to the correspondence (in the sense of 2-categories, see [5] for details) between degree two non-abelian cohomology classes and equivalence classes of standard gerbes there exists a standard gerbe corresponding to the non-abelian 2-cocycle (43)–(45). We briefly describe this corresponding standard gerbe.

The collection of data consisting of an open covering $\{U_\alpha\}$ of M, local rings $C^\infty(U_\alpha)[[\hbar]]$, isomorphisms $\mathcal{D}_{\alpha\beta}$ and invertible sections $\Lambda_{\alpha\beta\gamma}$ satisfying 2-cocycle relations (43)–(45) (more precisely the data satisfying relations (34–36)) define up to equivalence an algebroid stack $\mathfrak{C}$ in the terminology of [32] (see also [27]) such that $\mathfrak{C}|_\alpha$ is equivalent to the stack of locally free $\star_\alpha$-modules of rank 1. If we think about this algebroid stack in terms of the corresponding pseudofunctor $U \mapsto \mathfrak{C}(U)$, we can consider in each category $\mathfrak{C}(U)$ its maximal subgroupoid. The associated stack to the corresponding substack is a standard gerbe. Hence noncommutative gerbes, that we introduced in this article as the deformation quantization of abelian gerbes and related to the obstruction of defining a global $\star$-product on M, can be seen as non-abelian gerbes in the standard sense of Giraud, Deligne, Breen, and Brylinski.

We finish with a short remark concerning the relation to the later paper [23], where a more general question of deformation of (descent data for) a special kind of stacks is considered. Results of [23] concerning deformations of gerbes (see Section 2.1. of [23] for a definition of gerbe used there) and the classification of deformations of gerbes (see Section 4 of [23]) apply to the deformation quantization of Hitchin gerbes as well. We then notice that similarly to [30] and also to the present paper, the deformation quantization of a gerbe leads to a "stack of algebras". It would be interesting to compare the approaches of [23] and of the present paper in more detail.

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